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Numerical measure of strong stability and strong quasistability in the vector problem of integer linear programming *

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Abstract

In this paper we consider a vector integer programming problem with the linear partial criteria. Numerical evaluations of two types of stability of the Pareto set have been found.

Usually the stability (quasistability) of a vector optimization problem (see [1-10]) is understood as the property of nonappearance of new optimal solutions (preservation of initial) under small perturbations of the problem's parameters. When we relax these demands we get the concepts of the strong stability and strong quasistability accordingly (see definitions below), that were introduced first by V.K. Leontev for mono-criterion trajectorial problem in [11]. Later lower and upper bounds (in some cases formulas) for evaluation of radii of the strong stability and strong quasistability in the vector trajectorial problem of lexicographic optimization were obtained in [12].

In this paper we consider a vector integer programming problem with the linear partial criteria. Lower bound of radius of the strong stability and formula for evaluation of radius of the strong quasistability have been found for the case where Chebyshev norm was defined in the space of vector criterion parameters.

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1 Base definitions and properties

Let $= [c_{ij}]_{n\times m} \in \mathbf{R}^{n\times m},\ n \geq 1,\ m \geq 2$. We assign the linear vector criterion Cx on a bounded set $X \subset Z^m, |X| > 1$. Without loss of generality, suppose that the partial criteria are minimized:

$$C_i x \to \min_X,\ i \in N_n = \{1, 2, \dots, n\}.$$

Here and henceforth the subscript i at a matrix points to the corresponding string of the matrix.

We consider the problem of finding the Pareto set (the set of efficient solutions) [13]:

$$P(X, C) = \{x \in X : \pi(x, C) = \emptyset\},$$

$$\pi(x, C) = \{x' \in X : C(x - x') \geq \mathbf{0}, C(x - x') \neq \mathbf{0}\},$$

$$\mathbf{0} = (0, 0, \dots, 0) \in \mathbf{R}^n.$$

We call this problem the vector problem of integer linear programming and write $Z^n(X, C)$.

For $n = 1$, our problem is the scalar problem of integer linear programming and $P(X, C)$ is the set of optimal solutions (C is m-vector).

For any $k \in \mathbf{N}$ we assign the norm

$$\| z \| = \max\{|z_i| : i \in N_k\}$$

in the k-dimensional space $\mathbf{R}^k$ and the norm

$$\| z \|^* = \sum_{i \in N_k} |z_i|$$

in the space conjugate to $\mathbf{R}^k$.

Let $\epsilon > 0$. As usually [1-12, 14-18], we will perturb the matrix $C \in \mathbf{R}^{n\times m}$ adding it with perturbing matrices of the set

$$\Re(\epsilon) = \{B \in \mathbf{R}^{n\times m} : \| B \| < \epsilon\}.$$

Let $B \in \Re(\epsilon)$. The problem $Z^n(X, C+B)$ obtained from the initial problem $Z^n(X, C)$ by addition of matrices C and B is called perturbed, and the matrix B is called perturbing.

Henceforth we will use the notification

$$\bar{P}(X, C) = X \backslash P(X, C).$$

It is easy to see that the following properties are true.

Property 1 *For any solution $x \in \bar{P}(X, C)$ and $x' \in P(X, C)$ there exists the subscript $i \in N_n$ such that the inequality $C_i(x - x') > 0$ holds.*

Property 2 *Let $x,\ x' \in X$. Let the inequalities*

$$C_i(x - x') > 0$$

hold for any subscript $i \in N_n$. Then $x \in \bar{P}(X, C)$.

Property 3 *Let $\epsilon > 0,\ x' \in P(X, C)$ and the formula*

$$\forall\ x \in \bar{P}(X, C)\ \exists\ i \in N_n\ \forall\ B \in \Re(\epsilon)\ ((C_i + B_i)(x - x') > 0)$$

be true. Then the equality

$$\pi(x', C + B) \cap \bar{P}(X, C) = \emptyset$$

holds for any perturbing matrix $B \in \Re(\epsilon)$.

Lemma 1 *Let $i \in N_n,\ x,\ x' \in X$ be vectors such that the inequality*

$$C_i(x - x') > 0$$

holds. Than the inequality

$$(C_i + b)(x - x') > 0$$

is true for a vector $b \in \mathbf{R}^m$ satisfying the inequality

$$C_i(x - x') > \| b \| \| x - x' \|^*.$$

The proof is easy: using the evident inequality

$$b(x - x') \geq - \parallel b \parallel \parallel x - x' \parallel^{*}$$

we obtain

$$(C_i{+}b)(x{-}x') = C_i(x{-}x'){+}b(x{-}x') \geq C_i(x{-}x'){-} \parallel b \parallel \parallel x - x' \parallel^{*} > 0.$$

2 Strong stability

The stability of optimization problem is usually understood as the property of upper and lower semicontinity by Hauzdorf of the optimal mapping, i.e. the many-valued mapping that defines the choice function. If the set of admissible solutions is finite then the property of upper semicontinity can be replaced by an equivalent property of nonappearance of new optimal solutions under small perturbations of the problem's parameters [1-3,8,10]. The limit of such perturbations is called the stability radius. When we relax the demand of nonappearance of new optimal solutions we obtain the notion of radius of the strong stability.

According to [11] the number

$$\rho^n(X, C) = \begin{cases} \sup \Omega_1(X, C), & \Omega_1(X, C) \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\Omega_1(X, C) = \{\epsilon > 0 : \ P(X, C + B) \cap P(X, C) \neq \emptyset \ \forall B \in \Re(\epsilon)\},$$

is called the radius of the strong stability of the vector problen $Z^n(X, C)$.

It is clear that

$$\rho_0^n(X, C) \leq \rho^n(X, C), \qquad (2.1)$$

where $\rho_0^n(X, C)$ is the stability radius of the problem $Z^n(X, C)$ [see 10].

The strong stability radius is infinite in the case $P(X, C) = X$. So we exclude this case from the consideration. If the set $\bar{P}(X, C)$ is non-empty, then we say that the problem $Z^n(X, C)$ is non-trivial.

By definition, put

$$\phi^n(X,C) = \max_{x' \in P(X,C)} \min_{x \in \bar{P}(X,C)} \max_{i \in N_n} \frac{C_i(x-x')}{\| x - x' \|^*},$$

$$\kappa^n(X,C) = \min_{x \in \bar{P}(X,C)} \max_{x' \in P(X,C)} \max_{i \in N_n} \frac{C_i(x-x')}{\| x - x' \|^*}.$$

Theorem 1 *Let the problem $Z^n(X,C), n \geq 1$, be non-trivial. Then the strong stability radius $\rho^n(X,C)$ has the following lower bound*

$$\rho^n(X,C) \geq \phi^n(X,C) > 0.$$

If the problem is Boolean ($X \subseteq \{0,1\}^m$), then

$$\kappa^n(X,C) \geq \rho^n(X,C) \geq \phi^n(X,C).$$

Proof. The inequality $\phi := \phi^n(X,C) > 0$ follows from property 1.

Further let us prove that $\rho^n(X,C) \geq \phi$. According to definition of the number ϕ we get

$$\exists\ x' \in P(X,C)\ \forall\ x \in \bar{P}(X,C)\ \exists\ p \in N_n\ \ \frac{C_p(x-x')}{\| x - x' \|^*} \geq \phi.$$

So for any perturbing matrix $B \in \Re(\phi)$ we have

$$C_p(x-x') > \| B_p \| \ \| x - x' \|^* \geq 0.$$

Using the lemma, we obtain

$$(C_p + B_p)(x-x') > 0\ \forall B \in \Re(\phi).$$

Then, according to property 3, we obtain

$$\pi(x', C+B) \cap \bar{P}(X,C) = \emptyset\ \forall B \in \Re(\phi),$$

and

$$P(X, C+B) \cap P(X,C) \neq \emptyset\ \forall B \in \Re(\phi).$$

Hence, $\rho^n(X,C) \geq \phi^n(X,C)$.

Now let us prove that $\rho^n(X,C) \leq \kappa := \kappa^n(X,C)$ under the assumption $(X \subseteq \{0,1\}^m)$.

Let $\epsilon > \kappa$. By definition of the number κ, we obtain

$$\exists\, x \in \bar{P}(X,C)\ \forall\, x' \in P(X,C)\ \forall\, i \in N_n \quad \epsilon > \kappa \geq \frac{C_i(x - x')}{\| x - x' \|^*}.$$

Consider a perturbing matrix $B = \{b_{ij}\}_{n \times m} \in \Re(\epsilon)$, obtained by setting for any subscripts $i \in N_n,\ j \in N_m$

$$b_{ij} = \begin{cases} \beta, & \text{if } x_j = 0, \\ -\beta, & \text{if } x_j = 1\ , \end{cases}$$

where $\epsilon > \beta > \kappa$. Then we obtain

$$\forall i \in N_n \quad (C_i + B_i)(x - x') = C_i(x - x') - \beta \| x - x' \|^* < 0.$$

Hence $\pi(\mathbf{x}, C + B) \cap P(X,C) = \emptyset$.

Thus, for any $\epsilon > \kappa$ there exists a perturbating matrix $B \in \Re(\epsilon)$ such that $P(X, C+B) \cap P(X,C) = \emptyset$, i.e. $\rho^n(X,C) \leq \kappa^n(X,C)$.

Theorem 1 has been proved.

We say that problem $Z^n(X,C)$ is strongly stable, if there exists a number $\epsilon > 0$ such that the inequalities

$$P(X, C+B) \cap P(X,C) \neq \emptyset\ \forall B \in \Re(\epsilon)$$

are true, and if $\rho^n(X,C) > 0$.

The next corollary follows from theorem 1.

Corollary 1 *Any problem $Z^n(X,C),\ n \geq 1$, is strongly stable.*

From the corollary it follows (see [11]) that any mono-criterion ($n = 1$) linear trajectorial optimization problem is strongly stable.

The next well-known result [11] follows from theorem 1, theorem 1 from [10] and inequality (2.1).

Corollary 2 *The equality*

$$\rho_0^1(X,C) = \rho^1(X,C)$$

holds for any scalar Boolean problem $Z^1(X,C)$.

Consider an example that illustrates the attainability of the lower bound of the strong stability radius.

Example 1. Let $n = 2,\ m = 2,\ X = \{x' = (1,2),\ x'' = (2,1),\ x''' = (2,3),\ x^{IV} = (3,2)\}$,

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $P(X,C) = \{x', x''\},\ \phi^2(X,C) = \frac{3}{2}$.
If $\frac{3}{2} < \beta < \epsilon$, then $P(X,C+B) = \{x''', x^{IV}\}$, where

$$B = \begin{pmatrix} -\beta & -\beta \\ -\beta & -\beta \end{pmatrix}.$$

Hence, we have

$$\forall\ \epsilon > \frac{3}{2}\ \exists\ B \in \Re(\epsilon)\ (P(X,C+B) \cap P(X,C) = \emptyset).$$

By theorem 1 we obtain $\rho^2(X,C) = \phi^2(X,C) = \frac{3}{2}$.
The following example illustrates that the strong stability radius $\rho^n(X,C)$ may be greater than the number $\phi^n(X,C)$.

Example 2. Let $n = 2,\ m = 2,\ X = \{x' = (2,4),\ x'' = (4,2),\ x''' = (2,5),\ x^{IV} = (5,2)\}$,

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $P(X,C) = \{x', x''\},\ \phi^2(X,C) = \frac{1}{5}$.
It is easy to show that $\rho^2(X,C) = 1$.

3 Strong quasistability

Under the radius of quasistability $\theta_0^n(X,C)$ of the vector optimization problem $Z^n(X,C)$ (see [3-6, 10]) we understand the limit of independent perturbations of the parameters of vector criterion such that all the optimal solutions of the initial problem preserves their optimality in a pertubed problem. When we relax the demand of preservation of all the Pareto optimal solutions, we obtain the notion of radius $\theta^n(X,C)$ of the strong quasistability of the vector problem $Z^n(X,C)$.

Thus,

$$\theta^n(X,C) = \begin{cases} \sup \Omega_2(X,C), & \Omega_2(X,C) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Omega_2(X,C) = \{\epsilon > 0 : \ \exists\, x \in P(X,C)\ \forall B \in \Re(\epsilon)\ (x \in P(X,C+B))\}.$$

This type of stability means existing of a stable solution.

It is clear that

$$\theta_0^n(X,C) \leq \theta^n(X,C).$$

Theorem 2 *The strong quasistability radius of the problem $Z^n(X,C)$, $n \geq 1$, is expressed by the formula*

$$\theta^n(X,C) = \max_{x' \in P(X,C)} \ \min_{x \in X \backslash \{x'\}} \max_{i \in N_n} \frac{C_i(x-x')}{\parallel x-x' \parallel^*}. \tag{3.1}$$

Proof. Let ψ denote the right part of equality (3.1). First let us prove the inequality $\theta^n(X,C) \geq \psi$.

For $\psi = 0$, there is nothing to prove.

Let $\psi > 0$ and $B \in \Re(\psi)$. Then

$$\exists\, x' \in P(X,C)\ \forall\, x \in X \backslash \{x'\}\ \exists\, i \in N_n\ \frac{C_i(x-x')}{\parallel x-x' \parallel^*} \geq \psi > \parallel B_i \parallel .$$

Hence, using the lemma we have $(C_i + B_i)(x - x') > 0$. Thus, it follows that the solution x' is Pareto optimal in any perturbated problem $Z^n(X, C+B),\ B \in \Re(\psi)$. Hence, the inequality $\theta^n(X,C) \geq \psi$ is true.

Further let us prove that $\theta^n(X,C) \leq \psi$. To make this, it is enough to prove that

$$\forall \epsilon > \psi\ \forall x' \in P(X,C)\ \exists B \in \Re(\epsilon)\ (x' \in \bar{P}(X, C+B)). \tag{3.2}$$

Let $\epsilon > \psi,\ \mathbf{x}' \in P(X,C)$. By definition of the number ψ, there exists a solution $x \in X \backslash \{x'\}$ such that for any subscript $i \in N_n$ the inequalities

$$\epsilon > \psi \geq \frac{C_i(x - x')}{\| x - x' \|^*}$$

are true.

Consider a perturbing matrix $B = \{b_{ij}\}_{n \times m} \in \Re(\epsilon)$, obtained by setting for any subscripts $i \in N_n,\ j \in N_m$

$$b_{ij} = \begin{cases} \beta, & \text{if } x_j \leq x'_j, \\ -\beta, & \text{if } x_j > x'_j\ , \end{cases}$$

where $\epsilon > \beta > \psi$. Then we obtain

$$(C_i + B_i)(x - x') = C_i(x - x') - \beta \| x - x' \|^* < 0 \ \ \forall i \in N_n.$$

Hence, according to property 2, we get $x' \in \bar{P}(X, C+B)$. It follows that formula (3.2) is true, i.e. $\theta^n(X,C) \leq \psi$.

Combining it with the inequality $\theta^n(X,C) \geq \psi$ we obtain the equality $\theta^n(X,C) = \psi$.

Theorem 2 has been proved.

It follows from theorem 2 that the strong quasistability radius of the vector problem $Z^n(X,C)$ is always finite.

So in terms of the work [19], the number $\theta^n(X,C)$ is the radius of stability Kernel of the problem. Formula (see [19]) for evaluation of this radius follows from theorem 2.

The problem $Z^n(X,C)$ is called strongly quasistable, if $\theta^n(X,C) > 0$.

Consider the Smale set (i.e. the set of strictly efficient solutions) [13]:

$$S(X,C) = \{x \in X : \ \sigma(x,C) = \emptyset\},$$

where

$$\sigma(x,C) = \{x' \in X \backslash \{x\} : \ C(x - x') \geq \mathbf{0}\}.$$

From theorem 2, we obtain

Corollary 3 *The vector problem $Z^n(X,C)$, $n \geq 1$, is strongly quasistable, iff the Smale set $S(X,C)$ is not empty.*

In particular, we have

Corollary 4 [11] *Monocriterion linear trajectorial problem is strongly quasistable iff it has a unique optimal solution.*

Besides that, from theorem 1 and theorem 2, we obtain

Corollary 5 $\rho^n(X,C) \geq \theta^n(X,C)$, *i.e. it follows that any strongly quasistable problem $Z^n(X,C)$ is strongly stable.*

From theorems 1–2, theorems 1-2 [10] and Corollary 2 we get the next well-known result

Corollary 6 [11] *The formulas*

$$\rho_0^1(X,C) = \rho^1(X,C) = \theta_0^1(X,C) = \theta^1(X,C) > 0$$

hold for any scalar Boolean problem $Z^1(X,C)$ with a unique optimal solution.

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