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A sensitivity measure of the Pareto set in a vector l_∞-extreme combinatorial problem

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Abstract

We consider a vector minimization problem on system of subsets of finite set with Chebyshev norm in a space of perturbing parameters. The behavior of the Pareto set as a function of parameters of partial criteria of the kind MINMAX of absolute value is investigated.

1. Base definitions and lemma

The traditional [1 – 11] statement of vector (n-criteria) trajectorial problem is following. A system of nonempty subsets $T \subseteq 2^E \backslash \emptyset$, $\mid T \mid > 1$ of the set $E = \{e_1, e_2, ..., e_m\}$ is given. A vector criterion

$$f(t, A) = (f_1(t, A_1), f_2(t, A_2), ..., f_n(t, A_n)) \to \min_{t \in T}$$

is defined, where $n \geq 1$, $m \geq 2$, A_i is the row of a matrix $A = [a_{ij}]_{n \times m} \in \mathbf{R}^{nm}$. The elements of set T are called trajectories.

We consider the case, where partial criteria are given by

$$f_i(t, A_i) = \max_{j \in N(t)} \mid a_{ij} \mid, \; i \in N_n,$$

where $N_n = \{1, 2, ..., n\}$, $N(t) = \{j \in N_m : e_j \in t\}$. By that, the value $f_i(t, A_i)$ is Chebyshev norm l_∞ of vector, formed by those elements of matrix A, which correspond to the trajectory t.

We define the Pareto set (the set of efficient trajectories) by traditional way [12,13]:

$$P^n(A) = \{t \in T : \pi(t, A) = \emptyset\},$$

where

$$\pi(t,A) = \{t' \in T : q(t,t',A) \geq 0_{(n)},\ q(t,t',A) \neq 0_{(n)}\},$$

$$q(t,t',A) = (q_1, q_2, ..., q_n),$$

$$q_i = q_i(t,t',A_i) = f_i(t,A_i) - f_i(t',A_i),\ i \in N_n,$$

$$0_{(n)} = (0,0,...,0) \in \mathbf{R}^n.$$

It is natural to call the problem of finding the set $P^n(A)$ the vector l_∞-extreme trajectorial problem. If E and T are fixed, we denote the problem by $Z^n(A)$. Let us assign the norm l_∞ for any natural number $k \in N$ in the space $\mathbf{R}^k$:

$$\| z \| = \max\{| z_i | : i \in N_k\},\ z = (z_1, z_2, ..., z_k) \in \mathbf{R}^k.$$

Under the norm of a matrix we understand the norm of the vector, formed by all its elements. For any number $\varepsilon > 0$, let us define the set of perturbing matrices

$$\mathcal{B}(\varepsilon) = \{B \in \mathbf{R}^{nm} : \| B \| < \varepsilon\}.$$

By analogy with [1,2,9,14 – 19], we call the problem $Z^n(A)$ stable (on vector criterion), if

$$\exists \varepsilon > 0\ \forall B \in \mathcal{B}(\varepsilon)\ (P^n(A) \supseteq P^n(A+B)).$$

It is easy to see that the property of stability of the problem $Z^n(A)$ is a discrete analogue of upper semicontinuity property by Hauzdorf in a point $A \in \mathbf{R}^{nm}$ of the optimal mapping (see., for example, [15])

$$P^n : \mathbf{R}^{nm} \to 2^E.$$

This point-set (many-valued) mapping assign the Pareto set to any set of parameters (any matrix A).

As usual [1,2,16,20], we say that the value

$$\rho_1^n(A) = \begin{cases} \sup \Omega_1(A), & \text{if } \ \Omega_1(A) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega_1(A) = \{\varepsilon > 0 : \ \forall B \in \mathcal{B}(\varepsilon) \ (P^n(A) \supseteq P^n(A+B))\}$, is the stability radius of the problem $Z^n(A)$.

In other words, the stability radius is the limit level (in Chebyshev norm) of independent perturbations of matrix A elements, where new efficient trajectories do not appear.

It is natural to say that $\rho_1^n(A) = \infty$ in the case $\Omega_1(A) = \mathbf{R}_+$. Evidently, the problem $Z^n(A)$ is stable and its stability radius is infinite when equality $P^n(A) = T$ holds.

The problem $Z^n(A)$, is called nontrivial, if $\overline{P}^n(A) = T \backslash P^n(A) \neq \emptyset$.

As in [2,3,5,6,16,17,8,21], we call the problem $Z^n(A)$ quasistable, if the formula

$$\exists \varepsilon > 0 \ \forall B \in \mathcal{B}(\varepsilon) \ (P^n(A) \subseteq P^n(A+B))$$

is valid.

Note, that the property of quasistability of the problem $Z^n(A)$ is a discrete analogue of lower semicontinuity property (by Hauzdorf) of the many-valued mapping, that assign the Pareto set $P^n(A)$ to any matrix $A \in \mathbf{R}^{nm}$.

The value

$$\rho_2^n(A) = \begin{cases} \sup \Omega_2(A), & \text{if } \ \Omega_2(A) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega_2(A) = \{\varepsilon > 0 : \ \forall B \in \mathcal{B}(\varepsilon) \ (P(A) \subseteq P(A+B))\}$, is called the quasistability radius of the problem $Z^n(A), \ n \geq 1$.

By that, the quasistability radius defines the limit of independent perturbations, that retain all the efficient trajectories of initial problem and allow the appearance of new trajectories.

Let $t = \{e_{j_1}, e_{j_2}, ..., e_{j_s}\} \in T, \ s = \mid t \mid, \ j_1 < j_2 < ... < j_s$. If we put

$$t[A_i] = (a_{ij_1}, a_{ij_2}, ..., a_{ij_s})$$

for any index $i \in N_n$, then we obtain

$$f_i(t, A_i) = \parallel t[A_i] \parallel \leq \parallel A_i \parallel \leq \parallel A \parallel . \quad (1)$$

Consequently, the next inequalities are correct:

$$f_i(t, A_i) - f_i(t, B_i) \leq f_i(t, A_i + B_i) \leq f_i(t, A_i) + f_i(t, B_i). \qquad (2)$$

Really, the right inequality is the axiom (triangle inequality)

$$\| a + b \| \leq \| a \| + \| b \|, \ a, \ b \in \mathbf{R}^n,$$

true for any norm. Setting $a = a' - b'$, $b = b'$, we get

$$\| a' \| - \| b' \| \leq \| a' - b' \|, \ a', \ b' \in \mathbf{R}^n,$$

which proves the left inequality of (2).

Lemma 1 *If trajectories t and t' and a matrix $B \in \mathbf{R}^{nm}$ are such that the inequality*

$$q_i(t, t', A_i) > 2 \| B \| \qquad (3)$$

holds for some index $i \in N_n$, then the inequality

$$q_i(t, t', A_i + B_i) > 0$$

is true.

Really, taking into account (1) and (2) combining them with (3), we easily obtain

$$q_i(t, t', A_i + B_i) = f_i(t, A_i + B_i) - f(t', A_i + B_i) \geq$$

$$\geq f_i(t, A_i) - f(t, B_i) - (f_i(t', A_i) + f_i(t', B_i)) \geq$$

$$\geq q_i(t, t', A_i) - 2 \| B_i' \| \geq q_i(t, t', A_i) - 2 \| B \| > 0.$$

2. Stability

Denote

$$\varphi_1^n(A) = \frac{1}{2} \min_{t \in \overline{P}^n(A)} \max_{t' \in T \setminus \{t\}} \min_{i \in N_n)} q_i(t, t', A_i).$$

Theorem 1 *For the stability radius $\rho_1^n(A)$ of vector nontrivial l_∞-extreme problem $Z^n(A),\ n \geq 1,$ the estimates*

$$\varphi_1^n(A) \leq \rho_1^n(A) \leq \frac{1}{2} \parallel A \parallel \qquad (4)$$

are valid.

Proof. It is clear that $\varphi := \varphi_1^n(A) \geq 0$. We prove the left inequality of (4) at first. If $\varphi = 0$, then inequality $\rho_1^n(A) \geq \varphi$ is evident.

Let $\varphi > 0,\ B \in \mathcal{B}(\varphi)$. Then, by definition of the number φ, for any trajectory $t \in \overline{P}^n(A)$ (the existence of such a trajectory is guaranteed by nontriviality of our problem), there exists a trajectory $t \neq t'$, such that the inequalities

$$q_i(t, t', A_i) \geq 2\varphi > 2 \parallel B \parallel$$

are valid for any index $i \in N_n$.

Therefore using the lemma we obtain, that inequality

$$q_i(t, t', A_i + B_i) > 0$$

holds for any index $i \in N_n$, i.e. $t' \in \pi(t, A + B)$. Consequently $t \in \overline{P}^n(A + B)$. Thus we have

$$\forall B \in \mathcal{B}(\varphi)\ (P^n(A) \supseteq P^n(A + B)).$$

Hence, $\rho_1^n(A) \geq \varphi_1^n(A)$.

Now we prove that the right inequality of (4) is valid. If we take the matrix $B^* = [b_{ij}]_{n \times m}$ with elements

$$b_{ij} = \begin{cases} \frac{1}{2} \parallel A \parallel - a_{ij}, & \text{if } a_{ij} \geq 0, \\ -\frac{1}{2} \parallel A \parallel - a_{ij}, & \text{if } a_{ij} < 0, \end{cases}$$

as perturbing, then it is easy to see, that

$$\forall i \in N_n\ \forall t \in T\ (f_i(t, A_i + B_i^*) = \frac{1}{2} \parallel A \parallel).$$

Consequently, taking into account the nontriviality of our problem, we obtain

$$P^n(A+B^*) = T \not\subseteq P^n(A).$$

This implies that the upper estimate

$$\rho_1^n(A) \leq \frac{1}{2} \parallel A \parallel$$

is valid.

Theorem 1 has been proved.

Corollary 1 *Stability radius of any nontrivial problem* $Z^n(A)$, $n \geq 1$, *is finite.*

The following two examples show, that lower and upper bounds of stability radius, stated by theorem 1, are accessible.

Example 1. Let $n = m = 2$,

$$A = \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}, \; T = \{t_1, t_2\}, \; t_1 = \{e_1\}, \; t_2 = \{e_2\}.$$

Then $f(t_1, A) = (1, 2)$, $f(t_2, A) = (2, 3)$, $P^2(A) = \{t_1\}$, $\varphi_1^2(A) = \frac{1}{2}$, $\parallel A \parallel = 3$. If $\frac{1}{2} < \beta < \varepsilon$, then $P^2(A+B^*) = \{t_2\}$, where

$$B^* = \begin{bmatrix} \beta & \beta \\ -\beta & \beta \end{bmatrix}.$$

Therefore we have

$$\forall \varepsilon > \frac{1}{2} \; \exists B^* \in \mathcal{B}(\varepsilon) \; (P^2(A+B^*) \not\subseteq P^2(A)).$$

Consequently $\rho_1^2(A) \leq \frac{1}{2}$.

Hence, taking into account theorem 1 we get $\rho_1^2(A) = \varphi_1^2(A) = \frac{1}{2}$.

Example 2. Let $n = m = 2$,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \; T = \{t_1, t_2\}, \; t_1 = \{e_1, e_2\}, \; t_2 = \{e_2, e_3\}.$$

Then we obtain $f(t_1, A) = (2,2)$, $f(t_2, A) = (0,2)$, $P^2(A) = \{t_2\}$, $\varphi_1^2(A) = 0$, $\| A \| = 2$.

It is easy to see that

$$q_1(t_1, t_2, A_1 + B_1) > 0, \; q_2(t_1, t_2, A_2 + B_2) = 0$$

for any matrix $B \in \mathcal{B}(1)$, i.e. $\rho_1^2(A) \geq 1$. Taking into account (4), we have $\rho_1^2(A) = 1 = \frac{1}{2} \| A \|$.

Let us also show, that the value of stability radius can differ from the upper and lower bounds from (4).

Example 3. Let $n = m = 2$,

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}, \; T = \{t_1, t_2\}, \; t_1 = \{e_1, e_2\}, \; t_2 = \{e_2, e_3\}.$$

In this case, $f(t_1, A) = (2,2)$, $f(t_2, A) = (1,2)$, $P^2(A) = \{t_2\}$, $\varphi_1^2(A) = 0$, $\| A \| = 2$ and, by theorem 1, we get $0 \leq \rho_1^2(A) \leq 1$. Let us show, that the stability radius of the problem is equal to $\frac{1}{2}$.

For any $\varepsilon > \frac{1}{2}$, there exists a matrix $B^* \in \mathcal{B}(\varepsilon)$, such that $t_2 \notin P^2(A + B^*)$. For example

$$B^* = \begin{bmatrix} \beta & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix},$$

where $\frac{1}{2} < \beta < \varepsilon$. Hence, $\rho_1^2(A) \leq \frac{1}{2}$.

On the other hand, for any matrix $B \in \mathcal{B}(\frac{1}{2})$ we have

$$q_1(t_1, t_2, A_1 + B_1) > 0, \; q_2(t_1, t_2, A_2, B_2) = 0.$$

Hence, $\{t_2\} \subseteq P^2(A + B) \; \forall B \in \mathcal{B}(\frac{1}{2})$, i.e. $\rho_1^2(A) \geq \frac{1}{2}$.

Thus, $\rho_1^2(A) = \frac{1}{2}$.

Let us assign for our problem $Z^n(A)$ the traditional Slater set (the set of weakly efficient trajectories) [12,13]:

$$S_1^n(A) = \{t \in T : \sigma_1(t, A) \neq \emptyset\},$$

where

$$\sigma_1(t, A) = \{t' \in T : \; q_i(t, t', A_i) > 0, i \in N_n\}.$$

It is obvious, that $P^n(A) \subseteq S_1^n(A)$.

From theorem 1 we easily derive

Corollary 2 *If $P^n(A) = S^n_1(A)$, then nontrivial problem $Z^n(A)$, $n \geq 1$, is stable.*

Let us illustrate that the equality $P^n(A) = S^n(A)$ is not a necessary condition for stability of a nontrivial problem $Z^n(A)$.

Example 4. Let $n = 2$, $m = 3$, $T = \{t_1, t_2, t_3\}$, $t_1 = \{e_1\}$, $t_2 = \{e_2\}$, $t_3 = \{e_3\}$,

$$A = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 2 & 4 \end{bmatrix}.$$

Then $f(t_1, A) = (1, 2)$, $f(t_2, A) = (2, 2)$, $f(t_3, A) = (3, 4)$, $P^2(A) = \{t_1\}$, $S^2_1(A) = \{t_1, t_2\}$, i.e. $P^2(A) \neq S^2_1(A)$. But nontrivial problem $Z^2(A)$ is stable, because $\rho^2_1(A) \geq \varphi^2_1(A) = \frac{1}{2}$.

3. Quasistability

Suppose

$$\varphi^n_2(A) = \frac{1}{2} \min_{t' \in P^n(A)} \min_{t \in T \setminus \{t'\}} \max_{i \in N_n} q_i(t, t', A_i).$$

It is easy to see, that $\varphi^n_2(A) \geq 0$ for any matrix $A \in \mathbf{R}^{nm}$.

Theorem 2 *For the quasistability radius $\rho^n_2(A)$ of vector l_∞-extreme problem $Z^n(A)$, $n \geq 1$, the estimate*

$$\rho^n_2(A) \geq \varphi^n_2(A) \tag{5}$$

is valid.

Proof. Let $\varphi := \varphi^n_2(A) > 0$ (inequality (5) is trivial in the other case). Then, by definition of the number φ, for any trajectory $t' \in P^n(A)$ and any trajectory $t \neq t'$ there exists a number $p \in \arg\max\{q_i(t, t', A_i) : i \in N_n\}$, such that

$$\forall B \in \mathcal{B}(\varphi)\ (q_p(t, t', A_p) \geq 2\varphi > 2 \parallel B \parallel).$$

Thus, by the lemma, we get

$$\forall B \in \mathcal{B}(\varphi)\ (q_p(t, t', A_p + B_p) > 0).$$

Therefore $t' \in P(A+B)$ for any matrix $B \in \mathcal{B}(\varphi)$. Thereby we have

$$\forall B \in \mathcal{B}(\varphi)\ (P^n(A) \subseteq P^n(A+B)).$$

Hence, $\rho_2^n(A) \geq \varphi_2^n(A)$.

Theorem 2 has been proved.

Next example shows that the lower bound of quasistability radius, stated by theorem 2, is accessible.

Example 5. Let $n = m = 2$, $T = \{t_1, t_2\}$, $t_1 = \{e_1\}$, $t_2 = \{e_2\}$,

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}.$$

Then we easily obtain

$$f(t_1, A) = (1, 3),\ f(t_2, A) = (2, 4),\ P^2(A) = \{t_1\},\ \varphi_2^2(A) = \frac{1}{2}.$$

If the perturbing matrix is of the kind

$$B^* = \begin{bmatrix} -\beta & -\beta \\ \beta & \beta \end{bmatrix},$$

where $\frac{1}{2} < \beta < \varepsilon$, then $P^2(A + B^*) = \{t_2\}$, i.e. $P^2(A) \not\subseteq P^2(A + B^*)$ for some perturbing matrix $B^* \in \mathcal{B}(\varepsilon)$. Hence, $\rho_2^2(A) = \varphi_2^2(A) = \frac{1}{2}$.

Let us show that the quasistability radius can be greater then lower bound, stated by theorem 2.

Example 6. Let $n = 2,\ m = 3$,

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix},$$

$T = \{t_1, t_2, t_3\}$, $t_1 = \{e_1, e_2\}$, $t_2 = \{e_2\}$, $t_3 = \{e_3\}$. In this case we have $\| A \| = 3$, $f(t_1, A) = f(t_2, A) = (1, 1)$, $f(t_3, A) = (3, 3)$, $P^2(A) = \{t_1, t_2\}$. It is easy to see that, for any matrix $B \in \mathcal{B}(\frac{1}{2})$,

$$q_i(t_1, t_2, A_i + B_i) = 0,\ q_i(t_1, t_3, A_i + B_i) < 0,\ q_i(t_2, t_3, A_i + B_i) < 0,$$

$$i \in N_2.$$

Hence, $\{t_1, t_2\} \subseteq P^2(A+B)$. Thus the problem $Z^2(A)$ is quasistable, i.e. $\rho_2^2(A) > 0$. On the other hand $q(t_1, t_2, A) = (0,0)$, and therefore $\varphi_2^2(A) = 0$.

For the problem $Z^n(A)$ we assign also the Smale set (or the set of strictly efficient trajectories) [12,13]:

$$S_2^n(A) = \{t \in T : \sigma_2(t, A) = \emptyset\},$$

where

$$\sigma_2(t, A) = \{t' \in T : f(t, A) \geq f(t, A)\}.$$

It is clear, that $S_2^n(A) \subseteq P^n(A)$, and the Pareto set is always not empty, but the Smale set can be empty.

From theorem 2 we obtain

Corollary 3 *If equality $S_2^n(A) = P^n(A)$ holds, then the problem $Z^n(A)$, $n \geq 1$, is stable.*

Let us show, that the inequality $P^n(A) = S_2^n(A)$ is not a necessary condition of quasistability of the problem $Z^n(A)$.

Example 7. Let $n = 2$, $m = 4$, $T = \{t_1, t_2, t_3, t_4\}$, $t_i = \{e_i\}$, $i \in N_4$,

$$A = \begin{bmatrix} -1 & -2 & 2 & 2 \\ -2 & -1 & -1 & 3 \end{bmatrix}.$$

Then $f(t_1, A) = (1,2)$, $f(t_2, A) = (2,1)$, $f(t_3, A) = (2,1)$, $f(t_4, A) = (2,3)$, $P^2(A) = \{t_1, t_2, t_3\}$, $S_2^2(A) = \{t_1\}$, i.e. $P^2(A) \neq S_2^2(A)$. But, $\rho_2^2(A) \geq \varphi_2^2(A) = \frac{1}{2} > 0$, i.e. the problem $Z^2(A)$ is quasistable.

Remark 1. It is easy to see, that all essentials, illustrated by examples 1–7, can be shown for any number of criteria ($n > 2$).

Theorem 3 *For stability and quasistability radii of the vector l_∞-extreme problem $Z^n(A)$, $n \geq 1$, the next estimate is true:*

$$\rho_s^n(A) \geq \frac{1}{2} \min_{i \in N_n} \min_{1 \leq j < k \leq m} || \, a_{ij} \mid - \mid a_{ik} \, ||, \; s = 1, 2. \tag{6}$$

Proof. Let us consider the nontrivial case, i.e. where the right part of (6) is a positive number. Then, evidently, in any row of the matrix A, absolute values of elements are different in pairs. Therefore, taking into account partial the criteria definition, for any trajectories $t \neq t'$ and index $i \in N_n$ we obtain

$$q_i(t, t', A_i) = 0 \implies \forall B \in \mathcal{B}(\varphi_i) \ (q_i(t, t', A_i + B_i) = 0),$$

$$q_i(t, t', A_i) > 0 \implies \forall B \in \mathcal{B}(\varphi_i) \ (q_i(t, t', A_i + B_i) > 0),$$

where

$$\varphi_i = \frac{1}{2} \min_{1 \leq j < k \leq m} \| a_{ij} | - | a_{ik} \| .$$

Thereby, for any matrix $B \in \mathcal{B}(\varphi_i)$

$$sign \ q_i(t, t', A_i + B_i) = const.$$

Consequently, setting

$$\varphi = \min\{\varphi_i : i \in N_n\},$$

by definition of the Pareto set we get

$$\forall B \in \mathcal{B}(\varphi) \ (P^n(A) = P^n(A + B)).$$

Hence, inequalities (6) are valid.

Theorem 3 has been proved.

Examples 1 and 5 indicate that lower bound, stated by theorem 3, is accessible.

Theorem 3 implies

Corollary 4 *If, in any row of a matrix A, absolute values of elements are different in pairs, then the problem $Z^n(A)$, $n \geq 1$, is stable and quasistable.*

Remark 2. Evidently, that relations (1) and (2) are true for any norm l_k, $k \in N$ (not only for l_∞). Therefore it is easy to see, that theorems 1 and 2 hold for a vector l_k-extreme trajectorial problem with any index $k \in N$, i.e. for any $n-$criteria problem with partial criteria

$$\| t[A_i] \|_k \rightarrow \min_{t \in T},\ i \in N_n.$$

The norms must be the same, i.e. in the space of matrices $\mathbf{R}^{nm}$ it is also needed to use the norm l_k. In partial case, where $k = 1$, the partial criteria take the form

$$\sum_{j \in N(t)} | a_{ij} | \rightarrow \min_{t \in T},\ i \in N_n,$$

and in the space of perturbing parameters $\mathbf{R}^{nm}$ we have to assign the norm l_1 :

$$\| A \|_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} | a_{ij} | \ .$$

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