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On scalarization of vector discrete problems of majority choice*

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Abstract

A relationship between the set of majority efficient solutions of a multicriteria problem which has a finite set of vector estimators, and solutions of a scalar problem with the objective being the linear convolution of criteria ore transformed criteria, is investigated.

1 Introduction

Scalarization of a vector problem is one of the central methods in vector (multicriteria) optimization. The essence of scalarization consists in reduction of a vector problem to a scalar one with an aggregated (generalized) objective which is a convolution of criteria. It is known that, using the linear convolution of criteria (LCC), one is able to find the whole Pareto set (the set of efficient solutions) in the multicriteria problems of linear and convex programming (theorems due to Koopmans [1] and Karlin [2]). For the Slater set (the set of weakly efficient solutions), an analogous result was obtained by Hurwicz [3], Yu [4] and others; for the properly efficient solutions, it was established by Geoffrion [5], Hartley [6], and others. The mentioned results and also a series of other assertions concerning the conditions of effectiveness and solvability of the multicriteria problems by the LCC algorithm can be found in [7–26].

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Note that the linear convolution of transformed criteria and, in particular, the linear convolution of criteria in degrees [27–30] are used also to find the Pareto set of vector discrete problems.

The interest to the problem of scalarization is explained by the fact that reduction of a vector optimization problem to a scalar problem or a family of scalar optimization problems reveals possibilities to use scalar optimization methods well developed in vector optimization.

A natural question arises: can all the majority efficient solutions of a vector discrete problem be found by the LCC method? Such possibility is investigated in this paper. A necessary and sufficient condition, under which this problem is solvable by the LCC algorithm, is indicated. A series of sufficient conditions of a similar solvability is proposed.

2 Basic concepts

The mathematical formulation of a vector optimization problem assumes that a vector criterion

$$y = (y_1(x), y_2(x), ..., y_n(x)) : X \to \mathbf{R}^n,\ \ n \geq 2,$$

on a set of alternatives X is given. Its components, (partial) criteria, are considered to be minimized without loss of generality

$$y_i(x) \to \min_{x \in X},\ \ i \in N_n = \{1, 2, ..., n\}.$$

We denote by Y the set of vector estimations, i.e. the image of the set X in the criterial space $\mathbf{R}^n$

$$Y = \{y = y(x) :\ x \in X\} \subseteq \mathbf{R}^n.$$

In this paper, we assume Y to be a finite set containing $|Y| \geq 2$ elements. We will consider vector optimization problems out of dependence on the peculiarity of the vector criterion y. Therefore, we will consider a vector problem

$$y \to \min_{y \in Y},$$

which, taking into account the assumption of the finiteness of the set Y, is natural to call discrete.

The set of efficient solutions (the Pareto set) is defined [7–12] by the equality

$$P^n(Y) = \{y \in Y :\ \pi(y) = \emptyset\},$$

where

$$\pi(y) = \{y' \in Y :\ y \geq y' \ \&\ y \neq y'\}.$$

The set of majority efficient solutions [31–36] is given by the equality

$$M^n(Y) = \{y \in Y :\ \mu(y) = \emptyset\},$$

where

$$\mu(y) = \{y' \in Y :\ \sum_{i=1}^{n} \mathrm{sign}(y_i - y'_i) > 0\},$$

and, as usual,

$$\mathrm{sign} z = \begin{cases} 1 \text{ if } z > 0, \\ 0 \text{ if } z = 0, \\ -1 \text{ if } z < 0. \end{cases}$$

It is easy seen that $M^n(Y) \subseteq P^n(Y)$ for any $n \geq 2$, moreover,

$$M^2(Y) = P^2(Y). \tag{1}$$

Note that the Pareto set is always nonempty since the set Y is finite. However, it does not guarantee the non-emptiness of the set $M^n(Y)$ (see, e.g., [32–36]).

If the set $\mu(y)$ is written in the form

$$\mu(y) = \{y' \in Y :\ y \succ y'\},$$

where the binary relation $\succ$ in $\mathbf{R}^n$ is defined by the rule

$$y \succ y' \Longleftrightarrow \sum_{i=1}^{n} \mathrm{sign}(y_i - y'_i) > 0,$$

then the possible emptiness of the set $M^n(Y)$ can be explained by the fact that this binary relation is not always transitive for $n \geq 3$.

The famous voting paradox of J.C.Borda [37] and M.Condorcet [38] is connected with this fact. This paradox concerns XVIII century. An analysis of the collective choice and individual value proposed in 1952 by K.J.Arrow (see [31]) have become the foundation of a new axiomatic approach to understanding and investigation of the voting problem and the mechanism of taking social decisions. Note that the history of collective choice theory as the science is expounded in [39] (see also [35]) in detail.

It is evident that the majority relation introduced characterizes the procedure of making decisions by the majority of voices: a vector y is "preferred" to a vector y', if y surpasses y' in more components than y' surpasses y.

From now on we assume that $M^n(Y) \neq \emptyset$.

Following [36], we denote for any vector $z = (z_1, z_2, ..., z_n) \in \mathbf{R}^n$

$$k^+(z) = \mid N_n^+(z) \mid,$$

$$k^-(z) = \mid N_n^-(z) \mid,$$

$$N_n^+(z) = \{i \in N_n : \ z_i > 0\},$$

$$N_n^-(z) = \{i \in N_n : \ z_i < 0\},$$

$$N_n^0(z) = \{i \in N_n : \ z_i = 0\}.$$

Therefore, $N_n^+(z) \cup N_n^-(z) \cup N_n^0(z) = N_n$.

In this terms, it is easy to give the following equivalent definition of the majority efficient solutions

$$M^n(Y) = \{y \in Y : \ \forall y' \in Y \ (k^-(y - y') \geq k^+(y - y'))\}. \qquad (2)$$

Here, as usual, a vector difference $y - y'$ denotes the vector $(y_1 - y'_1, y_2 - y'_2, ..., y_n - y'_n)$.

Any vector problem (not necessary discrete) is widely known to satisfy the inclusion [9,10]

$$\Lambda^n(Y) \subseteq P^n(Y), \qquad (3)$$

where

$$\Lambda^n(Y) = \bigcup_{\lambda \in \Lambda_n} \Lambda^n(Y, \lambda),$$

$$\Lambda^n(Y, \lambda) = \arg\min\{\sum_{i=1}^{n} \lambda_i y_i : \ y \in Y\},$$

$$\Lambda_n = \{\lambda \in R^n : \ \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i > 0, \ i \in N_n\}.$$

The existence of vector discrete problems, for which the inclusion (3) is strict (see for instance [13–19,21,22]), is established. In that case, the vector problem of finding the Pareto set is said not solvable by the LCC algorithm. If $\Lambda^n(Y) = P^n(Y)$, then this problem is called solvable by the LCC algorithm. The problem of finding the set $M^n(Y)$ is naturally to call solvable by the LCC if $M^n(Y) \subseteq \Lambda^n(Y)$. Note that the relation $\Lambda^n(Y) \neq \emptyset$ is always valid for the vector discrete problems.

3 Solvability criterion

Let

$$C^n(Y) = \{y \in Y : \ \xi(y) = \emptyset\},$$

where

$$\xi(y) = \{y' \in \text{conv}Y : \ y \geq y' \ \& \ y \neq y'\},$$

conv$Y$ is the convex hull of the set Y in $\mathbf{R}^n$. Then the following lemma is valid.

Lemma 1 [19,20]. $C^n(Y) = \Lambda^n(Y)$.

Using this lemma, it is easy to prove the next criterion of solvability by the LCC algorithm of the problem of finding the set $M^n(Y)$.

Theorem 1 . $M^n(Y) \subseteq \Lambda^n(Y) \Longleftrightarrow \forall y \in M^n(Y) \ (\xi(y) = \emptyset)$.

The next example indicates that for $n \geq 3$ there exist problems such that $M^n(Y) \not\subseteq \Lambda^n(Y)$. For these problems there exists a solution $y \in M^n(Y)$ such that $\xi(y) \neq \emptyset$.

Example 1. Let $Y = \{y^{(1)}, y^{(2)}, y^{(3)}\} \subset \mathbf{R}^n,\ n \geq 3,$

$$y_i^{(1)} = \begin{cases} 2 & \text{if } i = 1, \\ 8 & \text{if } i \in \{2, 3, ..., n\}, \end{cases}$$

$$y_i^{(2)} = \begin{cases} 7 & \text{if } i \in \{1, 2\}, \\ 8 & \text{if } i \in \{3, 4, ..., n\}, \end{cases}$$

$$y_i^{(3)} = \begin{cases} 2 & \text{if } i = 2, \\ 8 & \text{if } i \in \{1, 3, 4, ..., n\}. \end{cases}$$

It is easy to check that $M^n(Y) = Y$.

Consider a vector $\tilde{y}$ with the elements

$$\tilde{y}_i = \begin{cases} 5 & \text{if } i \in \{1, 2\}, \\ 8 & \text{if } i \in \{3, 4, ..., n\}. \end{cases}$$

Since $\tilde{y} = \frac{1}{2}(y^{(1)} + y^{(3)})$, we have $\tilde{y} \in \mathrm{convY}$. By the obvious relations $\tilde{y} \leq y^{(2)}$ and $\tilde{y} \neq y^{(2)}$, we obtain $\tilde{y} \in \xi(y^{(2)})$, i.e. $\xi(y^{(2)}) \neq \emptyset$. Consequently, by theorem 1, we conclude that $M^n(Y) \not\subseteq \Lambda^n(Y)$ for $n \geq 3$.

Since equality (1) holds in the case of two criteria (n=2), in view of the result of the work [40], there exist bicriteria discrete problems such that

$$M^2(Y) \not\subseteq \Lambda^2(Y).$$

It is clear that a unique efficient solution ($|P^n(Y)| = 1$) can be found by the linear convolution, i.e., this solution belongs to the set $\Lambda^n(Y)$ that has only one element.

The next example shows that a unique majority efficient solution ($|M^n(Y)| = 1$) does not always belong to the set $\Lambda^n(Y)$.

Example 2. Let $n = 3,\ Y = \{y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\},\ y^{(1)} = (1, 8, 8),$ $y^{(2)} = (8, 1, 8),\ y^{(3)} = (8, 8, 1),\ y^{(4)} = (7, 7, 7).$

It is easy to check that $M^3(Y) = \{y^{(4)}\}$, $P^3(Y) = Y$.

Let $\tilde{y} = (\frac{17}{3}, \frac{17}{3}, \frac{17}{3})$. Since $\tilde{y} = \frac{1}{3}(y^{(1)} + y^{(2)} + y^{(3)})$, we see that $\tilde{y} \in$ convY. Therefore, by the inequalities $\tilde{y} \leq y^{(4)}$ and $\tilde{y} \neq y^{(4)}$, we have $\tilde{y} \in \xi(y^{(4)})$. So, $\xi(y^{(4)}) \neq \emptyset$, and in view of theorem 1, we obtain $y^{(4)} \notin \Lambda^3(Y)$. Consequently, the solution $y^{(4)}$ can not be found by the LCC.

4 Sufficient condition of solvability

As usual, we use the notation

$$\mathbf{R}^n_+ = \{z \in \mathbf{R}^n : \ z_i \geq 0, \ i \in N_n\}.$$

Theorem 2 . *Let $Y \subset \mathbf{R}^n_+$. If*

$$\forall y, y' \in Y \ \forall i \in N_n \quad (y_i > y'_i \Longrightarrow y_i \geq 2y'_i), \tag{4}$$

then $M^n(Y) \subseteq \Lambda^n(Y)$.

Proof. Let $\hat{y} = (\hat{y}_1, \hat{y}_2, ..., \hat{y}_n) \in M^n(Y)$. Consider a vector λ with the components

$$\lambda_i = \frac{L}{\zeta_i}, \ i \in N_n, \tag{5}$$

where

$$L = \frac{1}{\sum\limits_{i=1}^{n} \frac{1}{\zeta_i}},$$

$$\zeta_i = \begin{cases} \hat{y}_i & \text{if } i \in N^+_n(\hat{y}), \\ \gamma & \text{if } i \notin N^+_n(\hat{y}), \end{cases}$$

$$\gamma = \min\{y_i : \ y \in Y, \ i \in N^+_n(y)\}. \tag{6}$$

The existence of the value γ is guaranteed by the conditions $|Y| \geq 2$ and $Y \subset \mathbf{R}^n_+$. It is obvious that $\lambda \in \Lambda_n$.

We will show that $\hat{y} \in \Lambda^n(Y, \lambda)$. To do this, let us divide the set Y into two disjoint subsets

$$Y_1 = \{y \in Y : \ N^-_n(\hat{y} - y) \cup N^0_n(\hat{y} - y) = N_n\},$$

$$Y_2 = \{y \in Y : \ N_n^-(\hat{y} - y) \cup N_n^0(\hat{y} - y) \neq N_n\}.$$

It is easy to see that the equation

$$\sum_{i=1}^{n} \lambda_i \hat{y}_i \leq \sum_{i=1}^{n} \lambda_i y_i \tag{7}$$

holds for any solution $y \in Y_1$.

Let $y \in Y_2$. It is clear that $N_n^+(\hat{y} - y) \neq \emptyset$. The set $N_n^-(\hat{y} - y) \neq \emptyset$ since $\hat{y} \in M^n(Y)$. Therefore,

$$\sum_{i=1}^{n} \lambda_i(\hat{y}_i - y_i) = \sum_{i \in N_n^+(\hat{y}-y)} \lambda_i(\hat{y}_i - y_i) + \sum_{j \in N_n^-(\hat{y}-y)} \lambda_j(\hat{y}_j - y_j). \tag{8}$$

Let us evaluate the terms in the right part of this equality.

Taking into account (5), we obtain

$$\sum_{i \in N_n^+(\hat{y}-y)} \lambda_i(\hat{y}_i - y_i) \leq \sum_{i \in N_n^+(\hat{y}-y)} \lambda_i \hat{y}_i = Lk^+(\hat{y} - y). \tag{9}$$

Then let us evaluate the second term of the right part of (8).

Let $j \in N_n^-(\hat{y} - y)$. Then either $j \in N_n^+(\hat{y})$ or $j \notin N_n^+(\hat{y})$.

If $j \in N_n^+(\hat{y})$, then the inequality $\hat{y}_j - y_j \leq -\hat{y}_j$ holds due to condition (4). Therefore, we get by (5)

$$\lambda_j(\hat{y}_j - y_j) \leq -\lambda_j \hat{y}_j = -L.$$

If $j \notin N_n^+(\hat{y})$, then $j \in N_n^0(\hat{y})$. Therefore, in view of (5), we have

$$\lambda_j(\hat{y}_j - y_j) = -\lambda_j y_j = -L\frac{y_j}{\gamma}. \tag{10}$$

Since $j \in N_n^-(\hat{y} - y)$, it follows that $y_j > \hat{y}_j = 0$. Consequently, according to (6), the inequality $y_j \geq \gamma$ is true. Therefore, the inequality

$$\lambda_j(\hat{y}_j - y_j) \leq -L$$

follows from (10) and the obvious inequality $L > 0$.

Thus the inequality

$$\lambda_j(\hat{y}_j - y_j) \leq -L$$

holds for any index $j \in N_n^-(\hat{y} - y)$. From this, we deduce that

$$\sum_{j \in N_n^-(\hat{y}-y)} \lambda_j(\hat{y}_j - y_j) \leq -Lk^-(\hat{y} - y).$$

As a result, taking into account (8) and (9), we have

$$\sum_{i=1}^{n} \lambda_i(\hat{y}_i - y_i) \leq L(k^+(\hat{y} - y) - k^-(\hat{y} - y)).$$

From this, in view of (2) and $L > 0$, we had checked the correctness of inequality (7) for $y \in Y_2$.

Summarizing what has already been proved, we see that inequality (7) holds for any solution $y \in Y = Y_1 \cup Y_2$, i.e. $\hat{y} \in \Lambda^n(Y, \lambda)$. Consequently, $M^n(Y) \subseteq \Lambda^n(Y)$.

Theorem 2 is proved.

Taking into account (3), we obtain the next corollary directly from theorem 2.

Corollary 1 . *Let $Y \subset \mathbf{R}^n_+$. If $M^n(Y) = P^n(Y)$,then condition (4) is sufficient for the equality $M^n(Y) = \Lambda^n(Y)$ to be true.*

We show that formula (4) is not necessary for the inclusion $M^n(Y) \subseteq \Lambda^n(Y)$ to be true.

Example 3. Let $n = 3$, $Y = \{y^{(1)}, y^{(2)}, y^{(3)}\}$, $y^{(1)} = (8, 2, 8)$, $y^{(2)} = (8, 8, 2)$, $y^{(3)} = (7, 3, 3)$.

That $M^3(Y) = \{y^{(3)}\}$, $y^{(3)} \in \Lambda^3(Y, \lambda)$ where $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Thus $M^3(Y) \subseteq \Lambda^3(Y)$, but condition (4) is not valid.

Let us show that not any element of a set $\Lambda^n(Y)$ is majority efficient, i.e. the including $M^n(Y) \subseteq \Lambda^n(Y)$ can be strict.

Example 4. Let $Y = \{y^{(1)}, y^{(2)}, y^{(3)}\} \subset \mathbf{R}^3$, $y^{(1)} = (1, 1, 8)$, $y^{(2)} = (2, 2, 2)$, $y^{(3)} = (8, 1, 1)$.

It is clear that $P^3(Y) = Y$, $M^3(Y) = \{y^{(1)}, y^{(3)}\}$. The inclusion $M^3(Y) \subseteq \Lambda^3(Y)$ follows from theorem 2.

Let $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then by the obvious inequalities

$$\sum_{i=1}^{3} \lambda_i y_i^{(2)} < \sum_{i=1}^{3} \lambda_i y_i^{(1)} = \sum_{i=1}^{3} \lambda_i y_i^{(3)},$$

we conclude that $y^{(2)} \in \Lambda^3(Y, \lambda) \subseteq \Lambda^3(Y)$. Therefore, $y^{(2)} \in \Lambda^3(Y) \backslash M^3(Y)$.

Let us show that, in general, theorem 2 does not true for an arbitrary set $Y \subset \mathbf{R}^n$.

Example 5. Let $n = 3$, $Y = \{y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$, $y^{(1)} = (-9, -2, -2)$, $y^{(2)} = (-2, -9, -2)$, $y^{(3)} = (-2, -2, -9)$, $y^{(4)} = (-3, -3, -3)$.

It is easy to check that $M^3(Y) = \{y^{(4)}\}$, $P^3(Y) = Y$ and the set Y satisfies formula (4).

Let $\tilde{y} = (-\frac{13}{3}, -\frac{13}{3}, -\frac{13}{3})$. Since $\tilde{y} = \frac{1}{3}(y^{(1)} + y^{(2)} + y^{(3)})$, we have $\tilde{y} \in$ convY. Therefore, according to the inequalities $\tilde{y} \leq y^{(4)}$ and $\tilde{y} \neq y^{(4)}$, we see that $\tilde{y} \in \xi(y^{(4)})$. Thus $\xi(y^{(4)}) \neq \emptyset$. Due to theorem 1, we conclude that $M^3(Y) \not\subseteq \Lambda^3(Y)$.

The next example indicates that condition (4) of theorem 2, generally speaking, is not sufficient for any efficient solution to belong to the set $\Lambda^n(Y)$.

Example 6. Let $n = 3$, $Y = \{y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$, $y^{(1)} = (1, 1, 8)$, $y^{(2)} = (1, 8, 1)$, $y^{(3)} = (8, 1, 1)$, $y^{(4)} = (4, 4, 4)$.

It is easy to see that $P^3(Y) = Y$ and the set Y satisfies the conditions of theorem 2. But, since the vector

$$\tilde{y} = (\frac{10}{3}, \frac{10}{3}, \frac{10}{3}) = \frac{1}{3}(y^{(1)} + y^{(2)} + y^{(3)})$$

satisfies the inequalities

$$\tilde{y} \leq y^{(4)} \text{ and } \tilde{y} \neq y^{(4)},$$

we obtain $\tilde{y} \in \xi(y^{(4)})$, i.e., $\xi(y^{(4)}) \neq \emptyset$. From this, by lemma 1, it follows that $y^{(4)} \notin \Lambda^3(Y)$, but $y^{(4)} \in P^3(Y)$.

Only tightening of the demands to sufficient condition (4) leads to the aim. Namely, the next theorem holds.

Theorem 3 [28]. *Let $Y \subset \mathbf{R}^n_+$. For $P^n(Y) = \Lambda^n(Y)$, it is sufficient for the formula*

$$\forall y, y' \in Y \ \forall i \in N_n \quad (y_i > y'_i \Longrightarrow y_i \geq n y'_i)$$

to be true.

From theorem 2 and theorem 1 [19], we obtain the following corollary.

Corollary 2 . *Under the conditions of theorem 2, the inclusion*

$$M^n(Y) \subseteq P(\mathrm{convY}) \cap Y$$

holds.

5 Linear convolution of transformed criteria

As we have established above, some vector discrete problems have majority efficient solutions that can not be found by the linear convolution of criteria. So it is necessary to use the linear convolution of transformed criteria, as it was done in [28] for finding Pareto optima.

Let $\alpha > 0$, $y = (y_1, y_2, ..., y_n)$. We will consider expressions $y[\alpha]$ and $Y[\alpha]$ as the corresponding notations for the vector $(y_1^\alpha, y_2^\alpha, ..., y_n^\alpha)$ and the set $\{z \in \mathbf{R}^n : \ z = y[\alpha], \ y \in Y\}$.

It is clear that $Y[\alpha] \subset \mathbf{R}^n_+$ for $Y \subset \mathbf{R}^n_+$.

Since the equations

$$k^+(y - \tilde{y}) = k^+(y[\alpha] - \tilde{y}[\alpha]),$$

$$k^-(y - \tilde{y}) = k^-(y[\alpha] - \tilde{y}[\alpha]),$$

hold for any $y, \tilde{y} \in Y \subset \mathbf{R}^n_+$, according to definition (2), the following lemma is obvious.

Lemma 2 . *If $Y \subset \mathbf{R}^n_+$, then for any number $\alpha > 0$ a solution y belongs to the set $M^n(Y)$ if and only if $y[\alpha] \in M^n(Y[\alpha])$.*

Theorem 4 . *Let $Y \subset \mathbf{R}^n_+$. If there exists a number*

$$\alpha^* = \log 2 / \log \min\{\frac{y_i}{\hat{y}_i} > 1 : \; y, \hat{y} \in Y, \; i \in N_n\}, \tag{11}$$

then for any solution $\tilde{y} \in M^n(Y)$ and any number $\alpha \geq \alpha^*$ there exists a vector $\lambda \in \Lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_i^\alpha = \min\{\sum_{i=1}^{n} \lambda_i y_i^\alpha : \; y \in Y\}. \tag{12}$$

Thereby, this theorem states that the problem of finding the set $M^n(Y)$ is solvable by the linear convolution of criteria under the mentioned conditions, i.e. $M^n(Y) \subseteq \Lambda^n(Y)$. At the same time, this theorem (as the theorem 6 stated below) can be interpreted as a necessary condition of majority efficiency of a solution in the indicated class of vector problems.

Proof. Directly from formula (11) we easily obtain

$$\forall \alpha \geq \alpha^* \; \forall i \in N_n \; \forall y[\alpha], \hat{y}[\alpha] \in Y[\alpha] \; (y_i^\alpha > \hat{y}_i^\alpha \Longrightarrow y_i^\alpha \geq 2\hat{y}_i^\alpha).$$

From this, by theorem 2, we obtain

$$M^n(Y[\alpha]) \subseteq \Lambda^n(Y[\alpha]). \tag{13}$$

Let $\tilde{y} \in M^n(Y)$. Then, in view of lemma 2, we have $\tilde{z} = \tilde{y}[\alpha] \in M^n(Y[\alpha])$, i.e., by (13), it follows that $\tilde{z} \in \Lambda^n(Y[\alpha])$. This mean that there exists a vector $\lambda \in \Lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \tilde{z}_i = \min\{\sum_{i=1}^{n} \lambda_i z_i : \; z \in Y[\alpha]\}.$$

This equality is equal to (12).

Theorem 4 is proved.

Remark 1 . *Consider the set Y defined in example 4. Then, by (11), we obtain $\alpha^* = 1$. Under $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\alpha \geq 1$, the relations*

$$\sum_{i=1}^{3} \lambda_i (y_i^{(2)})^\alpha < \sum_{i=1}^{3} \lambda_i (y_i^{(1)})^\alpha = \sum_{i=1}^{3} \lambda_i (y_i^{(3)})^\alpha$$

are obvious. Therefore, $y^{(2)} \in \Lambda^3(Y) \backslash M^3(Y)$. Thus the conditions of theorem 4 are not sufficient for the majority efficiency of a solution.

A necessary addition to this theorem is following.

Theorem 5 . *Let $Y \subset \mathbf{R}^n_+$. If there does not exist number α^* satisfying equation (11), then the problem of finding the set $M^n(Y)$ is solvable by the linear convolution of criteria.*

Actually, the number α^* does not exists only if

$$\forall y \in Y \ \forall i \in N_n \ (y_i \in \{0, a_i\}),$$

where $a_i > 0, i \in N_n$. But then the conditions of theorem 2 are hold for the set Y, and, consequently, $M^n(Y) \subseteq \Lambda^n(Y)$.

Note that the results analogous to theorem 4 were obtained in [27,28] for the vector problems of boolean and discrete programming respectively, which consist in finding the Pareto set.

For any number $\beta > 0$ and any solution $y = (y_1, y_2, ..., y_n) \in Y$, we put

$$y\langle\beta\rangle = (\beta^{y_1}, \beta^{y_2}, ..., \beta^{y_n}),$$

$$Y\langle\beta\rangle = \{z \in \mathbf{R}^n : \ z = y\langle\beta\rangle, \ y \in Y\}.$$

It is clear that $Y\langle\beta\rangle \subset \mathbf{R}^n_+$ for $Y \subset \mathbf{R}^n_+$.

It is obvious that the equalities

$$k^+(y - \tilde{y}) = k^+(y\langle\beta\rangle - \tilde{y}\langle\beta\rangle),$$

$$k^-(y - \tilde{y}) = k^-(y\langle\beta\rangle - \tilde{y}\langle\beta\rangle)$$

hold for any solutions $y, \tilde{y} \in Y$. Thus the next lemma is valid.

Lemma 3 . *If $Y \subset \mathbf{R}^n_+$, then for any number $\beta > 0$ a solution y belongs to the set $M^n(Y)$ if and only if $y\langle\beta\rangle \in M^n(Y\langle\beta\rangle)$.*

Theorem 6 . *Let $Y \subset \mathbf{R}^n_+$,*

$$\beta^* = 2^{\frac{1}{\gamma}}, \tag{14}$$

where

$$\gamma = \min\{y_i - \hat{y}_i > 0 : \ y, \hat{y} \in Y, \ i \in N_n\}.$$

If $\tilde{y} \in M^n(Y)$, then for any number $\beta \geq \beta^*$ there exists a vector $\lambda \in \Lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \beta^{\tilde{y}_i} = \min\{\sum_{i=1}^{n} \lambda_i \beta^{y_i} : \ y \in Y\}. \tag{15}$$

Proof. We first note that all the conditions mentioned above ($Y \subset \mathbf{R}^n_+, 1 < |Y| < \infty$) are sufficient for the existence of the number γ and hence for the existence of the number β^*.

Further, using formula (14), it is easy to deduce the statement

$$\forall \beta \geq \beta^* \ \forall i \in N_n \ \forall y\langle\beta\rangle, \dot{y}\langle\beta\rangle \in Y\langle\beta\rangle$$

$$(\beta^{y_i} > \beta^{\dot{y}_i} \Longrightarrow \beta^{y_i} \geq 2\beta^{\dot{y}_i}).$$

From it, by theorem 2, it follows that

$$M^n(Y\langle\beta\rangle) \subseteq \Lambda^n(Y\langle\beta\rangle). \tag{16}$$

Let $\tilde{y} \in M^n(Y)$. Then, in view of lemma 3, the vector $\tilde{z} = \tilde{y}\langle\beta\rangle \in M^n(Y\langle\beta\rangle)$. Hence, by (16), we have $\tilde{z} \in \Lambda^n(Y\langle\beta\rangle)$. Therefore, there exists a vector $\lambda \in \Lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \tilde{z}_i = \min\{\sum_{i=1}^{n} \lambda_i z_i : \ z \in Y\langle\beta\rangle\}.$$

This equality is equal to (15).

Theorem 6 is proved.

Remark 2 . *Consider the set Y defined in example 4. Then by (14) we obtain $\beta^* = 2$. Under the conditions $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\beta \geq 2$, the relations*

$$\sum_{i=1}^{3} \lambda_i \beta^{y_i^{(2)}} < \sum_{i=1}^{3} \lambda_i \beta^{y_i^{(1)}} = \sum_{i=1}^{3} \lambda_i \beta^{y_i^{(3)}}$$

are obvious. Consequently, $y^{(2)} \in \Lambda^3(Y) \backslash M^3(Y)$, i.e., the conditions of theorem 6 are not sufficient for the majority efficiency of a solution.

Note that a result that is analogous to theorem 6 was obtained in [28] for the vector discrete problem of finding the Pareto set.

In conclusion, we note that for the bicriteria discrete problem (n=2) the next theorem is followed from (1) and corollary 1 [40].

Theorem 7 . *The equality $M^2(Y) = \Lambda^2(Y)$ holds if and only if*

$$\forall y \in M^2(Y)\ \forall y', y'' \in Y\ \Bigg((y'_1 < y_1 < y''_1) \& (y''_2 < y_2 < y'_2) \Longrightarrow$$

$$\Longrightarrow \left| \begin{array}{cc} y_1 - y'_1 & y_2 - y'_2 \\ y''_1 - y'_1 & y''_2 - y'_2 \end{array} \right| \geq 0 \Bigg).$$

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