An algorithm for solving the transport problem on network with concave cost functions of flow on edges

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Abstract

We study the transport problem on network with concave cost function of flow on edges. An heuristic algorithm for solving this problem is proposed, and an implementation for solving allocation problem is given.

1 Introduction

In this paper we study the transport problem on network with concave cost functions depending on edges flow. This problem has an important implementation for solving the synthesis network problems and allocation problem. We propose an heuristic algorithm for solving the problem with concave cost functions on edges of the network which is based on results from [1]. This algorithm can be used for solving classic allocation problems.

2 Problem formulation and main results

Let us consider the transport network [2] which has a structure of directed graph $G = (V, E)$, with vertex set V, $|V| = n$, and edge set E, $|E| = m$. On V a bounded supply and demand functions $q : V \longrightarrow R$ are defined. To each edge e from set E a concave function $\phi_e(x(e))$ is associated which depends on flow $x(e)$ on edge. In addition to each arc $e \in E$ a communication capacity $d(e)$ and the transportation cost $\phi_e(x(e))$ of flow are given. By [3,4] we have the following definition:

Definition 1: The flow in $G = (V, E)$ is a real function $x : E \to R$ which satisfies the condition:

$$\begin{cases} \sum\limits_{e \in E^-(v)} x(e) - \sum\limits_{e \in E^+(v)} x(e) & = & q\,(v)\,,\ \forall v \in V \\ x\,(e) & \geq & 0, \forall e \in E \end{cases} \tag{1}$$

where $E^+(v) = \{(u, v) \mid (u, v) \in E\}$, $E^-(v) = \{(v, u) \mid (v, u) \in E\}$.

The transport problem on network G with given supply and demand function $q : V \longrightarrow R$ and the cost functions $\phi_e(x(e))$ of flow on the arcs $e \in E$ consists in finding the flow x^* which minimizes the function

$$F(x) = \sum_{e \in E} \phi_e(x(e)) \ , \tag{2}$$

i.e.

$$F(x^*) = \min_{x \in X} F(x) \ . \tag{3}$$

Here X represents the set of solutions of the system (1), i.e. X is a set of admissible flows in G.

In addition to the classical transport model the following constraints

$$x(e) \leq d(e), \forall e \in E \tag{4}$$

are given, which reflect the admissible capabilities of arcs, where $d(e)$, $\forall e \in E$ – the real given numbers. It is known [5] that the problem (1)–(4) on network G with linear functions $\phi_e(x(e))$, $\forall e \in E$ can be reduced to the problems of the type (1)–(3) for another network $G^/$, which contains $m \times n$ vertexes and $2m$ arcs [4]. Such method of reducing of the problem can be used in general case with bounded arcs capabilities. That is why we shall study the transport problem on network of the type (1)–(4).

For an arbitrary flow x in G we denote by $G_x = (V_x, E_x)$ the subgraph of G generated by edges $e \in E$ for $x(e) \geq 0$, i.e. $E_x = \{e \in E \mid x(e) > 0\}$.

Theorem 1 *Let (G, q, Φ) be an arbitrary network for which a flow x exists and the following conditions are satisfied:*

a. $\phi_e(x)$ *is a concave function,* $\forall e \in E$,

b. for any oriented cycle C with the set of arcs E_C, the function $\Phi_e(\theta) \sum\limits_{e \in E} \phi_e(a_e + \theta)$

is nondecreasing one for every $a_e \geq 0$; $e \in E$,

Then there exists the optimal flow x^* such that the graph G_x doesn't contain directed cycles.

The proof of the theorem is given in [1].

Corollary 1. *If $\phi_e(x(e))$, $\forall e \in E$ are strictly concave functions and the condition a) is satisfied then any optimal flow x^* has the property that respective graph G_{x^*} doesn't contain the cycles.*

Corollary 2. *If $\phi_e(x(e))$, $\forall e \in E$ are non-decreasing and concave functions then there exists the flow x^*, for which graph G_{x^*} doesn't contain the cycles.*

Corollary 3. *If $|X^+| = 1$, $X^+ = \{x \in X \mid q(x) > 0\}$ then for the concave and increasing functions $\phi_e(x(e))$, $\forall e \in E$, there exists the flow x^* for which graph G_{x^*} has a structure of directed tree with root $x_0 \in X^+$.*

Theorem 2 *The transport problem on network with concave function $\phi_e(x(e))$, $\forall e \in E$ is NP-hard problem.*

Proof: Let us assume that for any given $a_e \geq 0$, $e \in E$, the functions

$$\psi_e(x) = \begin{cases} a_e & , \quad x > 0 \\ 0 & , \quad x = 0 \end{cases} \tag{5}$$

are concave and nondecreasing ones. If $|X^+| = 1$, $X^+ = \{x \in X \mid q(x) > 0\}$, then the optimal solution has the property that the corresponding graph G_x has a structure of directed tree with root. Therefore, the transport problem on network in this case became the Steiner tree problem in weighted directed graph, which is NP–complete.□

Here we propose an approximative algorithm for solving the problem with concave cost function on edges of the networks.

3 Algorithm for solving the nonlinear transport problem

1. In G we find an admissible flow. For finding of such a flow we can use the algorithm from [1]. This means that we find values $x^0(e)$, $\forall e \in E$ which satisfy the conditions:

$$\begin{cases} \sum\limits_{e\in E^-} x^0(e) - \sum\limits_{e\in E^+} x^0(e) & = & q\,(v)\,, \forall v \in V \\ x\,(e) & \geq & 0 \end{cases} \tag{6}$$

2. We find the values $F(x^0) = \sum\limits_{e\in E} \phi_e(x^0(e))$.

   For this we find the values $\phi_e(x^0(e))$ and the coefficients $C_e = = \frac{\phi_e(x^0(e))}{x^0(e)}$, $\forall e \in E$, $x\,(e) \geq 0$. If $x^0\,(e) = 0$ then we set $C_e = = F^{/}_{e}(0)$.

3. We solve the linear transport problem $\min \to z(x) = \sum\limits_{e\in E} C_e x$ which satisfies the conditions

$$\begin{cases} \sum\limits_{e\in E^-} x^0(e) - \sum\limits_{e\in E^+} x^0(e) & = & q\,(v) \\ x\,(e) & \geq & 0 \end{cases} \tag{7}$$

and find the optimal solution $x^1 = (x^1(e_1), x^1(e_2), , x^1(e_m))$.

4. We compare the values $z(x^1)$ and $F(x^0)$. If $z(x^1) \leq F(x^0)$ we substitute x^0 with x^1 and pass to point 3. If $z(x^1) > F(x^0)$ then we consider x^0 the solution of the problem.

Remark: *The algorithm for solving the transport problem can be used also in the case when the capacities of edges* are bounded. *In this case we add the conditions* $C(e) \geq x\,(e) \geq d\,(e),\ \forall e \in E$ *to (5) and (6).*

4 Example

We give an example based on this algorithm. In the figure we have a graph which represents a transport problem on network with concave cost functions on the edges.

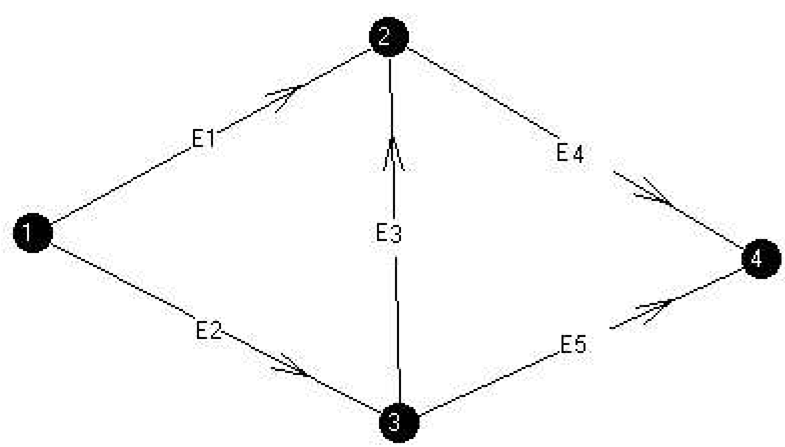

Figure 1

The costs on edges are described by the functions:

$$\phi_{e_1}(x) = \begin{cases} x & , \quad x \leq 1 \\ 1 & , \quad x > 1 \end{cases}$$

defined on the edges e_1, e_3,e_5 and

$$\phi_{e_2}(x) = \begin{cases} 2x & , \quad x \leq 2 \\ 2 & , \quad x > 2 \end{cases}$$

defined on the edges e_2, e_4.

We take $q(1) = -4$, $q(2) = 0$, $q(3) = 0$, $q(4) = 4$ and the flow $x^0(e) = (2, 2, 0, 2, 2)$.

First of all we find the values:

$\phi_{e_1}(2) = 1$, $\phi_{e_2}(2) = 4$, $\phi_{e_3}(2) = 0$, $\phi_{e_4}(2) = 4$, $\phi_{e_5}(2) = 1 \Rightarrow F(x^0) = \phi_{e_1}(2) + \phi_{e_2}(2) + \phi_{e_3}(2) + \phi_{e_4}(2) + \phi_{e_5}(2) = 10$.

We find the coefficients $C_e^0 = \frac{\phi_e(x^0(e))}{x^0(e)} : C_{e_1}^0 = \frac{\phi_{e_1}(x^0(e))}{x^0(e)} = \frac{1}{2}$, $C_{e_2}^0 = \frac{\phi_{e_2}(x^0(e))}{x^0(e)} = 2$, $C_{e_3}^0 = 0$, $C_{e_4}^0 = \frac{\phi_{e_4}(x^0(e))}{x^0(e)} = 2$, $C_{e_5}^0 = \frac{\phi_{e_5}(x^0(e))}{x^0(e)} = \frac{1}{2}$, then $c^0 = \left(\frac{1}{2}, 2, 0, 2, \frac{1}{2}\right)$

We solve the linear transport problem $\min \rightarrow z(x) = \sum\limits_{e \in E} C_e^0 x = \frac{1}{2}x_1 + 2x_2 + 2x_4 + \frac{1}{2}x_5$ which satisfies the conditions (6) and obtain $x^1(e) = (4, 0, 0, 4, 0)$ and respectively $z(x^1) = 10$.

Since $z(x^1) = F(x^0)$ then x^0 is the solution. Then we pass to point 2.

We find the values:

$\phi_{e_1}(4) = 1$, $\phi_{e_2}(0) = 0$, $\phi_{e_3}(0) = 0$, $\phi_{e_4}(4) = 1$, $\phi_{e_5}(0) = 0 \Rightarrow F(x^1) = \phi_{e_1}(4) + \phi_{e_2}(0) + \phi_{e_3}(0) + \phi_{e_4}(4) + \phi_{e_5}(0) = 2$.

We find the coefficients $C_e^1 = \frac{\phi_e(x^1(e))}{x^1(e)} : C_{e_1}^1 = \frac{\phi_{e_1}(x^1(e))}{x^1(e)} = \frac{1}{4}$, $C_{e_2}^1 = F'_{e_2}(0) = 2$, $C_{e_3}^1 = F'_{e_3}(0) = 1$, $C_{e_4}^1 = \frac{\phi_{e_4}(x^1(e))}{x^1(e)} = \frac{1}{4}$, $C_{e_5}^1 = F'_{e_5}(0) = 2$, then $c^1 = \left(\frac{1}{4}, 2, 1, \frac{1}{4}, 2\right)$

We solve the linear transport problem $\min \rightarrow z(x) = \sum\limits_{e \in E} C_e^1 x = \frac{1}{4}x_1 + 2x_2 + x_3 + \frac{1}{4}x_4 + 2x_5$ which satisfies the conditions (6) and obtain $x^2(e) = (0, 4, 0, 0, 4)$ and respectively $z(x^2) = 16$.

Since $z(x^2) > F(x^1)$ then we consider x^1 as optimal solution of the problem. $\square$

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