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Lower and upper chromatic numbers for BSTSs(2^h - 1) *

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Abstract

In [*Discrete Math.* **174**, (1997) 247-259] an infinite class of STSs(2^h-1) was found with the upper chromatic number $\bar{\chi} = h$. We prove that in this class, for all STSs($2^h - 1$) with $h < 10$, the lower chromatic number coincides with the upper chromatic number, i.e. $\chi = \bar{\chi} = h$; and moreover, there exists a infinite sub-class of STSs with $\chi = \bar{\chi} = h$ for any value of h.

1 Introduction

A *mixed hypergraph* [9, 10] is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the *vertex set* and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges* respectively. If $\mathcal{C} = \emptyset$, then $\mathcal{H}$ is called a $\mathcal{D}$-hypergraph; if $\mathcal{D} = \emptyset$, then $\mathcal{H}$ is called a $\mathcal{C}$-hypergraph; if $\mathcal{C} = \mathcal{D}$, then $\mathcal{H}$ is called a bi-hypergraph. A proper k-coloring of a mixed hypergraph is a mapping from the vertex set into a set of colors $\{1, 2, \ldots, k\}$ so that each $\mathcal{C}$-edge has at least two vertices with $\mathcal{C}$ommon color and each $\mathcal{D}$-edge has at least two vertices with $\mathcal{D}$istinct colors. A mixed hypergraph is *k-colorable (uncolorable)* if it has a proper coloring with at most k colors (it admits no colorings). A *strict* k-coloring is a proper coloring using all k colors. The minimum number of colors in a coloring of $\mathcal{H}$ is

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the *lower chromatic number* $\chi(\mathcal{H})$; the maximum number of colors in a strict coloring is the *upper chromatic number* $\bar{\chi}(\mathcal{H})$. From coloring view point, $\mathcal{D}$-hypergraphs coincide with classical hypergraphs [1] and their lower chromatic number coincides with the chromatic number introduced by Erdös and Hajnal in 1966 [3].

For each $k, 1 \leq k \leq n$, let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each $\mathcal{C}$- and each $\mathcal{D}$-edge. In fact, r_k is the number of different strict k-colorings of $\mathcal{H}$ if we ignore permutations of colors. The vector $R(\mathcal{H}) = (r_1, \ldots, r_n)$ is the *chromatic spectrum* of $\mathcal{H}$.

A Steiner Triple System STS(v) (also denoted as S$(2, 3, v)$) is defined as a pair $(X, \mathcal{B})$, where X is a finite set of v vertices and $\mathcal{B}$ is a family of subsets of X called blocks, (i.e. it is a hypergraph) and such that it has the following two properties: each block contains three vertices, and each two vertices of X belong to a single block in $\mathcal{B}$. As it is known, the value of v in an STS(v) cannot be arbitrary, but must be $v \equiv 1 \; or \; 3 \; (mod \; 6 \;)$.

In [6] STSs(v) are studied as two particular cases of mixed hypergraphs, the first called CSTSs(v) (Co-Steiner Triple Systems), in which all the blocks are $\mathcal{C}$-edges (also called co-edges), and the second called BSTSs(v) (Bi-Steiner Triple Systems) in which all the blocks are biedges (i.e. $\mathcal{C}$- and $\mathcal{D}$-edges at the same time). For both CSTSs and BSTSs two particular infinite subclasses of order $2^h - 1$ with upper chromatic number h have been studied.

A strict coloring using h colors may be viewed as a partition (*h-coloring*) of size h of the vertex set X, such that each $b \in \mathcal{B}$ meets exactly two color classes.

Determining the lower and upper chromatic numbers of BSTS is a complex problem. But even if known, they do not assure the existence of colorings using any intermediate number of colors because as it was discovered recently [7, 8], the chromatic spectra of mixed hypergraphs (even of 3-uniform bi-hypergraphs) may have gaps. We have no information about the gaps in the chromatic spectra of BSTSs.

In this paper we attempt to determine the lower chromatic number for BSTSs($2^h - 1$) with $\bar{\chi} = h$. In Section 2 we give preliminary results regarding BSTSs. In Section 3 we investigate about necessary conditions for the existence of the smallest value of h for which there exists a BSTS(2^h-1) with $\bar{\chi}=h$ and $\chi=h-1$. In Section 4 we prove that $\bar{\chi}=\chi=h$ for any BSTS(2^h-1) with $h<10$. Finally in Section 5 we show that all the point - line designs associated with a projective geometry over GF(2) are a subclass of BSTS(2^h-1) with $\bar{\chi}=h$, and we prove that for this subclass $\bar{\chi}=\chi=h$ for any value of h.

When $h\geq 10$ the problem to find out if the upper and lower chromatic numbers for these structures coincide, remains open.

2 Background

2.1 "$2v+1$ construction"

In this subsection we will illustrate a technique for the recurrent construction of STS(v). It is called $2v+1$ construction and allows to obtain an STS($2v+1$) from an STS(v).

Let $(X',\mathcal{B}')$ be an STS(v) with $|X'|=v$, X'' be a set of vertices disjoint from X' with a cardinality $|X''|=v+1$. Obviously, $v+1$ is an even number and so it is possible to consider a 1-factorization $\mathcal{F}=\{F_1,F_2,\ldots,F_v\}$ of a complete graph K_{v+1} on the set of vertices X''. Let us now define the family of triples $\mathcal{B}$ on the set $X'\cup X''$ as follows:

1. each triple belonging to $\mathcal{B}'$ belongs to $\mathcal{B}$;
2. if $z_i\in X'$ $(i=1,2,\cdots,v)$ and $y_1,\ y_2\in X''$, then $\{z_i,y_1,y_2\}\in\mathcal{B}$ if and only if $\{y_1,y_2\}\in F_i$.

It is easy to prove that $(X,\mathcal{B})$ is an STS($2v+1$) and that $(X',\mathcal{B}')$ is a subsystem of it, whereas X'' is a stable set.

Among the various possible 1-factorization of a complete graph on the vertex set X'' let us consider $\bar{\mathcal{F}}$, called "$K_{2k,2k}$", in which $X''=\tilde{\mathrm{X}}\cup\bar{\mathrm{X}}$ with $|\tilde{\mathrm{X}}|=|\bar{\mathrm{X}}|=2k$ and it consists of $4k-1$ 1-factors. The last $2k$ 1-factors of $\bar{\mathcal{F}}$ are identified by a 1-factorization of $K_{2k,2k}$ on $\tilde{\mathrm{X}}$ and $\bar{\mathrm{X}}$, whereas all the remaining 1-factors are the union of a 1-factor of

K_{2k} on $\tilde{X}$ and a 1-factor of K_{2k} on $\bar{X}$. *Table* 1 gives an example of a 1-factorization of the type "$K_{2k,2k}$" in which $k = 2$, $\tilde{X} = \{1, 2, 3, 4\}$ and $\bar{X} = \{5, 6, 7, 8\}$.

1 2	1 3	1 4	1 5	1 8	1 7	1 6
3 4	2 4	2 3	2 6	2 5	2 8	2 7
5 6	5 7	5 8	3 7	3 6	3 5	3 8
7 8	6 8	6 7	4 8	4 7	4 6	4 5

Table 1

2.2 Preliminary Results

The following theorem [5] gives the information about the parity of color classes in a strict coloring $\mathcal{P}$ of a BSTS.

Theorem 1 *If $\mathcal{P}$ is a strict coloring for BSTS(v), then there is precisely one coloring class having an odd number of vertices.*

□

The next three results [6] allow to determine the sharp upper bound for the upper chromatic number.

Theorem 2 *If $\mathcal{H}$ is a BSTS(v) or CSTS(v) such that $v \leq 2^h - 1$, then $\bar{\chi}(\mathcal{H}) \leq h$.*

□

Corollary 1 *If $\mathcal{H}$ is a BSTS(v) or CSTS(v) with $v \leq 2^h - 1$ and $\bar{\chi}(\mathcal{H}) = h$, then:*

1. $v = 2^h - 1$;
2. *in a strict coloring of $\mathcal{H}$ with h colors, the color classes have cardinalities:*

$$2^0,\ 2^1,\ 2^2,\ \cdots,\ 2^{h-1};$$

3. $\mathcal{H}$ is obtained from STS(3) by repeated application of $2v+1$ construction.

□

Theorem 3 *The upper chromatic number of BSTS or CSTS of order $2^h - 1$ is equal to h if and only if BSTS or CSTS is obtained from STS(3) by a sequence of $2v+1$ constructions. In this case the system contains $h-2$ subsystems of cardinality $2^i - 1$, $i = 2, 3, \dots, h-1$.*

□

Theorem 3 gives a possibility to identify two infinite classes of BSTSs and CSTSs with cardinality $2^h - 1$ and an upper chromatic number $\bar{\chi} = h,\ h = 2, 3, \dots$.

3 Lower chromatic number for BSTSs($2^h - 1$)

In this section we deal with the problem of determining the lower chromatic number for BSTSs characterized by Theorem 3. Evidently, $\chi = 1$ for any CSTSs .

Let us consider now the class of BSTSs($2^h - 1$) obtained from STS(3) by a sequence of h recursive $2v+1$ constructions. For the trivial system BSTS($2^2 - 1$), $\bar{\chi} = \chi = 2$. In all the next theorems, let $\bar{h}$ be the smallest value of h for which there exists a BSTS($2^{\bar{h}} - 1$) denoted by $\bar{\mathcal{H}}$=(X,$\mathcal{B}$), with $\bar{\chi}(\bar{\mathcal{H}}) > \chi(\bar{\mathcal{H}})$.

By Theorem 3 and the hypotheses concerning $\bar{h}$, there exists a subsystem $\bar{\mathcal{H}}'$=(X',$\mathcal{B}'$), which is a BSTS($2^{\bar{h}-1} - 1$), and for which $\bar{\chi}(\bar{\mathcal{H}}') = \chi(\bar{\mathcal{H}}') = \bar{h} - 1$. Therefore $\chi(\bar{\mathcal{H}}) = \bar{h} - 1$. This means that there exist a strict coloring $\mathcal{P}$ of $\bar{\mathcal{H}}$ with $\bar{h} - 1$ colors. In $\mathcal{P}$, the vertices of $X \setminus X'$ receive the colors from the set of $\{1, 2, \dots, \bar{h} - 1\}$ colors. Notice that $\mid X - X' \mid = 2^{\bar{h}-1}$ and the $\bar{h} - 1$ color classes are defined by the subsystem $\bar{\mathcal{H}}'$. In addition, in this subsystem the cardinalities of color classes are $2^0,\ 2^1,\ 2^2,\ \cdots,\ 2^{\bar{h}-2}$. Let us number the colors in this order and use $[i]$, $1 \le i \le \bar{h} - 1$, to denote the i-th color from $\bar{h} - 1$

colors of $\mathcal{P}$. Denote also by X_i, $1 \leq i \leq \bar{h}-1$, the sets of vertices of $X - X'$ colored with the color $[i]$ and let $\mid X_i \mid = x_i$. In this notation, the color classes of $\mathcal{P}$ have the cardinalities $2^{i-1} + x_i$. Further we use the vector $(x_1, x_2, \cdots, x_{\bar{h}-1})$ to identify the strict coloring $\mathcal{P}$. The next theorem gives a necessary condition for the existence of $\mathcal{P}$.

Theorem 4 *If $\bar{\mathcal{H}}$ is the BSTS$(2^{\bar{h}}-1)$ with the smallest value of $\bar{h}$ such that $\bar{\chi} = \bar{h}$ and $\chi = \bar{h}-1$, then all the strict colorings with $\bar{h}-1$ colors satisfy the following equalities:*

$$\begin{cases} \sum_{i=1}^{\bar{h}-1} x_i^2 + \sum_{i=1}^{\bar{h}-1} 2^i x_i = 2^{2\bar{h}-2} \\ \\ \sum_{i=1}^{\bar{h}-1} x_i = 2^{\bar{h}-1}. \end{cases} \tag{1}$$

Proof. Since $|X - X'| = \sum_{i=1}^{\bar{h}-1} x_i = 2^{\bar{h}-1}$ it remains to show that the first equality of (1) holds. We have that $\bar{\mathcal{H}}$=(X,$\mathcal{B}$) which is a BSTS$(2^{\bar{h}}-1)$ is obtained from $\bar{\mathcal{H}}'$=(X',$\mathcal{B}'$) which is a BSTS$(2^{\bar{h}-1}-1)$ by a $2v+1$ construction. Let $\bar{\mathcal{F}}$ be the 1-factorization of $X - X'$ needed to obtain $\bar{\mathcal{H}}$ and let $F_l \in \bar{\mathcal{F}}$ be a generic 1-factor corresponding to the vertex $z_l \in X'$ colored with the color $[i]$.

We count the numbers of monochromatic and non monochromatic pairs in $X - X'$ using the factorization $\bar{\mathcal{F}}$. The total number of monochromatic pairs is $\sum_{i=1}^{\bar{h}-1} \binom{x_i}{2}$ because any two vertices of the same color from $X - X'$ are contained in one block with the third vertex from X'.

Further, if $x_i = 0$, then in F_l all the pairs are monochromatic (all these colors are different from $[i]$; no other x_j is odd). If $x_i > 0$, then in F_l there are x_i non monochromatic pairs of vertices (colored with two colors, one vertex being colored with the color $[i]$), while the remaining $2^{\bar{h}-2} - x_i$ pairs are again monochromatic.

Since the i-th color class in the subsystem $\bar{\mathcal{H}}'$ contains precisely 2^{i-1} vertices there are 2^{i-1} 1-factors in $\bar{\mathcal{F}}$ colored like F_l. Thus the total number of non monochromatic pairs is $\sum_{i=1}^{\bar{h}-1} 2^{i-1} x_i$. The number of all pairs in $\bar{\mathcal{F}}$ is $2^{\bar{h}-2}(2^{\bar{h}-1}-1)$. We therefore have:

$$\sum_{i=1}^{\bar{h}-1} \binom{x_i}{2} + \sum_{i=1}^{\bar{h}-1} 2^{i-1} x_i = 2^{\bar{h}-2}(2^{\bar{h}-1} - 1).$$

By simple calculation we obtain the first equality of (1) and the theorem follows.

□

Observation 1 It is important to point out that if there exists an $x_j = 0$, then all the $x_i > 0$ are even.

Numerical analysis of the system (1) shows that it does not admit integer solutions when $h \leq 5$, so with these values of h we have $\bar{\chi} = \chi = h$. With values of $h \geq 6$ the system does admit integer solutions, for example with $h = 6$ and 7 the vectors $(3, 8, 0, 4, 17)$ and $(0, 1, 11, 5, 15, 32)$ are two solutions of system (1), but by Observation 1 they do not determine the values of x_i in a strict coloring.

Another important necessary condition for the existence of a strict coloring of a BSTS$(2^{\bar{h}} - 1)$ is given by the following theorem.

Theorem 5 *If $\mathcal{P}$ is a strict coloring of BSTS$(2^{\bar{h}} - 1)$ with $x_i > 0$ and $x_j > 0$ for some $1 \leq i, j \leq \bar{h} - 1$, then $x_i \leq 2^{i-1} + 2^{j-1}$ and $x_j \leq 2^{i-1} + 2^{j-1}$.*

Proof. Let us fix an element $x' \in X_i$. It forms x_j bichromatic pairs colored with the colors $[i]$ and $[j]$. These pairs belong to separate 1-factors corresponding to the vertices $z_l \in X'$ colored either with the color $[i]$ or with the color $[j]$. The number of these factors is $2^{i-1}+2^{j-1}$, so $x_j \leq 2^{i-1} + 2^{j-1}$. In the same way starting with $x'' \in X_j$ we obtain that $x_i \leq 2^{i-1} + 2^{j-1}$.

□

The following two theorems give important characterizations of the possible colorings of the system BSTS$(2^{\bar{h}} - 1)$ and are of fundamental importance in determining the main results reported on in this paper.

Theorem 6 *If $\mathcal{P}$ is a strict coloring of the system $BSTS(2^{\bar{h}}-1)$ using $\bar{h}-1$ colors, then there exists at least one $x_i = 0$ and all the $x_j > 0$ are even.*

Proof. It is obvious that if some $x_i = 0$, then by Observation 1, all the $x_j > 0$ are even. Let us assume from the contrary that $x_i > 0$ for all $1 \leq i \leq \bar{h}-1$. Recall that $\bar{h}$ is the smallest value of h for which there exists a system BSTS($2^{\bar{h}}-1$) denoted by $\bar{\mathcal{H}}$=(X,$\mathcal{B}$) with $\chi(\bar{\mathcal{H}}) = \bar{h}-1$. By Theorem 3, $\bar{\mathcal{H}}$ contains a sub-system BSTS($2^{\bar{h}-1}-1$) denoted by $\bar{\mathcal{H}}'$= (X',$\mathcal{B}'$), for which $\bar{\chi}(\bar{\mathcal{H}}') = \chi(\bar{\mathcal{H}}') = \bar{h}-1$. Let $X'' = X - X'$, with $\mid X'' \mid = 2^{\bar{h}-1}$.
If t is the number of vertices of X'' colored with the color $[\bar{h}-1]$, then $t \leq 2^{\bar{h}-2}$ holds. The vertex x' belonging to the generic X_i must be present in each of the $2^{\bar{h}-2}$ 1-factors corresponding to the vertices of X' colored with the color $[\bar{h}-1]$. Therefore it can form in them at most t bichromatic pairs of vertices of the type $[i]-[\bar{h}-1]$ and at least $2^{\bar{h}-2}-t$ monochromatic pairs of the type $[i]-[i]$. Thus we have $x_i \geq 2^{\bar{h}-2}-t+1$ where $1 \leq i \leq \bar{h}-2$.

Let us suppose $t \leq 2^{\bar{h}-2}-3$; thus we have

$$x_i \geq 2^{\bar{h}-2}-t+1 \geq 2^{\bar{h}-2}-2^{\bar{h}-2}+3+1 = 4,$$

that is, $x_i \geq 4$ where $1 \leq i \leq \bar{h}-2$. But by Theorem 5 we have $x_1 \leq 3$ and $x_2 \leq 3$ and this is a contradiction. Therefore, since $\mathcal{P}$ is a strict coloring, one of the following three cases remains to consider for t.

Case 1: $t = 2^{\bar{h}-2}-2$**,** so $x_i \geq 3$ and by Theorem 5 we have $x_1 = 3$ and $x_2 = 3$. All the bichromatic pairs of the type $[1]-[3]$ and $[2]-[3]$ belong to the 1-factors corresponding to the vertices of X' colored with the color [3], so $x_3 \geq 6$, but by Theorem 5 as $x_1 > 0$, we have $x_3 \leq 2^0 + 2^2 = 5$ and this is a contradiction.

Case 2: $t = 2^{\bar{h}-2}-1$, so $x_i \geq 2$. By Theorems 1 and 5 $x_1 = 3$ and $x_2 = 2$, but in this case in the 1-factors corresponding to the vertices of X' colored with the color [2] there exists a bichromatic pair in which the color [1] is present, which makes impossible to color the corresponding block correctly.

Case 3: $t = 2^{\bar{h}-2}$. In this case the last $2^{\bar{h}-2}$ 1-factors will be formed by bichromatic pairs of the type $[\bar{h}-1]-[i]$ with $i \neq \bar{h}-1$, while in the remaining 1-factors there are $2^{\bar{h}-4}$ monochromatic pairs of the type $[\bar{h}-1]-[\bar{h}-1]$ which cover all the pairs of $K_{X_{\bar{h}-1}}$.

$\bar{\mathcal{H}}'$ is not colorable with $\bar{h}-2$ colors, so this value of t is not acceptable either and the Theorem follows. □

Theorem 7 *Let $\mathcal{P}$ be a strict coloring of the system $BSTS(2^{\bar{h}}-1)$ using $\bar{h}-1$ colors, $x_i > 0$ and $x_l = 0$ for $l < i$ ($i = 1$ is possible), and let $x_j > 0$ and $x_k = 0$ for $i+1 \leq k < j$. Then $x_{j+t} > 0$ for all $j+t > i+1$.*

Proof. Let us assume from the contrary that $x_{j+t} = 0$. Observation 1 implies that all the $x_m > 0$ are even. Let us consider a vertex $x' \in X_i$ and a vertex $z' \in X'$ colored with the color $[l]$ or $[k]$: The pair $\{x', z'\}$ will be found in a block $\{x', x'', z'\}$ where $x'' \in X_i$. All this holds for each of the 2^{n-1} vertices colored with a color $[n]$ to which a $x_n = 0$ corresponds. Therefore we have

$$2^{j-1} - 1 - 2^{i-1} + 2^{j+t-1} \leq x_i \leq 2^{i-1} + 2^{j-1},$$

where the inequality on the right is true according to Theorem 5, x_i is even and so

$$2^{j-1} - 2^{i-1} + 2^{j+t-1} \leq x_i \leq 2^{i-1} + 2^{j-1},$$

After simple calculation we obtain:

$$2^{j+t-1} \leq 2^i,$$

or $j+t \leq i+1$, a contradiction. □

Corollary 2 *Let $\mathcal{P}$ be a strict coloring of the system $BSTS(2^{\bar{h}}-1)$ using $\bar{h}-1$ colors, and let $x_i > 0$ where $i > 1$ and $x_l = 0$ with $1 \leq l < i$. Then $x_i \geq 2^{l-1}$.* □

The following Proposition allows us to determine relations between the values $x_{\bar{h}-1}$ and $x_{\bar{h}-2}$.

Proposition 1 *If $\mathcal{P}$ is a strict coloring of a BSTS*$(2^{\bar{h}}-1)$ *using* $\bar{h}-1$ *colors, then the following inequality holds:*

$$2^{\bar{h}-2} \geq \lceil \frac{x_{\bar{h}-1}(x_{\bar{h}-1}-1)}{2^{\bar{h}-1}-2} \rceil + x_{\bar{h}-2}. \quad (2)$$

Proof.
In a factorization of X'' each 1-factor possesses $2^{\bar{h}-2}$ pairs of vertices. In the 1-factors corresponding to the vertices of X' colored with the color $[\bar{h}-2]$ there are at least $\lceil \frac{x_{\bar{h}-1}(x_{\bar{h}-1}-1)}{2^{\bar{h}-1}-2} \rceil + x_{\bar{h}-2}$ pairs, and this proves (2). □

4 BSTSs(2^h-1) with $h<10$

Numerical analysis (by exhaustive computer search) of system (1), taking into account the conditions determined by Theorems 5 and 6, Corollary 2 and Proposition 1, did not give solutions $(x_1, x_2, \cdots, x_{h-1})$ corresponding to strict colorings when $h \leq 8$. When $h=9$ the following solutions were found:

1)	$(0,0,0,18,14,30,68,126)$
2)	$(0,0,0,16,20,22,70,126)$
3)	$(0,0,0,22,10,30,68,126)$
4)	$(0,0,6,6,20,30,68,126)$
5)	$(0,0,10,12,6,34,68,126)$
6)	$(0,4,4,10,18,26,66,128)$
7)	$(0,6,2,10,18,26,66,128)$

Table 2

The first five solutions in *Table* 2 do not satisfy Proposition 1, while it can be proved, using the same technique as used in Case 3, Theorem 6, that the last two do not determine strict colorings.

We can now enunciate the following Theorem.

Theorem 8 *For BSTSs*$(2^h - 1)$ *obtained by a sequence of* $2v + 1$ *constructions starting from STS(*3*) we have* $\bar{\chi} = \chi = h$ *for all* $h < 10$. □

5 BSTSs$(2^h - 1)$ from projective geometries

Here we prefer to consider a strict coloring of a BSTS which uses h colors as a partition $\mathcal{C} = \{X_1, X_2, \cdots, X_h\}$ of vertex set X. We say that $\mathcal{C}$ has size h. Note that any h-coloring $\mathcal{C}$ of a BSTS induces a coloring in any subsystem of it. If W is the vertex set of the subsystem, the induced coloring, denoted by $\mathcal{C}_{|W}$, is the set of all non-empty intersections $X_i \cap W$ with $X_i \in \mathcal{C}$. In this section we considered point-line designs associated with $PG(h, 2)$ as a BSTSs$(2^{h+1} - 1)$ obtained from particular sequences of "$2v + 1$ constructions" i.e., the point line design associated with $PG(h, 2)$ [2]. For these systems we have both lower and upper chromatic numbers equal to h for any value of h.

We prove in fact that, up to isomorphism, there exists exactly one BSTS$(2^{h+1} - 1)$ whose underlying STSs is $PG_1(h, 2)$. By "isomorphism" between two BSTSs we mean an isomorphism between their underlying STSs mapping color classes into color classes.

5.1 Uniqueness of a colouring of $PG_1(h, 2)$

First, we show that $PG_1(h, 2)$ admits at least one colouring.

Lemma 9 *For any flag* $f = (\pi_0, \pi_1, ..., \pi_{h-1}, \pi_h)$ *of* $PG(h, 2)$ *we have that* $\mathcal{C}(f) = \{\pi_0, \pi_1 - \pi_0, ..., \pi_i - \pi_{i-1}, ..., \pi_h - \pi_{h-1}\}$ *is a colouring of* $PG_1(h, 2)$.

Proof. It is easily seen by induction on h. □

We are going to show that any other colouring of $PG_1(h, 2)$ is of the form $\mathcal{C}(f)$ for some flag f.

Lemma 10 *Any colouring of* $PG_1(h,2)$ *is of the form* $\mathcal{C}(f) = \{\pi_0, \pi_1 - \pi_0, ..., \pi_i - \pi_{i-1}, ..., \pi_h - \pi_{h-1}\}$ *where* $f = (\pi_0, \pi_1, ..., \pi_{h-1}, \pi_h)$ *is a flag of* $PG(h,2)$.

Proof. By induction on h. The theorem is obviously true for $h = 0$ and $h = 1$. Assume the theorem true for $h = k$.

We have to show that a colour class of any colouring of $PG_1(k+1,2)$ is the complement of a hyperplane.

Let $\mathcal{C}$ be a colouring of $PG_1(k+1,2)$. By induction, $\mathcal{C}_{|\pi}$ is a $(k+1)$-colouring for any hyperplane π of $PG(k+1,2)$. Let α be a hyperplane and let $\mathcal{C}_{|\alpha} = \{C_1 \cap \alpha, ..., C_{k+1} \cap \alpha\}$ where the C_i's are colour-classes of $\mathcal{C}$. By induction, one class of $\mathcal{C}_{|\alpha}$, say $C_{k+1} \cap \alpha$, is $\alpha - \sigma$ where σ is a suitable $(k-1)$-dimensional subspace of α. Thus $\mathcal{C}_{|\sigma} = \{C_1 \cap \sigma, ..., C_k \cap \sigma\}$

Let β and γ be the other two hyperplanes containing σ.

If $\mathcal{C} = \{C_1, ..., C_{k+1}\}$, then at least one point $P \in \beta - \sigma$ is coloured $k+1$ otherwise $\mathcal{C}_{|\beta}$ would be a k-colouring. Analogously, at least one point $Q \in \gamma - \sigma$ is coloured $k+1$ otherwise $\mathcal{C}_{|\gamma}$ would be a k-colouring. On the other hand, the line through P and Q obviously has its third point in $\alpha - \sigma$ and hence all its points are coloured $k+1$, absurd.

By the above paragraph $\mathcal{C}$ has one more colour-class C_{k+2}. Let P be a point of C_{k+2}. Of course $P \in (\beta - \sigma) \cup (\gamma - \sigma)$.

Assume, for instance, that $P \in \beta - \sigma$. In this case no point of β is coloured $k+1$ otherwise $\mathcal{C}_{|\beta}$ would have size at least $k+2$.

Also, each point $Q \in \gamma - \sigma$ is coloured $k+1$ or $k+2$ otherwise the line through P and Q (whose third point is in $\alpha - \sigma$) would have points of pairwise distinct colours. On the other hand colours $k+1$ and $k+2$ cannot be present together in γ otherwise $\mathcal{C}_{|\gamma}$ would be a $(k+2)$-colouring. It follows that either $\gamma - \sigma \subset C_{k+1}$ or $\gamma - \sigma \subset C_{k+2}$.

If $\gamma - \sigma \subset C_{k+1}$, then C_{k+1} is the complement of β.

If $\gamma - \sigma \subset C_{k+2}$, then $\beta - \sigma$ is also contained in C_{k+2} otherwise we would have a line with points of pairwise distinct colours. So, C_{k+2} is the complement of α. The assertion follows. □

Considering that the automorphism group $PGL(h+1,2)$ of $PG_1(h,2)$ acts transitively on the flags of $PG(h,2)$, the above two lemmas allow to state

Theorem 11 *Up to isomorphism, there is exactly one colouring of* $PG_1(h, 2)$.

Corollary 3 *For any* h *the lower and upper chromatic numbers of* $PG_1(h, 2)$ *coincide and they are equal to* $\{h + 1\}$.

Observation 2 The full automorphism group of the only BSTS($2^{h+1} - 1$) associated with $PG(h, 2)$ obviously is the stabilizer of a flag of $PG(h, 2)$ under the action of $PGL(h + 1, 2)$.

Hence, it is isomorphic to the group of nonsingular $(h+1) \times (h+1)$ upper-triangular matrices over GF(2). Its order is $2^{h(h+1)/2}$.

6 Concluding remarks

Theorem 8 and 11 reinforce the importance of the strict colorings determined in [6] for BSTSs($2^h - 1$). These systems can, in fact, only be colored with strict colorings using h colors whose color classes have cardinalities of $2^0,\ 2^1,\ 2^2,\ \cdots,\ 2^{h-1}$.

The problem of determining whether Theorem 8 is true for any value of h, or whether there exists an $\bar{h}$ such that $\chi = \bar{h} - 1$ remains an open problem.

If it were true, the strict coloring using $\bar{h} - 1$ colors would be identified by the vector $(0, 0, \cdots, x_i, 0, \cdots, 0, x_j, x_{j+1}, x_{j+2}, \cdots, x_{\bar{h}+1})$, where $x_i \geq 0$ and $x_t > 0$ for $j \leq t \leq \bar{h} - 1$, and each x_k is even.

If it were possible to determine an $\bar{h}$, then we would have $\bar{\chi} \neq \chi$ for each $h \geq \bar{h}$. Starting from the system BSTS($2^{\bar{h}} - 1$), in fact, it would be possible to use a sequence of $2v + 1$ constructions based on 1-factorization of the type $K_{2^{k-1},2^{k-1}}$ on the disjoint sets $\tilde{X}$ and $\bar{X}$ with $X''=\tilde{X} \cup \bar{X}$ and $\mid \tilde{X} \mid=\mid \bar{X} \mid= 2^{k-1}$.

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