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On the upper chromatic index of a multigraph *

Mario Gionfriddo, Lorenzo Milazzo, Vitaly Voloshin †

Abstract

We consider the colorings of the edges of a multigraph in such a way that every non-pendant vertex is incident to at least two edges of the same color. The maximum number of colors that can be used in such colorings is the upper chromatic index of a multigraph G, denoted by $\bar{\chi}'(G)$. We prove that if a multigraph G has n vertices, m edges, p pendant vertices and maximum number c disjoint cycles, then $\bar{\chi}'(G) = c + m - n + p$.

1 Introduction

In this paper we consider multigraphs without loops, i.e. undirected graphs having multiple (repeated) edges [1, 8]. The basic facts concerning the colorings of the edges of multigraphs are related to the results of V. Vizing, see [4, 5, 6]. These colorings, which we will call *"classical",* define a mapping from the set of edges into a finite set of colors such that two adjacent edges never have the same color. Given a multigraph $G = (X, E)$, the *chromatic index* $\chi'(G)$ is the minimum number of colors for which the classical coloring of G exists.

The following theorem [4] relates the chromatic index $\chi'(G)$ to the maximum degree $\Delta(G)$ and the maximum multiplicity (the number of multiple edges) h of a multigraph G:

Theorem 1 (Vizing) *If G is a multigraph without loops then:*

$$\chi'(G) \leq \Delta(G) + h.$$

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□

In a multigraph, a vertex is called pendant if its degree is 1; we also consider that two repeated edges form a cycle of length 2. In this paper, we define a coloring of the edges of a multigraph $G = (X, E)$ by a correspondence between E and the set of colors $\{1, 2, \ldots, k\}$, in which each non-pendant vertex $x \in X$ is incident to at least two edges of the same color.

We determine the maximum number of colors for a general multigraph $G = (X, E)$ which has n vertices, m edges, p pendant vertices and the maximum number of disjoint cycles c. This value is determined for particular classes of graphs and multigraphs. We also present two algorithms. The first determines an optimal edge coloring for trees. The second determines a lower bound for the number of colors in an arbitrary multigraph.

It is worth mentioning that the problem is motivated by the coloring theory of mixed hypergraphs and in a more general setting was first formulated in [7, Problem 13].

2 An edge coloring of multigraphs

Let $G = (X, E)$ be an arbitrary multigraph and $\{1, 2, \ldots, k\}$ be a set of colors.

Definition 1 A *proper edge k-coloring* of the multigraph G is a mapping $f : E \to \{1, 2, \ldots, k\}$ such that every non-pendant vertex of G is incident to at least two edges of the same color, and in addition, each of k colors is used.

It is evident that following this definition, the minimum number of colors is one. This is a complementary fashion of the fact that in classical edge coloring the maximum number of colors is trivially equal to $|E|$. Henceforward the term "coloring" always refers to an edge coloring of the type specified by Definition 1.

Definition 2 The maximum number k for which there exists a proper edge k-coloring of a multigraph G is called the *upper chromatic index* and denoted by $\bar{\chi}'(G)$.

Theorem 2 *For a connected multigraph G, $\bar{\chi}'(G) = 1$ if and only if $\Delta(G) \leq 2$.*

Proof. Let $\bar{\chi}'(G) = 1$. For a contradiction, suppose $\Delta(G) \geq 3$. If G contains a cycle, then color the edges of the cycle with color 1, and all the other edges with color 2. If G is a tree, color any maximal path with color 1 and all the other edges with color 2. In both cases we obtain a proper edge 2-coloring, a contradiction.

The converse is evident since $\Delta(G) \leq 2$ implies that G is either a cycle or a path.

□

For a connected multigraph G, theorem 2 immediately implies that $\bar{\chi}'(G) \geq 2$ if and only if $\Delta(G) \geq 3$.

Let f be a proper edge k-coloring of $G = (X, E)$; it partitions E into k *color classes* $\{C_1, C_2, \ldots, C_k\}$ where each C_i is the set of edges colored with color i.

Let $G_i = (X_i, C_i)$ be the partial subgraph [1] of G with a set of vertices X_i determined by the endpoints of the edges of C_i. Note that generally G_is are not necessarily connected multigraphs and may have common vertices.

Definition 3 A vertex x of a subgraph G_i is said to be *satisfied* by the subgraph G_i if it is not a pendant vertex in G_i.

In the next part of this section, we deal with colorings that use $\bar{\chi}'(G)$ colors.

Theorem 3 *If f is a coloring of G using $\bar{\chi}'(G)$ colors, then $\bar{\chi}'(G_i) = 1$, $i = 1, \ldots, \bar{\chi}'(G)$.*

Proof. In the coloring f, every pendant vertex of G_i is satisfied by some other subgraph. Therefore any proper coloring of G_i with at least two colors leads to a proper $(\bar{\chi}'(G)+1)$-coloring of G, a contradiction.

$\square$

Corollary 1 *Let f be a coloring of G using $\bar{\chi}'(G)$ colors. Then each G_i, where $1 \leq i \leq \bar{\chi}'(G)$, is either a cycle or a path.*

Proof. Apply Theorem 2. $\square$

Theorem 4 *Let f be a coloring of G using $\bar{\chi}'(G)$ colors. Then for every vertex x of G the following implications hold:*

1) if x is satisfied by a cycle, then x is satisfied by no path;

2) if x is satisfied by two cycles, say G_i and G_j, then x is the only common vertex for G_i and G_j.

Proof. It follows immediately, since otherwise we can easily construct a coloring using at least $\bar{\chi}'(G)+1$ colors. $\square$

Corollary 2 *Let f be a coloring of G using $\bar{\chi}'(G)$ colors. If a vertex x of G is satisfied by more than one G_k, then all G_ls which satisfy x, are all cycles with one vertex in common.*

$\square$

3 Formula and some applications

Given a $\bar{\chi}'(G)$-coloring of an arbitrary multigraph G, we now partition the subgraphs G_i of G into the following three parts.

Part A: contains all G_i forming the maximum number of (vertex) disjoint cycles in f. Clearly, the number c' of subgraphs in A satisfies $c' \leq c$ where c is the maximum number of disjoint cycles in G.

Part B: contains the remaining cycles and paths with length at least two each.

Part C: contains G_i which represent a separate edge each.

If the part B contains cycles, then given the maximum number of disjoint cycles in A, and by Theorem 3, each of these cycles has a single vertex in common with only one of the cycles in A. The cycles in B may be considered as paths whose endpoints coincide.

The next theorem determines the formula for the upper chromatic index.

Theorem 5 *If $G = (X, E)$ is an arbitrary multigraph with $|X| = n$, $|E| = m$, the number of pendant vertices p, and the maximum number of disjoint cycles c, then*

$$\bar{\chi}'(G) = c + m - n + p.$$

Proof.

We prove first that $\bar{\chi}'(G) \leq c+m-n+p$. By Corollary 1, if a coloring f uses $\bar{\chi}'(G)$ colors, then the partial subgraphs G_i, $1 \leq i \leq \bar{\chi}'(G)$, are either cycles or paths, while Corollary 2 states that if a vertex x is satisfied by more than one graph G_i, then all these graphs have precisely one vertex in common.

Let c' be the number of subgraphs, and v' be the number of satisfied vertices in part A. Clearly, v' coincides with the number of edges in A. If x is satisfied by some G_l from part A, then it is possible to look at all the other G_ks which satisfy x, like at paths with endpoints x.

If x is not satisfied by graphs from part A, then it is possible to consider one G_k which satisfies x like a cycle and all others like paths with endpoints x.

Let r be the number of G_i in part B. Then in part B, $n - v' - p$ vertices are satisfied, and there remains $n - v' - p + r$ edges.

Let m' be the number of edges in parts A and B. We then get:

$$m' = n - p + r.$$

The number of colors in f is:

$$\bar{\chi}'(G) \;=\; c' \;+\; r \;+\; m \;-\; m',$$

and so we have

$$\bar{\chi}'(G) \;=\; c' \;+\; m \;-\; n \;+\; p \leq c + m - n + p.$$

Next we prove that $\bar{\chi}'(G) \geq c + m - n + p$. Let us choose c disjoint cycles $C_1, \ldots, C_c$ and denote the partial subgraph obtained by A_0. Color the edges of these cycles with the colors $1, \ldots, c$ respectively. We have that $\bar{\chi}'(A_0) = c + m(A_0) - n(A_0) + p(A_0)$, so the formula holds.

Now we implement the following coloring procedure. Choose any satisfied vertex, say x, incident to an uncolored edge and construct a path along uncolored edges until the first satisfied vertex y is reached or we stick at a pendant vertex. Since A_0 constitutes the maximum number of disjoint cycles, the new vertices and edges represent either a cycle or a path (if they form a cycle, then $x = y$). Add these vertices and edges to A_0 and denote the new partial subgraph obtained by A_1. Color the added edges with a new color and declare respective vertices satisfied. Simple calculations show that

$$\bar{\chi}'(A_1) = \bar{\chi}'(A_0) + 1 = c + m(A_1) - n(A_1) + p(A_1),$$

and therefore the formula holds. We repeat this coloring procedure by constructing subgraphs $A_2, A_3, \ldots$ until all the edges of G are colored and all the vertices satisfied. Since at each coloring step the formula holds, the theorem follows.

□

Observe that in a multigraph, $c \leq \lfloor n/2 \rfloor$, so generally,

$$\bar{\chi}' \leq m - \lceil n/2 \rceil + p.$$

3.1 Upper chromatic index for particular graphs

Theorem 5 allows us to calculate the upper chromatic index for particular classes of multigraphs.

We can also easily see that the upper chromatic index does not depend on the subdivisions of edges, since the number of disjoint cycles, difference between the number of edges and the number of vertices and the number of pendant vertices remain the same.

Theorem 6 *If G is a tree then*

$$\bar{\chi}'(G) \ = \ p \ - \ 1.$$

Proof. Evident given that $c = 0$ and $m \ = \ n \ - \ 1$.

□

As we have seen, Theorem 5 relates $\bar{\chi}'(G)$ to the maximun number of disjoint cycles in a multigraph. Let us now consider graphs G that possess a 2-factor [1, 8], i.e. a family $\mathcal{F}$ of vertex disjoint cycles (partial subgraphs of G) such that each vertex of G belongs to one and only one element of $\mathcal{F}$. If $\mathcal{F}$ is a 2-factor, then let $|\mathcal{F}|$ means the number of connected components in $\mathcal{F}$.

Let us denote by $c_{max} \ = \ \max\{ \ | \ \mathcal{F} \ | \ : \ \mathcal{F} \ factor \ of \ G\}$, i.e. the maximum cardinality over all the families $\mathcal{F}$ of G.

Theorem 7 *If G is a graph with a 2-factor, then we have:*

$$\bar{\chi}'(G) \ \geq \ c_{max} \ + \ m \ - \ n.$$

□

Examples show that generally, the number of disjoint cycles in a multigraph can exceed the maximum number of components in 2-factor: in this case disjoint cycles do not cover all the vertices.

Corollary 3 *If G is a complete graph K_n then we have*

$$\begin{cases} if \quad n \ = \ 3k & \bar{\chi}' \ = \ \dfrac{9k^2 - 7k}{2} \\ if \quad n \ = \ 3k \ + \ 1 & \bar{\chi}' \ = \ \dfrac{9k^2 + k - 2}{2} \\ if \quad n \ = \ 3k \ + \ 2 & \bar{\chi}' \ = \ \dfrac{9k^2 + 5k - 2}{2}. \end{cases} \tag{1}$$

Proof.
For each K_n identified in (1), $c_{max} = k$ and the proof follows from Theorem 7. □

Let us consider a graph $G = (X, E)$ made up of l multigraphs $(G_1, G_2, \ldots, G_l)$ connected in sequence by a single edge. By connecting in sequence we mean that the graph G_i is connected by a single edge e to graph G_{i+1} and by a single edge e' to graph G_{i-1} and, in addition, G_1 and G_l are not connected in sequence by any edge. Cancelling e and e' disconnects G_i from all the other graphs G_k making up G.

Theorem 8 *Let G be a multigraph comprising l multigraphs $(G_1, G_2, \ldots, G_l)$ connected in sequence by a single edge. We have:*

$$\bar{\chi}'(G) = \sum_{i=1}^{l} \bar{\chi}'(G_i) + l - 1. \tag{2}$$

Proof.
The multigraphs $G_i = (X_i, E_i)$ that make up G have n_i vertices, m_i edges and p_i pendant vertices respectively, where $1 \leq i \leq l$. If e is the single edge connecting G_i with G_{i+1} then we have $e \notin E_i \cup E_{i+1}$ and the number of edges in G is $\sum_{i=1}^{l} m_i + l - 1$. So by Theorem 5 we have:

$$\bar{\chi}'(G) = \sum_{i=1}^{l}(c_i + m_i - n_i + p_i) + l - 1.$$

which is equivalent to (2). □

Finally, let us consider plane multigraphs.

Theorem 9 *If G is a plane multigraph with f faces, then*

$$c + p - \bar{\chi}'(G) + f = 2.$$

Proof. Apply Euler's formula [1, 8]. □

3.2 Algorithmic issues

In this subsection we present two algorithms. The first colors trees with exactly $\bar{\chi}'$ colors. The second greedily colors any multigraph G with some number of clors.

Algorithm 1 (edge-coloring of a tree)

Input: A tree $T = (X, E)$.
Idea: Consecutive coloring by adding pendant vertices.
Initialization: Color any edge with the first color.
Iteration: Implement reconstruction of T by adding pendant vertices. If pendant vertex x to be added is adjacent to pendant vertex y, then color the edge xy with the old color to satisfy y. Otherwise color the edge xy with a new color. Iterate.

It is clear that we get a coloring with $\bar{\chi}'(T) = p-1$ colors, moreover, if we have k colors and not all colors must be used, then there are k^{p-1} such colorings.

Algorithm 2 (greedy edge-coloring of a multigraph)

Input: An arbitrary multigraph $G = (X, E)$.
Idea: Shortest self-return.
Initialization: A vertex $x \in X$.
Iteration: Choose any satisfied vertex y (or x if none) and uncolored edge $yy_1(xy_1)$. Construct a path along uncolored edges taking each time a new vertex, until the first satisfied vertex z is reached or we stick at a vertex whose neighbors are not new. During this search, check for the uncolored "back" edges (connecting new vertex with old one of the current path). If there are back edges, then choose the shortest cycle, color its edges with the next color, declare its vertices satisfied. Otherwise color the edges of the path with the next color and declare its vertices except x satisfied. Iterate.

Theorem 10 *It is NP-complete to find the upper chromatic index of a multigraph.*

Proof. Since upper chromatic index does not depend on subdivisions of the edges, the problem for multigraphs is equivalent to the problem on ordinary graphs.

It is polynomial to verify that a family of cycles is disjoint, so the problem is in NP. It remains to show that it is NP-hard. We reduce the problem INDEPENDENT SET in 3-regular graphs [3] to the maximum number of disjoint cycles. Given an arbitrary 3-regular graph G, we construct a graph G' such that every maximal independent set of vertices in G corresponds to a maximal family of disjoint cycles in G' and vice versa.

Let $G = (X, E)$ be an arbitrary 3-regular graph. Consider n disjoint triangles, each of which corresponds to some vertex of G and vice versa. If vertices x and y are adjacent in G, then we connect the respective triangles by an edge in such a way that in each triangle no vertex is used twice. Since G is 3-regular, this is always possible. We now contract added edges to make respective triangles nondisjoint. Triangles corresponding to nonadjacent vertices of G remain disjoint. We claim that in G', for any maximal family of disjoint cycles other than consisting of initial triangles, there exists a family of disjoint triangles consisting of the initial triangles which has the same cardinality.

Indeed, if such a family does not intersect at least one initial triangle, then it can be augmented. If it contains cycles passing through all the initial triangles, then we make the following modification: replace passing cycle by the respective triangle from the initial family.

□

Remark. After the paper was submitted, Duchet communicated to the authors that essentially the same relation of Theorem 5 was found in [2].

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M.Gionfriddo, L.Milazzo, V.Voloshin,

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Mario Gionfriddo, Lorenzo Milazzo
Dipartimento di Matematica e Informatica,
Università di Catania,
Viale A.Doria, 6 95125 - Catania, Italia.
E-mail: gionfriddo@dmi.unict.it
milazzo@dmi.unict.it

Vitaly Voloshin
Institute of Mathematics and Computer Science
of Moldovan Academy of Sciences,
str. Academiei 5, Chişinău,
MD-2028, Moldova.
E–mail: voloshin@math.md