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Stability, pseudostability and quasistability of vector trajectorial lexicographic optimization problem*

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Abstract

Lower bounds of stability, pseudostability and quasistability radii of lexicographic set in vector combinatorial problem on systems of subsets of finite set with partial criteria of more general kinds have been found.

Many specialists are engaged in study of stability of discrete optimization problems to perturbations of their parameters (see [1-3]). Need for investigation of stability of optimization problems is connected with inaccuracy of the input data, inadequacy of mathematical models to real processes, mistakes of computations and other factors.

The stability of single-criterion trajectorial discrete optimization problems have been investigated in detail. Many well-known optimization problems on graphs, Boolean programming problems, and also scheduling problems, can be described as the special cases [3-9]. Analyzing stability of such problems the authors paid the main attention to the calculation of the stability radius. This notion for a single-criterion trajectorial problem was introduced by V.K.Leontiev [4].

Necessary and sufficient conditions of three types of stability (in our terms stability, quasistability and pseudostability) of Pareto set in vector integer programming problems were obtained in [10-13].

The papers [14-17] is devoted to stability of trajectorial problems with partial criteria of the three most widespread types: MINSUM,

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MINMAX and MINMIN. Lower bounds of the stability radius of Pareto set, and formulae in several cases were found. The stability and stability radii of efficient (Pareto optimal, Slater optimal and Smale optimal) trajectories were investigated in [18].

In this paper we consider a vector lexicographic optimization trajectorial problem with partial criteria of more general kinds. This criteria include three criteria named above. Lower bounds of stability, quasistability and pseudostability radii of lexicographic set, and formulae in several cases, have been found for the case where l_∞-norm is defined in the space of vector criterion parameters. The stability kernel, i.e. the set of all stable trajectories, has been investigated.

1 Statement of the problem

Let $E = \{e_1, e_2, \dots, e_m\}, m > 1$, be a given set, $T \subseteq 2^E \backslash \{\emptyset\}$ be a system of nonempty subsets (trajectories) of the set E , $|T| > 1$.

On the set E, we define a vector weight function

$$a(e) = (a_1(e), a_2(e), \dots, a_n(e)) \in \mathbf{R}^n, n \geq 1,$$

and, on the set T, a vector criterion

$$f(t) = (f_1(t), f_2(t), \dots, f_n(t)).$$

The partial criteria of the vector criterion are functions of the following two kinds:

$$\Sigma\text{−MINMAX} \quad f_i(t) =$$
$$= \max\left\{\sum_{e\in q} a_i(e): \ q \subseteq t, \ |q| = \min\{|t|, k_i\}\right\} \to \min_T, \tag{1.1}$$

$$\Sigma\text{−MINMIN} \quad f_i(t) =$$
$$= \min\left\{\sum_{e\in q} a_i(e): \ q \subseteq t, \ |q| = \min\{|t|, k_i\}\right\} \to \min_T, \tag{1.2}$$

where $k_i, i \in N_n = \{1, 2, \ldots, n\}$, are given natural numbers such that

$$1 \leq k_i \leq p = \max\{|t|: \ t \in T\} \ \forall i \in N_n.$$

When $k_i = p, i \in N_n$, both the criterion (1.1) and the criterion (1.2) turn into a linear criterion

$$\text{MINSUM} \quad f_i(t) = \sum_{e \in t} a_i(e) \to \min_T . \tag{1.3}$$

When $k_i = 1, i \in N_n$, the criterion (1.1) turns into a bottleneck criterion

$$\text{MINMAX} \quad f_i(t) = \max\{a_i(e): \ e \in t\} \to \min_T \tag{1.4}$$

and the criterion (1.2) turns into a criterion

$$\text{MINMIN} \quad f_i(t) = \min\{a_i(e): \ e \in t\} \to \min_T . \tag{1.5}$$

Note that the problems with Σ-MINMAX and Σ-MINMIN criteria are related to needs of optimal distribution [19].

When we say n-criteria trajectorial problem, we mean the problem of finding the lexicographic set. The lexicographic set is a subset of the Pareto set and is defined as follows [20-26].

Suppose S_n is the set of all $n!$ permutations of the numbers $1, 2, \ldots, n$. We say that a trajectory t is a lexicographic optimum if there exists a permutation $s = \{s_1, s_2, \ldots, s_n\} \in S_n$ such that one of the following conditions holds for any trajectory t':

1) $f(t) = f(t')$;

2) $\exists k \in N_n \ (f_{s_k}(t) < f_{s_k}(t'))\&(\forall i \in N_{k-1} \ f_{s_i}(t) = f_{s_i}(t'))$.

If $k = 1$, then the last equalities are absent ($N_o = \emptyset$).

Thus s orders the partial criteria by an impotence such that every previous criterion is more significant than all the consequent ones.

We say that the set of all lexicographic optimal trajectories defined by all $n!$ permutations is the lexicographic set and write L^n. The n-criteria trajectorial problem is denoted by Z^n.

The vector weight function $a(e)$ can be represented as the matrix $A = \{a_{ij}\}_{n\times m}$, where $a_{ij} = a_i(e_j)$. Let I_1 and I_2 be the sets of those numbers from N_n, which number the criteria (1.1) and (1.2) respectively ($I_1 \bigcup I_2 = N_n$). If the numbers $k_1, k_2, \dots, k_n$ and the sets E, T, I_1 , I_2 are fixed, then the individual n-criteria trajectorial lexicographic optimization problem is uniquely determined by the matrix A. Therefore we denote it by $Z^n(A)$. $L_n(A)$ denotes the lexicographic set, $f(t, A)$ denotes the vector criterion $f(t)$ and $f_i(t, A)$ denotes its partial criteria $f_i(t)$.

It is evident that the lexicographic optimization problem turns into the problem of finding the optimal set when $n = 1$. Its stability radius has been investigated by Leontev V. K. and Gordeev E. N. in the case of linear and bottleneck criteria (see [4-7]).

As usually, we will perturbate the matrix $A \in \mathbf{R}^{nm}$ by adding to A matrices from the set

$$\Re(\varepsilon) = \{B \in \mathbf{R}^{nm} : \|B\| < \varepsilon\},$$

where $\varepsilon > 0, \|.\|$ is the norm l_∞ (Chebyshev norm) in $\mathbf{R}^{nm}$, i.e.

$$\|B\| = \max\{|b_{ij}| : (i, j) \in N_n \times N_m\},\ B = \{b_{ij}\}_{n\times m}.$$

Let $A, B \in \mathbf{R}^{nm}$. If we add a matrix B to the matrix A of the problem $Z^n(A)$ we get a perturbed problem $Z^n(A + B)$. Here the matrix B is called perturbing.

As in [15,16], we say that the problem $Z^n(A)$ is

- stable if

$$\exists\ \varepsilon > 0\ \forall B \in \Re(\varepsilon)\ L^n(A) = L^n(A + B);$$

- pseudostable if

$$\exists\ \varepsilon > 0\ \forall B \in \Re(\varepsilon)\ L^n(A) \supseteq L^n(A + B);$$

- quasistable if

$$\exists\ \varepsilon > 0\ \forall B \in \Re(\varepsilon)\ L^n(A) \subseteq L^n(A + B).$$

It is evident that the properties of pseudostability and quasistability of the discrete problem $Z^n(A)$ are equivalent to the properties of upper and lower semicontinuity by Hauzdorf of the optimal mapping $L^n : \mathbf{R}^{nm} \to 2^T$ in a point $A \in \mathbf{R}^{nm}$ respectively (see [1,2,27-30]).

Let I_{sum}, I_{max} and I_{min} be the set of those numbers from N_n that number the partial criterion (1.3), (1.4) and (1.5) of the vector criterion $f(t)$ respectively.

2 Pseudostability

We say that the value

$$\rho_1^n(A) = \begin{cases} \sup \Omega_1(A) & \text{if} \quad \Omega_1(A) \neq \emptyset, \\ 0 & \text{if} \quad \Omega_1(A) = \emptyset, \end{cases}$$

where $\Omega_1(A) = \{\varepsilon > 0 : \ L^n(A) \supseteq L^n(A+B) \ \forall B \in \Re(\varepsilon)\}$, is the pseudostability radius of the problem $Z^n(A), \ n \geq 1$.

Thus the pseudostability radius of the problem $Z^n(A)$ is defined as the limit of perturbations for elements of matrix A that do not cause appearance of new lexicographic optimal trajectories.

Obviously, if $T = L^n(A)$, then the pseudostability radius $\rho_1^n(A)$ is equal to infinity. The problem $Z^n(A)$ is called nontrivial if $\bar{L}^n(A) = T \backslash L^n(A) \neq \emptyset$.

The next two properties follow directly from the definition of the pseudostability radius.

Property 2.1. Let $\varphi > 0$. If

$$\bar{L}^n(A) \subseteq \bar{L}^n(A+B) \quad \forall B \in \Re(\varphi),$$

then $\rho_1^n(A) \geq \varphi$.

Property 2.2. Let the problem $Z^n(A)$ be nontrivial, $\varphi \geq 0$. If, for any number $\varepsilon > \varphi$, there exist a perturbing matrix $B \in \Re(\varepsilon)$ and a trajectory $t \in \bar{L}^n(A)$ such that

$$t \in \bar{L}^n(A+B),$$

then $\rho_1^n(A) \le \varphi$.

Let us introduce the following notation for any two different trajectories t, t':

$$\tau_i(t, t', A) = f_i(t, A) - f_i(t', A);$$

$$\Delta_i(t, t') = \begin{cases} |t| + |t'| - 2|t \bigcap t'| & \text{if} \quad i \in Isum, \\ \\ \min\{|t|, k_i\} + \min\{|t'|, k_i\} & \text{if} \quad i \notin Isum. \end{cases}$$

It is obvious that the inequality $\Delta_i(t, t') > 0$ holds for any index $i \in N_n$ and any trajectories $t \neq t'$.

In order to find a lower bound of the pseudostability radius, we will formulate some evident properties and prove an auxiliary lemma.

Property 2.3. The trajectory t is a lexicographic optimum of the problem $Z^n(A)$ if there exists an index $i \in N_n$ such that

$$\tau_i(t, t', A) < 0 \ \forall t' \in T, \ t' \neq t.$$

Property 2.4. The trajectory t is not a lexicographic optimum of the problem $Z^n(A)$ if for any index $i \in N_n$ there exists a trajectory $t' \neq t$ such that $\tau_i(t, t', A) > 0$.

Further, for any subset $t \subseteq E$, let $N(t)$ be the set of indexes $j \in N_m$ such that $e_j \in t$:

$$N(t) = \{j \in N_m : \ e_j(t)\}.$$

Lemma 2.1. Let $t, t' \in T, \ t \neq t', \ i \in N_n, \ \varepsilon > 0$. If

$$\tau_i(t, t', A) \ge \varepsilon \Delta_i(t, t'), \tag{2.1}$$

then

$$\tau_i(t, t', A + B) > 0 \ \forall B \in \Re(\varepsilon).$$

Proof. Let us consider three cases.

Case 1. $k_i = p$. Then the function $f_i(t, A)$ is linear. Hence, using inequalities $\|B\| < \varepsilon$ and (2.1), we get

$$\begin{aligned}
\tau_i(t, t', A+B) &= f_i(t, A+B) - f_i(t', A+B) > \\
&> f_i(t, A) - \varepsilon|t \backslash t'| - (f_i(t', A) + \varepsilon|t' \backslash t|) = \\
&= \tau_i(t, t', A) - \varepsilon\Delta_i(t, t') \geq 0 \\
&\quad \forall B \in \Re(\varepsilon).
\end{aligned}$$

Case 2. $k_i < p,\ i \in I_1$. Then we have

$$\tau_i(t, t', A+B) = f_i(t, A+B) - f_i(t', A+B) =$$

$$= \max\left\{\sum_{j \in N(q)} (a_{ij} + b_{ij}) : q \subseteq t,\ |q| = \min\{|t|, k_i\}\right\}$$

$$- \max\left\{\sum_{j \in N(q)} (a_{ij} + b_{ij}) : q \subseteq t',\ |q| = \min\{|t'|, k_i\}\right\} >$$

$$> f_i(t, A) - \varepsilon \min\{|t|, k_i\} - (f_i(t', A) + \varepsilon \min\{|t'|, k_i\}) =$$

$$= \tau_i(t, t', A) - \varepsilon\Delta_i(t, t') \geq 0 \qquad \forall B \in \Re(\varepsilon).$$

Case 3. In the third case where $k_i < p, i \in I_2$ the proof is carried out by analogy.

This completes the proof of lemma 2.1.

By definition, put

$$\varphi^n(A) = \min_{t \in \bar{L}^n(A)} \min_{i \in N_n} \max_{t' \in T \backslash \{t\}} \frac{\tau_i(t, t', A)}{\Delta_i(t, t')}.$$

It is obvious that $\varphi^n(A) \geq 0$.

Theorem 2.1. Let $A \in \mathbf{R}^{nm}$. For any combination of partial criteria (1.1) and (1.2) of the nontrivial trajectorial problem $Z^n(A),\ n \geq 1$, we have

$$\rho_1^n(A) \geq \varphi^n(A), \tag{2.2}$$

moreover

$$\rho_1^n(A) = \varphi^n(A) \tag{2.3}$$

if $I_{sum} = N_n$.

Proof. Inequality (2.2) is evident if $\varphi^n(A) = 0$.

Let $\varphi = \varphi^n(A) > 0$. Then $\Re(\varphi) \neq \emptyset$ and, by definition of the number φ, for any trajectory $t \in \bar{L}^n(A)$ and any index $i \in N_n$, there exists a trajectory $t' \neq t$ such that

$$\tau_i(t, t', A) \geq \varphi \Delta_i(t, t').$$

Therefore, by lemma 2.1, we obtain

$$\tau_i(t, t', A + B) > 0\ \forall B \in \Re(\varphi).$$

Hence, using property 2.4, we get

$$t \in \bar{L}^n(A + B)\ \forall B \in \Re(\varphi).$$

Consequently, we obtain

$$\bar{L}^n(A) \subseteq \bar{L}^n(A + B)\ \forall B \in \Re(\varphi).$$

Thus, taking into account property 2.1, we have (2.2).

In order to prove equality (2.3) we will show that $\rho_1^n(A) \leq \varphi$ if $I_{sum} = N_n$.

Then by definition of the number φ there exist a trajectory $t \in \bar{L}^n(A)$ and an index $k \in N_n$ such that

$$\gamma = \max \left\{ \frac{\tau_k(t, t', A)}{\Delta_k(t, t')} : t' \in T,\ t' \neq t \right\} \leq \varphi. \tag{2.4}$$

Let $\varepsilon > \varphi$. Hence taking the perturbing matrix $B \in \Re(\varepsilon)$ with the elements

$$b_{ij} = \begin{cases} -b & \text{if} \quad i = k,\ j \in N(t), \\ b & \text{if} \quad i = k,\ j \notin N(t), \\ 0 & \text{if} \quad i \neq k,\ j \in N_m, \end{cases}$$

where $\gamma < b < \varepsilon$, and using (2.4), we have

$$\tau_k(t, t', A + B) = \tau_k(t, t', A) - b\Delta_i(t, t') < \tau_k(t, t', A) - \gamma\Delta_i(t, t') \leq 0$$

$$\forall t' \in T, t' \neq t.$$

Thus by property 2.3 t is a lexicographic optimal trajectory of the perturbed problem $Z^n(A{+}B)$. Consequently, taking into account property 2.2, we obtain

$$\rho_1^n(A) \leq \varphi.$$

Combining this with (2.2), we see that theorem 2.1 is true.

Remark 2.1. If $n = 1$, then formula (2.3) turns into the well-known formula for the pseudostability radius of the single-criterion linear trajectorial problem [7].

Since the problem $Z^n(A)$ is pseudostable only when $\rho_1^n(A) > 0$, the next corollary follows from theorem 2.1.

Corollary 2.1. In order for the nontrivial trajectorial problem $Z^n(A), n \geq 1$, to be pseudostable it is sufficient, and also necessary in the case $I_{sum} = N_n$, to have

$$\forall t \in \bar{L}^n(A)\ \forall i \in N_n\ \exists t' \in T \backslash \{t\}\ (\tau_i(t, t', A + B) > 0). \tag{2.5}$$

It follows from this corollary that any single-criterion linear problem $Z^1(A)$ is pseudostable (see [5]).

The next example shows that condition (2.5) is not necessary in general when $I_{sum} \neq N_n$.

Example 2.1. Suppose $n = 2;\ m = 3;\ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$

$$T = \{t_1, t_2\};\ t_1 = \{e_1, e_2\};\ t_2 = \{e_2, e_3\};$$

$$f_1(t, A) = \max\{a_{1j} :\ j \in N(t)\} \to \min_T;$$

$$f_2(t, A) = \sum_{j \in N(t)} a_{2j} \to \min_T.$$

Then $t_2 \in \bar{L}^2(A)$.

As for any matrix $B \in \Re(1/2)$ the expressions

$$\tau_1(t_1, t_2, A+B) = 0, \ \tau_2(t_1, t_2, A+B) < 0$$

are true, t_2 is not a lexicographic optimum of a perturbed problem $Z^2(A+B)$ $\forall B \in \Re(1/2)$. Consequently, the problem $Z^2(A)$ is pseudostable.

On the other hand we have $\tau_1(t_1, t_2, A) = 0$. Hence formula (2.5) is not true.

3 Quasistability

We say that the value

$$\rho_2^n(A) = \begin{cases} \sup \Omega_2(A) & \text{if} \quad \Omega_2(A) \neq \emptyset, \\ 0 & \text{if} \quad \Omega_2(A) = \emptyset, \end{cases}$$

where $\Omega_2(A) = \{\varepsilon > 0 : \ L^n(A) \subseteq L^n(A+B) \ \forall B \in \Re(\varepsilon)\}$, is the quasistability radius of the problem $Z^n(A)$, $n \geq 1$.

Thus the quasistability radius of the problem $Z^n(A)$ defines the limit of perturbations for elements of matrix A such that the initial lexicographic optimums are saved and the new ones may appear.

The next properties follow directly from the definition of the quasistability radius.

Property 3.1. Let $\varphi > 0$. If

$$L^n(A) \subseteq L^n(A+B) \ \forall B \in \Re(\varphi),$$

then $\rho_2^n(A) \geq \varphi$.

Property 3.2. Let $\varphi \geq 0$. If for any number $\varepsilon > \varphi$ there exist a trajectory $t \in L^n(A)$ and a perturbing matrix $B \in \Re(\varepsilon)$ such that

$$t \in \bar{L}^n(A+B),$$

then $\rho_2^n(A) \leq \varphi$.

By definition, put

$$\psi^n(A) = \min_{t \in L^n(A)} \max_{i \in N_n} \min_{t' \in T \setminus \{t\}} \gamma_i(t, t', A),$$

where

$$\gamma_i(t, t', A) = -\frac{\tau_i(t, t', A)}{\Delta_i(t, t')}. \tag{3.1}$$

It is obvious that $\psi^n(A) \geq 0$.

Theorem 3.1. Let $A \in \mathbf{R}^{nm}$. For any combination of partial criteria (1.1) and (1.2) of the trajectorial problem $Z^n(A), n \geq 1$, we have

$$\rho_2^n \geq \psi^n(A), \tag{3.2}$$

moreover

$$\rho_2^n = \psi^n(A)$$

if $I_{sum} = N_n$.

Proof. First we prove inequality (3.2). If $\psi^n(A) = 0$, then (3.2) is evident. Let $\psi = \psi^n(A) > 0$. Then $\Re(\psi) \neq \emptyset$ and by definition of the number ψ for any trajectory $t \in L^n(A)$ there exist an index $k \in N_n$ such that for any trajectory $t' \neq t$ we have

$$0 < \psi \leq \gamma_k(t, t', A),$$

i.e.

$$\tau_k(t', t, A) \geq \psi \Delta_k(t, t').$$

Therefore, by lemma 2.1, we obtain

$$\tau_k(t, t', A) > 0 \ \forall B \in \Re(\psi) \ \forall t' \in T \setminus \{t\}.$$

Hence, using property 2.3, we get

$$t \in L^n(A + B) \ \forall B \in \Re(\psi),$$

i.e.

$$L^n(A) \subseteq L^n(A + B) \quad \forall B \in \Re(\psi).$$

Consequently, taking into account property 3.1, we have (3.2).

Now we prove that $\rho_2^n(A) \leq \psi$ if $I_{sum} = N_n$. Suppose $0 \leq \psi < \varepsilon$. Then by definition of the number ψ there exists a trajectory $t \in L^n(A)$ such that for any index $i \in N_n$ there is a trajectory $t' \neq t$, for which

$$\gamma_i(t, t', A) \leq \psi \quad \forall i \in N_n.$$

Hence taking the perturbing matrix $B \in \Re(\varepsilon)$ with elements

$$b_{ij} = \begin{cases} \alpha, & \text{where} \quad i \in N_n,\ j \in N(t), \\ -\alpha, & \text{where} \quad i \in N_n,\ j \notin N(t), \end{cases}$$

and $\psi < \alpha < \varepsilon$, by virtue of linearity of $\tau_i(t, t', A)$ we have

$$\begin{aligned} \tau_i(t, t', A + B) &= \tau_i(t, t', A) + \alpha \Delta_i(t, t') > \\ &> \tau_i(t, t', A) + \gamma_i(t, t', A)\Delta_i(t, t') = 0 \\ &\forall i \in N_n. \end{aligned}$$

Thus by property 2.4 t isn't a lexicographic optimum of the perturbed problem $Z^n(A+B)$. Consequently, taking into account property 3.2, we obtain

$$\rho_2^n(A) \leq \psi.$$

Combining this with (3.2) we get theorem 3.1.

Since the problem $Z^n(A)$ is quasistable only when $\rho_2^n(A) > 0$, then the next corollary follows from theorem 3.1.

Corollary 3.1. In order for the trajectorial problem $Z^n(A), n \geq 1$, to be quasistable it is sufficient, and also necessary in the case $I_{sum} = N_n$, to have

$$\forall t \in L^n(A)\ \exists i \in N_n\ \forall t' \in T \backslash \{t\}\ (\tau_i(t, t', A + B) < 0). \tag{3.3}$$

By the above we get the next corollary.

Corollary 3.2. If $I_{sum} = N_n$, then the inequality $|L^n(A)| \leq n$ is a necessary condition for the problem $Z^n(A)$ to be quasistable.

The next example shows that condition (3.3) is not necessary in general when $I_{sum} \neq N_n$.

Example 3.1. Suppose $n = 2; m = 4; A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix};$

$$T = \{t_1, t_2, t_3\}; t_1 = \{e_1, e_2\}; t_2 = \{e_2, e_3\}; t_3 = \{e_2, e_4\};$$

$$f_1(t, A) = \max\{a_{1j} : \ j \in N(t)\} \to \min_T;$$

$$f_2(t, A) = \sum_{j \in N(t)} a_{2j} \to \min_T .$$

Then $L^2(A) = \{t_1\}$.

As for any matrix $B \in \Re(1/2)$ the expressions

$$\tau_1(t_1, t_2, A + B) = 0, \ \tau_2(t_1, t_2, A + B) < 0, \ \tau_1(t_1, t_3, A + B) < 0$$

are true, $t_1 \in L^2(A + B) \ \forall B \in \Re(1/2)$. Consequently, the problem $Z^2(A)$ is quasistable.

On the other hand we have $\tau_1(t_1, t_2, A) = 0$ and $\tau_2(t_1, t_2, A) = 0$. Hence formula (3.3) is not true.

Remark 3.1. By corollary 3.1, it follows that if an optimal trajectory is unique, then the problem $Z^1(A)$ is quasistable. The converse is true under assumption that the criterion is linear.

Remark 3.2. It is not difficult to give examples of matrices A_1, A_2, A_3 $(n \geq 1)$ such that

$$\rho_1^n(A) < \rho_2^n(A); \ \rho_1^n(A) > \rho_2^n(A); \ \rho_1^n(A) = \rho_2^n(A).$$

4 Stability

It is obvious that the problem $Z^n(A)$ is stable if it is pseudostable and quasistable simultaneously. Therefore the next theorem follows from corollary 2.1 and 3.1.

Theorem 4.1. In order for the nontrivial problem $Z^n(A), n \geq 1$, to be stable it is sufficient, and also necessary in the case $I_{sum} = N_n$, to satisfy formulae (2.5) and (3.3) simultaneously.

The value

$$\rho_3^n(A) = \begin{cases} \sup \Omega_3(A) & \text{if} \quad \Omega_3(A) \neq \emptyset, \\ 0 & \text{if} \quad \Omega_3(A) = \emptyset, \end{cases}$$

where $\Omega_3(A) = \{\varepsilon > 0 : \ L^n(A) = L^n(A+B) \ \forall B \in \Re(\varepsilon)\}$, is called the stability radius of the problem $Z^n(A)$. The next formula is obvious.

$$\rho_3^n(A) = \min\{\rho_1^n(A), \rho_2^n(A)\}.$$

Hence, taking into account theorems 2.1 and 3.1, we have

$$\rho_3^n(A) \geq \min\{\varphi^n(A), \psi^n(A)\},$$

moreover

$$\rho_3^n(A) = \min\{\varphi^n(A), \psi^n(A)\},$$

if $I_{sum} = N_n$.

Now let us introduce other lower estimates of the stability radius.

By definition, put

$$k_i^0 = \min\{k_i, \min\{|t| : \ t \in T\}\},$$

$$Q_i = \{q \subset E : \ k_i^0 \leq |q| \leq k_i\}$$

for any index $i \in N_n$. Since the set E is finite the sets $Q_i, i \in N_n$, are finite too. Let $q_1^i, q_2^i, \ldots, q_{p_i}^i$ where $p_i = |Q_i|$, are the elements of the set $Q_i, i \in N_n, V^i = (v_1^i, v_2^i, \ldots, v_{p_i}^i)$ be the vector such that

$$v_s^i = \sum_{j \in N(q_s^i)} a_{ij} \ \forall s \in N_{p_i},$$

where a_{ij} are the elements of the matrix A.

Theorem 4.2. If all the components of every vector $V^i, \ i \in N_n$, are different in pairs, then the following lower bound for stability radius of the nontrivial trajectorial problem $Z^n(A)$ holds:

$$\rho_3^n(A) \geq \frac{1}{2} \min_{i \in N_n} \ \min_{1 \leq j < s \leq p_i} \frac{|v_j^i - v_s^i|}{k_i}. \tag{4.1}$$

Proof. By virtue of the structure of vectors $V_i,\ i \in N_n$, the following inclusions are obvious

$$\{f_i(t, A):\ t \in T\} \subseteq \{v^i_j:\ j \in N_{p_i}\}\ \forall i \in N_n.$$

Hence, since all the components of every vector $V_i,\ i \in N_n$, are different in pairs, the next equivalencies are true for any index $i \in N_n$ and any trajectories $t \neq t'$:

$$f_i(t, A) = f_i(t', A) \Longleftrightarrow f_i(t, A + B) = f_i(t', A + B)\ \forall B \in \Re(\varepsilon),$$

$$f_i(t, A) < f_i(t', A) \Longleftrightarrow f_i(t, A + B) < f_i(t', A + B)\ \forall B \in \Re(\varepsilon),$$

if

$$0 < \varepsilon \leq \frac{1}{2} \min_{i \in N_n} \min_{1 \leq j < s \leq p_i} \frac{|v^i_j - v^i_s|}{k_i}.$$

Therefore the equality $L^n(A) = L^n(A+B)\ \forall B \in \Re(\varepsilon)$ holds for the number ε defined above. Consequently, the bound (4.1) is true.

Theorem 4.2. has been proved.

The next proposition follows directly from theorem 4.2.

Corollary 4.1. Let $I_{max} \bigcup I_{min} = N_n$. If the elements of every row of the matrix A are different in pairs, then the following lower estimate for stability radius of the nontrivial trajectorial problem $Z^n(A)$ holds

$$\rho^n_3(A) \geq \frac{1}{2} \min_{i \in N_n} \min_{1 \leq j < s \leq p_i} |a_{ij} - a_{is}|.$$

5 Stability kernel

A trajectory $t \in L^n(A)$ is called stable if there exists a number $\varepsilon > 0$ such that

$$t \in L^n(A + B)\ \forall B \in \Re(\varepsilon).$$

We say that the set of all stable trajectories of the trajectorial problem $Z^n(A)$ is a stability kernel and write $K^n(A)$.

Let us introduce the set of all strict lexicographic optimal trajectories of the trajectorial problem $Z^n(A)$. By definition, put

$$S^n(A) = \{t \in L^n(A) : \exists i = i(t) \in N_n \ \forall t' \in T, \ t' \neq t \ (\tau_i(t, t', A) < 0)\}.$$

Theorem 5.1. For any combination of partial criteria (1.1) and (1.2) of the trajectorial problem $Z^n(A), \ n \geq 1$, we have

$$S^n(A) \subseteq K^n(A), \ \forall A \in \mathbf{R}^{nm},$$

moreover

$$S^n(A) = K^n(A)$$

if $I_{sum} = N_n$.

Proof. Let $t \in S^n(A)$. Then there exists an index $i \in N_n$ such that

$$\tau_i(t, t', A) < 0 \ \forall t' \in T, \ t \neq t.$$

Hence by virtue of continuity of any function $f_i(t, A)$ on the set $\mathbf{R}^{nm}$, we can find a number $\varepsilon > 0$ such that

$$\tau_i(t, t', A + B) < 0 \ \forall B \in \Re(\varepsilon).$$

Therefore from property 2.3 the trajectory t is a lexicographic optimum of a perturbed problem $Z^n(A+B)$ and, consequently, is stable. Thus, the inclusion $S^n(A) \subseteq K^n(A)$ holds.

Now we turn to proof of equality $S^n(A) = K^n(A)$ in the case where $I_{sum} = N_n$.It is sufficient to show that any trajectory t such that

$$t \in L^n(A) \backslash S^n(A)$$

is not stable. Since t is not strict lexicographic optimal, for any index $i \in N_n$, there exists a trajectory $t' \neq t$ such that

$$\tau_i(t, t', A) \geq 0.$$

Therefore for any number $\varepsilon > 0$ and the perturbing matrix $B \in \Re(\varepsilon)$ with the elements

$$b_{ij} = \begin{cases} \varepsilon \backslash 2 & \text{if} \quad i \in N_n, \ j \in N(t), \\ -\varepsilon \backslash 2 & \text{if} \quad i \in N_n, \ j \notin N(t), \end{cases}$$

we have

$$\tau_i(t,t',A+B) = \tau_i(t,t',A) + \frac{\varepsilon}{2}\Delta_i(t,t') > 0 \ \forall i \in N_n.$$

Hence by property 2.4 the trajectory t is not a lexicographic optimum of the perturbed problem $Z^n(A{+}B)$, i.e. it is not stable. Theorem 5.1 has been proved.

The next corollary follows from theorem 5.1.

Corollary 5.1. If $I_{sum} = N_n$, then $|K^n(A)| \le n \ \forall A \in \mathbf{R}^{nm}$.

Remark 5.1. By virtue of equivalence of all the norms in a finite-dimentsional space (see [31]) corollaries 2.1, 3.1, 3.2, 5.1 and theorems 4.1, 5.1 are valid for any norm in the space $\mathbf{R}^{nm}$ of perturbing matrices.

Suppose $\varepsilon > 0$. Then the ε-stability kernel of the problem $Z^n(A)$ is the set

$$K^n_\varepsilon(A) = \{t \in L^n(A) : \ t \in L^n(A+B) \ \forall B \in \Re(\varepsilon)\}.$$

Theorem 5.2. In order for lexicographic optimal trajectory t to belong to the ε-stability kernel $K^n_\varepsilon(A)$ it is sufficient, and also necessary in the case $I_{sum} = N_n$, to satisfy the following inequality

$$\max_{i\in N_n} \min_{t'\in T\backslash\{t\}} \gamma_i(t,t',A) \ge \varepsilon, \tag{5.1}$$

where the value $\gamma_i(t,t',A)$ is calculated by formula (3.2).

Proof. Sufficiency. Let $t \in L^n(A)$. Then by virtue (5.1) there exists an index $k \in N_n$ such that

$$\tau_k(t',t,A) \ge \varepsilon\Delta_k(t,t') \ \forall t' \in T\backslash\{t\}.$$

Therefore, by lemma 2.1, we have

$$\tau_k(t',t,A+B) > 0 \ \forall B \in \Re(\varepsilon) \ \forall t' \in T\backslash\{t\}.$$

Hence, using property 2.3, we get

$$t \in L^n(A+B) \ \forall B \in \Re(\varepsilon).$$

Consequently, the trajectory t belongs to $K_\varepsilon^n(A)$.

Necessity. Let $I_{sum} = N_n,\ \varepsilon > 0,\ t \in K_\varepsilon^n(A)$. Suppose that (5.1) doesn't hold. Then for any index $i \in N_n$ there exists a trajectory $t' \neq t$ such that

$$\gamma_i(t, t', A) < \alpha,$$

where

$$\varepsilon > \alpha > \max_{i \in N_n} \min_{t' \in T \setminus \{t\}} \gamma_i(t, t', A)$$

.

Therefore for the matrix $B \in \Re(\varepsilon)$ with elements

$$b_{ij} = \begin{cases} \alpha & \text{where} \quad i \in N_n,\ j \in N(t), \\ -\alpha & \text{where} \quad i \in N_n,\ j \notin N(t), \end{cases}$$

we have

$$\begin{aligned} \tau_i(t, t', A + B) &= \tau_i(t, t', A) + \alpha \Delta_i(t, t') > \\ &> \tau_i(t, t', A) + \gamma_i(t, t', A) \Delta_i(t, t') = 0 \\ &\forall i \in N_n. \end{aligned}$$

Hence by property 2.4 the trajectory t is not lexicographic optimum of the problem $Z^n(A + B)$, i.e. $t \notin K_\varepsilon^n(A)$.

The contradiction proves theorem 5.2.

The value

$$\rho^n(A) = \sup\{\varepsilon > 0 : \ K_\varepsilon^n(A) \neq \emptyset\}$$

is called the radius of the stability kernel of the problem $Z^n(A)$. If $K_\varepsilon^n(A) = \emptyset\ \forall \varepsilon > 0$, then put $\rho^n(A) = 0$.

By definition, put

$$\xi^n(A) = \max_{t \in L^n(A)} \max_{i \in N_n} \min_{t' \in T \setminus \{t\}} \gamma_i(t, t', A).$$

The next theorem follows from theorem 5.2.

Theorem 5.3. Let $A \in \mathbf{R}^{nm}$. For the radius of the stability kernel of the problem $Z^n(A),\ n \geq 1$, with any combination of partial criteria (1.1) and (1.2), the next estimate holds

$$\rho^n(A) \geq \xi^n(A),$$

moreover

$$\rho^n(A) = \xi^n(A)$$

if $I_{sum} = N_n$.

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