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A generalization of the optimality for multicriterion problems

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Abstract

A new notion of the optimality for multicriterion problems which generalizes the well-known notions of optimality in Pareto and Nash senses is considered and some properties of the introduced notions are studied.

1 Introduction

It is well-known that the most important classes of multicriterion problems is the class of problems for which optimality is considered in Pareto sense and the class of problems for which the optimality is considered in Nash sense. These classes of the multicriterion problems have a large implementation in the models of decision-making systems with non-coincidence interests of the participants[1–4].

In our paper we introduce some new notions of the optimality for multicriterion problems which generalize the notions of optimality in Pareto sense and in Nash sense. We have studied some properties of the introduced notions and suggest some possible implementations.

2 Definitions and some properties

Let X be an arbitrary set in R^n and on X we consider p real functions

$$f_1 : X \to R^1;\ f_2 : X \to R^1;\ \dots;\ f_p : X \to R^1.$$

To each function f_i a set $M_i \subseteq R^n,\ \ i = \overline{1,p}$ is associated.

Definition 1. The point $x^* \in X$ is called the minimal efficient point for the set of functions $f_1,\ f_2,\ \dots,\ f_p$ on X with respect to given sets $M_1,\ M_2,\ \dots,\ M_p$ if

$$f_i(x^*) \le f_i(x^* + x),\ \ \forall x \in M_i,\ x^* + x \in X,\ i = \overline{1,p}.$$

Definition 2. The point $x^0 \in X$ is called the maximal efficient point for the set of functions $f_1,\ f_2,\ \dots,\ f_p$ on X with respect to given sets $M_1,\ M_2,\ \dots,\ M_p$ if

$$f_i(x^0) \ge f_i(x^0 + x),\ \ \forall x \in M_i,\ x^0 + x \in X,\ i = \overline{1,p}.$$

Note that a similar notion was introduced in [5] where the sets $M_1, M_2, \dots, M_p$ are the conical sets. Here we consider that $M_1, M_2, \dots,$ M_p are arbitrary sets from R^n. So, if $M_i,\ i = \overline{1,p}$ are conical sets then these notions become the optimal solutions of the multicriterion problems introduced in [5].

Let us show that the introduced notions generalize the optimality in Nash sense for the game theory problems. Let $X = X_1 \times X_2 \times \dots \times X_p$, $X_i \in R^{n_i},\ i = \overline{1,p},\ x = (x_1,\ x_2,\dots,\ x_{i-1},\ x_i,\ x_{i+1},\ \dots,\ x_p)$, where $x \in X,\ x_i \in X_i,\ \sum_{i=1}^{p} n_i = n$. It is easy to observe that if we define $M_i,\ i = \overline{1,p}$ in the following way

$$M_i = \{x = (0, 0, \dots, x_i, 0, 0, \dots, 0) | x_i \in R^{n_i}\},\ \ i = \overline{1,p}, \tag{1}$$

where 0 is the vector with zero components ($0 \in R_{n_i}$), then the minimal (maximal) efficient point for the set of functions $f_1,\ f_2,\ \ldots,\ f_p$ on X with respect to given sets $M_1, M_2, \ldots, M_p$ becomes the optimal point in Nash sense of players for the game $G = (X_1, X_2, \ldots, X_n, f_1, f_2, \ldots, f_p)$ in normal form. [2,4]

Indeed, let $x^* = (x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_n^*)$ be the optimal point for the set of functions $f_1,\ f_2,\ \ldots,\ f_p$ and $M_i,\ i = \overline{1,p}$ is determinated according (1).
Then

$$f_i(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_n^*) \le$$

$$\le f_i(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^* + x_i, x_{i+1}^*, \ldots, x_n^*),$$

$$\forall x_i \in M_i,\ x_i^* + x_i \in X,\ i = \overline{1,p}.$$

But that means

$$f(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_n^*) \le$$

$$\le f(x_1^*, x_2^*, \ldots, x_{i-1}^*, y, x_{i+1}^*, \ldots, x_n^*),$$

$$\forall y \in X_i,\ i = \overline{1,p}$$

i. e. x^* is the optimal solution in Nash sense of the given game $G = (X_1, X_2, \ldots, X_n, f_1, f_2, \ldots, f_p)$ in normal form.

It is easy to observe too that if we find the minimal (maximal) efficient point $z^* = (x^*, y^*)$ ($z^0 = (x^0, y^0)$) on

$$Z = \{z = (x, y) \in R^{n+p} | x \in X, y = (y_1, y_2, \ldots, y_p), y_i = f_i(x), i = \overline{1,p}\}$$

for the set of functions $f_1,\ f_2,\ \ldots,\ f_p$ with respect to given conical sets

$$M_1 = \{(\underbrace{0,\ 0,\ \ldots,\ 0}_{n},\ \underbrace{y_1,\ 0,\ 0,\ \ldots,\ 0}_{p}) \in R^{n+p} \mid y_1 \in R^1\};$$

$$M_2 = \{(\underbrace{0,\ 0,\ \ldots,\ 0}_{n},\ \underbrace{0,\ y_2,\ 0,\ \ldots,\ 0}_{p}) \in R^{n+p} \mid y_2 \in R^1\};$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$M_p = \{(\underbrace{0,\ 0,\ \ldots,\ 0}_{n},\ \underbrace{0,\ 0,\ 0,\ \ldots,\ y_p}_{p}) \in R^{n+p} \mid y_p \in R^1\},$$

then x^* (x^0) is a minimal(maximal) efficient point for the set of functions $f_1, f_2, \ldots, f_p$ on X in Pareto sense. Indeed, in X the point $x \neq x^*$ for which $f_i(x) \leq f_i(x^*)$ and $f_{i_0}(x) < f_{i_0}(x^*)$ for some $i_0 \in \{1, 2, \ldots, p\}$ does not exist. So, if $z^* = (x^*, y^*)$ is a minimal efficient point for the set of functions $f_1, f_2, \ldots, f_p$ on Z with respect to $M_1,\ M_2,\ \ldots,\ M_p$ then x^* is a minimal efficient point in Pareto sense for the set of functions $f_1, f_2, \ldots, f_p$ on X.

3 The main result

The main results of the introduced notion are connected with the existence of the optimal solutions for considered multicriterion problems.

Let $M_1, M_2, \ldots, M_p$ be the conical sets. The set $M_i \subseteq R^n$, $i = \overline{1,p}$ we call the conical set if for every $x \in M_i$ and $t \in R^i$ we have $tx \in M_i$. Each conical set M_i, $i = \overline{1,p}$ consists of two cones M_i^+ and M_i^- with common vertex $0 = (0, 0, \ldots, 0)$, where $M_i^- = (M_i \backslash M_i^+) \cup \{0\}$. Note that the set M_i^+ we call the cone if for every $x \in M_i^+$ and $\lambda > 0$ we have $\lambda x \in M_i^+$, $-\lambda x \notin M_i^+$. If each part M_i^+ and M_i^- of conical set M_i is convex set then we say that M_i is convex conical set. An arbitrary function $\varphi(x)$ we call convex (concave) function on convex conical set M_i if $\varphi(x)$ is convex (concave) function on each part of convex cone M_i^+ and M_i^-.

The following theorems hold.

Theorem 1. Let X be non-empty convex compact set in R^n and $M_1,\ M_2,\ \ldots,\ M_p$ are non-empty convex classed conical sets in R^n. Moreover let us consider that $int(M_i) \neq 0,\ i = \overline{1,p}\ (p \leq n)$ and $M_i \cap M_j = \{0\}$, for $i \neq j$, where $0 = (\underbrace{0, 0, \ldots, 0}_{n})$. If $\varphi_i(x) = f_i(y + x)$ is a convex function on conical set M_i for every fixed $y \in X,\ i = \overline{1,p}$, then the minimal efficient point $x^* \in X$ for the set of functions $f_1, f_2, \ldots, f_p$ on X with respect to conical sets $M_1,\ M_2,\ \ldots,\ M_p$ exists, i. e. there is the point $x^* \in X$ for which the following condition is satisfied

$$f_i(x^*) \leq f_i(x^* + x),\ \ \forall x \in M_i,\ x^* + x \in X,\ i = \overline{1,p}.$$

This theorem can be proved by analogy as proof of Nash theorem [2,4] using the fix-point method.

Theorem 2. Let X be non-empty convex compact set in R^n and $M_1,\ M_2,\ \ldots,\ M_p$ are non-empty convex classed conical sets in R^n. Moreover let us consider that $int(M_i) \neq 0,\ i = \overline{1,p}\ (p \leq n)$ and $M_i \cap M_j = \{0\}$, for $i \neq j$, where $0 = (\underbrace{0, 0, \ldots, 0}_{n})$. If $\varphi(x) = f_i(y + x)$ is a concave function on convex set M_i for every fixed $y \in X,\ i = \overline{1,p}$, then the maximal efficient point $x^0 \in X$ for the set of functions $f_1,\ f_2,\ \ldots,\ f_p$ on X with respect to conical sets $M_1,\ M_2,\ \ldots,\ M_p$ exists, i. e. there is the point $x^0 \in X$ for which the following condition is satisfied

$$f_i(x^0) \geq f_i(x^0 + x),\ \ \forall x \in M_i,\ x^0 + x \in X,\ i = \overline{1,p}.$$

This theorem also can be proved by analogy as the proof of Nash theorem by using the fix-point method [2,4]. If the condition of the theorem 1 is satisfied then the minimal efficient point x^* for the set of functions $f_1,\ f_2,\ \ldots,\ f_p$ on X with respect to given conical sets $M_1,\ M_2,\ \ldots,\ M_p$ can be found in the following way.

Let x_{s_0} be an arbitrary point from X. If x_{s_0} satisfies the condition

$$f_i(x_{s_0}) \leq f(x_{s_0}+x),\ \ \forall x \in M_i,\ x_{s_0}+x \in X,\ i=\overline{1,p}, \qquad (2)$$

then x_{s_0} is a minimal efficient point for the set of functions f_1, f_2, ..., f_p on X with respect to conical sets M_1, M_2, ..., M_p. If the condition (2) is not satisfied then we find $i_0 \in \{1,2,\ldots,p\}$ for which $y \in M_{i_0}$ exists, such that

$$f_{i_0}(x_{s_0}) > f_{i_0}(x_{s_0}+y).$$

Denote by I_{s_0} the set of numbers i for which the condition (2) is satisfied. Let y^* be the solution of the problem:

$$minimize : f_{i_0}(x_{s_0}+y)$$

to subject

$$f_i(x_{s_0}) \leq f(x_{s_0}+x),\ \ \forall x \in M_i,\ x_{s_0}+x \in X,\ i \in I_{s_0}.$$

After that we change x_{s_0} by $x_{s_0}+y^*$. This procedure we repeat while the condition (2) is satisfied.

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