Computer Science Journal of Moldova, vol.8, no.2(23), 2000

The numerical identification of the mechanical systems with respect to natural frequences

C. Popa A. Sharaniuk

1 Introduction

The problems of adeguate mechanical models construction and parametrical identification of these models have a great significance for structure design. The ignorance of geometrical and physical characteristics of designed structures explains discrepancy between the theoretical solutions obtained on the basis of mathematical models and experimental data. Therefore it is interesting to elaborate some methods which would take into consideration experimental data of a real structure behavior for its mathematical model construction. A possible modality to tackle consists in determining and specifying geometrical and mechanical characteristics of the model on the basis of minimizing discrepancy between the state functions and functionals which characterize the structure dynamic coresponding to some data obtained by experimental way. Details of these numerical methods for linear systems are given in [2,4]. In this work some parametrical identification problems for structure in function of natural frequencies values of its free oscillations will be examined.

2 Identification of the mechanical systems with respect to natural frequences. The problem formulation

To apply the finite element method, the mathematical model for free oscillations of certain structure can be writted in the form:

$$\mathbf{M}\frac{\mathbf{d^2u}}{\mathbf{dt^2}}+\mathbf{Ku}=\mathbf{0}, \tag{1}$$

where $\mathbf{M}$ is a square matrix of order $\mathbf{N}$ which characterizes inertial qualities of the structure (mass matrix), $\mathbf{K}$ is a square matrix of the same order which describes rigidity qualities (stiffnes matrix) and $\mathbf{u(t)}=[\mathbf{u_1(t)},\ldots,\mathbf{u_N(t)}]^\mathbf{T}$ is the vector of nodal deplacements [1,6]. Mechanical systems for which matrixes $\mathbf{M}$ and $\mathbf{K}$ are symmetrical and pozitively definited will be examined. Considering the structure oscillations to be harmonious, we will represent the deplacements vector in the form:

$$\mathbf{u(t)}=\mathbf{Ue^{i\omega t}}, \tag{2}$$

where ω is the free oscillations frequency, $\mathbf{U}$ is the amplitude deplacements vector, $\mathbf{t}$ is the time and $\mathbf{i}$ is the imaginary unit. Substituting (2) into the movement equation (1) we arrive at the generalized problem of eigenvalues and eigenvectors:

$$\mathbf{KU}=\omega^\mathbf{2}\mathbf{MU}. \tag{3}$$

The matrixes $\mathbf{K}$ and $\mathbf{M}$ depend on the geometrical characteristics of structure (cross sections, geometrical configuration) and on the mechanical characteristics of the material (modulus of elastisity). Including this set of parameters in the vector $\mathbf{h}=[\mathbf{h_1},\ldots,\mathbf{h_{Ne}}]^\mathbf{T}$ and considering $\mathbf{K}=\mathbf{K(h)},\mathbf{M}=\mathbf{M(h)}$, we will rewrite the eigenvalues problem (3) in the form:

$$\mathbf{K(h)U}=\omega^\mathbf{2}\mathbf{M(h)U}. \tag{3a}$$

To any vector $\mathbf{h}$ there are corresponding natural frequences spectrum $\omega_\mathbf{1}^\mathbf{2},\cdots,\omega_\mathbf{N}^\mathbf{2}$ and the respective mode shapes. We will formulate the structure parametrical identification problem in the following way: to determine the design vector $\mathbf{h}$ for which the natural frequences of free oscillations $\omega_{\mathbf{1}}^{\mathbf{2}}, \cdots, \omega_{\mathbf{L}}^{\mathbf{2}}$ ($\mathbf{L} \leq \mathbf{N}, \mathbf{N}-$ number of degrees of freedom) have the following values $\tilde{\omega}_{\mathbf{1}}^{\mathbf{2}}, \cdots, \tilde{\omega}_{\mathbf{L}}^{\mathbf{2}}$ (the case of the simple eigenvalues is studied). With a view to mathematical formulation of the identification problem, we will introduce the functional

$$\mathbf{J}(\mathbf{h}) = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{L}} (\omega_{\mathbf{i}}^{\mathbf{2}} - \tilde{\omega}_{\mathbf{i}}^{\mathbf{2}})^{\mathbf{2}}, \tag{4}$$

which will be minimized by rational choosing of the vector $\mathbf{h}$. The component parts of the vector $\mathbf{h}$ satisfy the inequality restrictions

$$\mathbf{0} < \mathbf{h}_{\mathbf{i}}^{\mathbf{min}} \leq \mathbf{h}_{\mathbf{i}} \leq \mathbf{h}_{\mathbf{i}}^{\mathbf{max}}, \qquad \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{Ne}, \tag{5}$$

(consequence of some structure limitation) and, in addition, the integral restriction:

$$\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{Ne}} \mathbf{h}_{\mathbf{j}} \beta_{\mathbf{j}} = \mathbf{m}, \tag{6}$$

where the constant $\mathbf{m}$ is the mass of structure and the constants $\beta_{\mathbf{j}}$ caracterize the fixed dimensions of the structure elements. Therefore the mathematical formulation of the identification problem is the following:

$$\mathbf{h}^* : \mathbf{J}(\mathbf{h}) \longrightarrow \min_{\mathbf{h}}, \tag{7}$$

where functional $\mathbf{J}(\mathbf{h})$ is defined by the relation (4), the restriction on the design variables are given by (5) and (6) and the squares of frequencies $\omega_{\mathbf{1}}^{\mathbf{2}}, \cdots, \omega_{\mathbf{N}}^{\mathbf{2}}$ are found by solving the eigenvalues problem (3a). An ample study of this kind of optimal design problems of structures is given in [2,3].

3 Analysis of free oscilations' frequencies sensibility for the mechanical system. Functional's gradient

In this paragraph we will deduce the free oscillations frequencies sensibility formula of a mechanical system to the design parameters $\mathbf{h_i}$ changement and on this basis we will construct the numerical calculus algorithm in order to minimize the functional $\mathbf{J}(\mathbf{h})$ (4). With this purpose we will formulate the free oscillations problem of some structure in variational terms:

$$\omega_{\mathbf{i}}^{\mathbf{2}}(\mathbf{h}) = \min_{\mathbf{U}} \frac{(\mathbf{U_i}, \mathbf{K}(\mathbf{h})\mathbf{U_i})}{(\mathbf{U_i}, \mathbf{M}(\mathbf{h})\mathbf{U_i})}, \qquad \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{N}, \tag{8}$$

where

$$(\mathbf{U_i}, \mathbf{M}(\mathbf{h})\mathbf{U_k}) = \mathbf{0}, \qquad \mathbf{k} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{i} - \mathbf{1}. \tag{8a}$$

The variational formulation of this problem and its description in the form of the eigenvalues generalized problem (3a) are equivalent, i.e. the squares of natural frequencies and theirs corresponding mode shapes $\mathbf{U_1}, \mathbf{U_2}, \ldots, \mathbf{U_N}$ obtained by achievement of both methods coincide [3]. But in order to obtain the sensibility relation, the utilization of representation (8) is more convenient. The restriction (5) on design variables $\mathbf{h_i}$ may be exluded by use of Valentin's substitution for the parameters $\varphi_{\mathbf{i}}$:

$$\mathbf{h_i} = \frac{\mathbf{h_i^{max}} + \mathbf{h_i^{min}}}{\mathbf{2}} + \frac{\mathbf{h_i^{max}} - \mathbf{h_i^{min}}}{\mathbf{2}} \sin \varphi_{\mathbf{i}}. \tag{5a}$$

The variation of natural frequencies squares $\omega_{\mathbf{i}}^{\mathbf{2}}$, determined by the variation of the design vector $\mathbf{h}$, in view of representation (8) for $\omega_{\mathbf{i}}^{\mathbf{2}}$, can be written in the form:

$$\delta\omega_{\mathbf{i}}^{\mathbf{2}} = \frac{\mathbf{1}}{(\mathbf{U_i}, \mathbf{M}\mathbf{U_i})} \Big(\sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{Ne}} \{ (\mathbf{U_i}, \frac{\partial \mathbf{K}(\mathbf{h})}{\partial \mathbf{h_j}} \mathbf{U_i}) - \omega_{\mathbf{i}}^{\mathbf{2}} (\mathbf{U_i}, \frac{\partial \mathbf{M}(\mathbf{h})}{\partial \mathbf{h_j}} \mathbf{U_i}) \} \delta \mathbf{h_j} \Big). \tag{9}$$

The expression in braces represents natural frequencies squares $\omega_{\mathbf{i}}^{\mathbf{2}}$ sensibility to the change of parameter $\mathbf{h_i}$ by $\delta\mathbf{h_i}$. To solve the problem of functional (4) minimizing, we apply the gradient method, and, in order to take into consideration the equality restriction (6), we introduce the Lagrange's functional

$$\mathbf{J_L} = \mathbf{J(h)} + \mu(\sum_{\mathbf{j=1}}^{\mathbf{Ne}} \beta_{\mathbf{j}}\mathbf{h_j} - \mathbf{m}), \tag{10}$$

where μ is the Lagrange multiplier. Equalizing the primary variation of the Lagrange's functional with zero, we will obtain the necessary condition of the optimum existence

$$\delta\mathbf{J_L} = \mathbf{0}, \tag{11}$$

where

$$\delta\mathbf{J_L} = \delta\mathbf{J} + \mu\sum_{\mathbf{i=1}}^{\mathbf{Ne}} \beta_{\mathbf{i}}\delta\mathbf{h_i}, \tag{11a}$$

and $\delta\mathbf{J}$ is the functional variation (4) in the vector $\mathbf{h}$ direction:

$$\delta\mathbf{J} = \mathbf{2}\sum_{\mathbf{i=1}}^{\mathbf{L}} (\omega_{\mathbf{i}}^{\mathbf{2}} - \tilde{\omega}_{\mathbf{i}}^{\mathbf{2}})\delta\omega_{\mathbf{i}}^{\mathbf{2}}. \tag{12}$$

The variation $\delta\omega^{\mathbf{2}}$ in (12) is given by the relation (9), which is true only in case of simple eigenvalues [4]. By virtue of differenciability by Frechet, we will represent the variation $\delta\mathbf{J_L}$ in the form:

$$\delta\mathbf{J_L} = (\mathbf{\Lambda(h, U)}, \delta\mathbf{h}), \tag{13}$$

where $\mathbf{\Lambda(h, U)}$ is the gradient of the Lagrange's functional (10). In concordance with introduced notations, the necessary condition of optimum existence can be written in the form:

$$\mathbf{\Lambda(h, U)} = \mathbf{0}, \tag{14}$$

where, taking into consideration (9) and (10), the component $\mathbf{\Lambda_j}$ is written in the followng manner:

$$\begin{aligned}\mathbf{\Lambda_j(h,U)} &= -\mathbf{2}\sum_{\mathbf{i=1}}^{\mathbf{L}}\frac{(\omega_\mathbf{i}^\mathbf{2}-\tilde{\omega}_\mathbf{i}^\mathbf{2})}{\mathbf{U_i^T,MU_i})}[(\mathbf{U_i},\frac{\partial\mathbf{K(h)}}{\partial\mathbf{h_j}}\mathbf{U_i})- \\ &\quad -\omega_\mathbf{i}^\mathbf{2}(\mathbf{U_i},\frac{\partial\mathbf{M(h)}}{\partial\mathbf{h_j}}\mathbf{U_i})]+\mu\beta_\mathbf{j}. \end{aligned} \tag{15}$$

For the convenience of subsequent exposition, we will represent the gradient in the form:

$$\mathbf{\Lambda(h,U)}=\mathbf{\Lambda_0(h,U)}+\mu\beta, \tag{16}$$

where the vector $\beta^\mathbf{T}=[\beta_\mathbf{1},\beta_\mathbf{2},\ldots,\beta_\mathbf{Ne}]$. The numerical solution is done by appling the gradient method, in which at the iteration $\mathbf{k}$ the calculus proceeding is represented in the following manner:

$$\mathbf{h^{k+1}}=\mathbf{h^k}-\alpha[\mathbf{\Lambda(h,U)}+\mu^\mathbf{k+1}\beta], \tag{17}$$

where the multiplier Lagrange $\mu^\mathbf{k+1}$ is obtained from the condition that the vector $\mathbf{h^{k+1}}$ verifies the restriction (6) and has the form:

$$\mu^\mathbf{k+1}=-\frac{(\mathbf{\Lambda(h^k,U^k)},\beta)}{(\beta,\beta)}. \tag{18}$$

4 The numerical solution of the identification problem in case of girder-console

In concordance with the numerical algorithm shown above, the identification problem of some girder-console consisted of one-dimensional bar elements and represented in fig. 3.1, was solved.

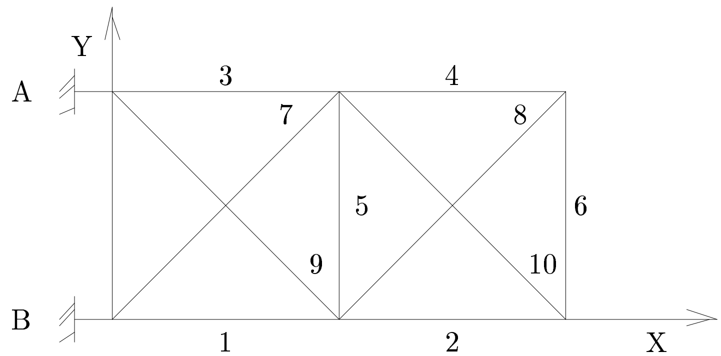

Fig. 1. Girder-console

The nodes A and B are fixed but the others are free. Bars with numbers 1,2,3,4,5,6 have the length $\mathbf{l}$ and bars with numbers 7,8,9,10 have the length $\mathbf{l}\sqrt{2}$. The values of natural frequencies obtained after the numerical solution of the eigenvalues problem (3a) for the girder represented in fig. 3.1 have been taken in function of experimental values of natural frequencies squares ω_i^2. In the course of stiffness and mass matrix formation for the bar element, the element deformation is described by the linear function

$$\mathbf{u_{ij}(x) = u_i(1 - \frac{x}{l}) + u_j\frac{x}{l}},$$

where $(\mathbf{u_i}, \mathbf{u_j})$ are element's deplacements of nodes $\mathbf{i}$ and $\mathbf{j}$ and $\mathbf{0 \leq x \leq l}$. The numerical calculus was realised with the following nondimensional values of the mechanical characteristics $\mathbf{E = 1, h_i = 2}$, $\mathbf{i = 1, 2, \ldots, Ne}, \quad \mathbf{l = 1}$, $\rho = \mathbf{1}$. In case of other values of the mechanical characteristics $\mathbf{E}, \mathbf{l}$ and ρ, the squares of natural frequencies are obtained by to the following rule:

$$\mathbf{\Omega_i^2} = \frac{\mathbf{E}}{\rho \mathbf{l^2}}\omega_\mathbf{i}^\mathbf{2}.$$

The eigenvalues problem (3a) was numericaly solved by iterations in subspace method [1], and, in consequence, the following values for the squares of natural frequencies were obtained.

$$\tilde{\omega}_1^2 = \mathbf{3.05541x10^{-2}};\quad \tilde{\omega}_4^2 = \mathbf{1.35194};\quad \tilde{\omega}_7^2 = \mathbf{2.10478};$$
$$\tilde{\omega}_2^2 = \mathbf{2.50090x10^{-1}};\quad \tilde{\omega}_5^2 = \mathbf{1.42123};\quad \tilde{\omega}_8^2 = \mathbf{3.09643};$$
$$\tilde{\omega}_3^2 = \mathbf{3.60189x10^{-1}};\quad \tilde{\omega}_6^2 = \mathbf{1.88496}.$$

These values were taken as experimental data and further the identification problem (7) was studied. The identification problem was numerically solved applying the gradient method described in the previous paragraph. The following vector

$$\mathbf{h^0} = [\mathbf{1., 1., 1., 1., 1., 1., 3.06, 3.06, 3.06, 3.06}],$$

which verifies condition (6) was taken as initial distribution of cross sections. The order numbers of the vector $\mathbf{h}$ components corespond to the numbers of elements girder-console representeted in fig. 3.1. For the selected distribution $\mathbf{h^0}$ the functional values (4) was equal to $\mathbf{J(h^0) = 7.419499x10^{-1}}$ and maximum values of the component part in the absolute value of the gradient (15) was achieved for the element with number $\mathbf{i}$ =1

$$\max_{\mathbf{j}} |\mathbf{\Lambda_j(h^0, U)}| = \mathbf{\Lambda_1(h^0, U)} = \mathbf{2.61143x10^{-2}}.$$

The value of the parameter α of movement in the gradient direction was calculated from the formula:

$$\alpha = \mathbf{0.1}\frac{\mathbf{h_j^0}}{\mathbf{\Lambda_j(h^0, U)}}, \tag{19}$$

where

$$\mathbf{\Lambda_j(h^0, U)} = \max_{\mathbf{i}} |\mathbf{\Lambda_i(h^0, U)}|.$$

After each 100 iteration this values α was calculated again, where in (19) $\mathbf{h^{100}}$ was taken instead of $\mathbf{h^0}$. The iterative process (17) continues till the functional values reaches the given precision $\mathbf{EPS} = \mathbf{10^{-12}}$. The functional value at the iteration 413 for the given precision is $\mathbf{J(h^{413})} = \mathbf{1.53779x10^{-14}}$ and the distribution of cross sections is:

$$\mathbf{h^{413}} = [\mathbf{2.000, 1.999, 2.000, 2.000, 2.000, 2.000, 2.000,}$$
$$\mathbf{2.000, 2.000, 2.000}].$$

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C.Popa, A.Sharaniuk,
The Informational and Technical Services Centre
The State University of Moldova
60, Mateevici str.,
277009, Chisinau, Moldova

The Mechanical Problems Institute
of Russian Federation Academy of Sciences
101, Vernadsky avenue,
117526, Moscow, Russia.

Received April 15, 2000