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The family of local C^2 splines with two free generating functions

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Abstract

The family of local generalised C^2 splines with two free generating functions is proposed. Cases of one-dimensional and bidimensional interpolation are discussed with numerical examples.

1 Introduction

The generation of interpolating curves using splines is a useful and powerful tool in computer-aided geometric design. Although various methods of spline interpolation were proposed (cf., e.g., [1]–[4], [6]–[8]) the problem of new splines generation is the actual one. In the present paper a spline family with two free generating functions is discussed. Let us assume that on the segment $[a, b]$ the mesh $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ is given and values $f_i = f(x_i), i = 0, (1), n,$ at the knots of the mesh are known. The interpolant $S(x)$ such that $S(x_i) = f_i, i = 0, (1), n$; and $S \in C^2[a, b]$, is to be constructed.

This problem can be solved [1] using polynomial C^2 cubic splines. But in many cases difficulties may appear when these splines are used. We refere to the cases when the set of initial data is too large or the initial data are dynamically complemented. Thus when complementary data appear the using of cubic splines leads to the necessity to solve once more the equations system upon the unknown coefficients of spline and to recompute the values of spline and its derivatives on the whole segment. The lack of the property of localness becomes more critical in the case of bidimensional interpolation. Because of these factors we sometimes give up this type of splines and use the well known local splines (see, for instance, [3]–[4]), which do not possess the smoothness required. At the same time the continuity of the second derivative may be essential requirement in many practical problems.

In the present paper the family of splines which hold both conditions - splines are local ones and are continuous with their derivatives of the first and the second order is proposed .

2 One-dimensional interpolation.

Let us introduce the following notations:

$$S'(x_i) = m_i, \quad i = 0,(1),n; \quad h_i = x_{i+1} - x_i, \quad t = (x - x_i)/h_i.$$

Let us define splines as follows: on $[x_i, x_{i+1}]$

$$\begin{aligned} S(x) \;\; &= \;\; f_i(1 - \nu_1(t)) + f_{i+1}\nu_1(t) + h_i m_i \nu_2(t) + \\ &\quad\; + h_i m_{i+1}(t - \nu_1(t) - \nu_2(t)), \end{aligned} \tag{1}$$

where $(\nu_1(t), \nu_2(t)) \in L$, and

$$\begin{aligned} L = \{(\nu_1(t), \nu_2(t)) : \nu_1(1) = 1, \quad \nu_1(0) = \nu'(0) = \nu'(1) = 0, \\ \nu_2(0) = \nu_2(1) = \nu_2'(1) = 0, \quad \nu_2'(0) = 1, \quad \nu_2''(1) = 0, \\ \nu_2''(0) + \nu_1''(0) = 0; \quad \nu_1(t), \nu_2(t) \in C^2[0,1]\}. \end{aligned} \tag{2}$$

Functions $\nu_1(t)$ and $\nu_2(t)$ will be called generating functions for spline (1) and the set L will be called the set of generating functions. It is easy to prove that interpolation conditions are held for every pair of generating functions $(\nu_1(t), \nu_2(t)) \in L$, namely $S(x_i) = f_i, \quad i = 0,(1),n$.

From (1) the next formulae for the first and the second derivatives of spline are obtained.

$$S'(x) = \delta_i^{(1)}\nu_1'(t) + m_i\nu_2'(t) + m_{i+1}(1 - \nu_1'(t) - \nu_2'(t)), \tag{3}$$

$$S''(x) = [\delta_i^{(1)}\nu_1''(t) + m_i\nu_2''(t) - m_{i+1}(\nu_1''(t) + \nu_2''(t))]/h_i, \tag{4}$$

where $\delta_i^{(1)} = (f_{i+1} - f_i)/h_i$.

Taking into account (3) and properties of generating functions it follows that the first derivative of spline is continuous on $[a, b]$. From (4) it follows

$$S''(x_i+) = (\delta_i^{(1)} - m_i)\nu_1''(0)/h_i, \tag{5}$$

$$S''(x_{i+1}-) = (\delta_i^{(1)} - m_{i+1})\nu_1''(1)/h_i. \tag{6}$$

and from the requirement of continuity of the second derivative of the spline at the knots of the mesh Δ $S''(x_i+) = S''(x_i-)$ we get

$$m_i = [\lambda_i \nu_1''(1)\delta_{i-1}^{(1)} - \mu_i \nu_1''(0)\delta_i^{(1)}]/[\lambda_i \nu_1''(1) - \mu_i \nu_1''(0)], \tag{7}$$

$$i = 1, (1), n-1,$$

where $\mu_i = h_{i-1}/(h_{i-1} + h_i), \lambda_i = 1 - \mu_i$.

As a result $(n-1)$ unknown coefficients of splines are determined. Values of m_0 and m_n remain unknown. There are two possibilities in this case - either to construct an interpolant on $[x_1, x_{n-1}]$ only or to determine values of m_0 and m_n in an appropriate way. So, one of the following methods of approximation can be used:

1. If at points a and b $f'(a) = f_0'$ and $f'(b) = f_n'$ are known then $m_0 = f_0'$ and $m_n = f_n'$ is an obvious choice.

2. In the case when at points a and b $f''(a) = f_0''$ and $f''(b) = f_n''$ are known then forcing $S''(a) = f_0''$ and $S''(b) = f_n''$, we get $m_0 = \delta_0^{(1)} + h_0 f_0''/\nu_1''(0)$ and $m_n = \delta_{n-1}^{(1)} + h_{n-1} f_n''/\nu_1''(1)$.

3. If the function $f(x)$ is periodic with period $(b-a)$, extending periodically the mesh Δ, values m_0 and m_n can be computed using formula (7).

Methods 1–3 above represent well known end conditions which are widely used in cases of nonlocal interpolation. It should be mentioned that any appropriate way of approximation of m_0 and m_n may be used. In the sequel it is supposed that the interpolant is constructed on the segment $[x_1, x_{n-1}]$ only.

So, the family of splines which solve the problem stated at the beginning of the paper is constructed. Splines from this family are local ones, which is very important in many practical situations.

In many cases an other representation of spline may be more convenient, namely the representation via the values of the second derivative of spline at the knots of the mesh. Let us denote $S''(x_i) = M_i$. Then from (5) and (6) it follows

$$m_i = \delta_i^{(1)} - h_i M_i/\nu_1''(0)$$

$$m_{i+1} = \delta_i^{(1)} - h_i M_{i+1}/\nu_1''(1).$$

Substituting the previous in (1) the following representations of splines and its derivatives are obtained

$$\begin{aligned}
S(x) &= f_i(1-t) + f_{i+1}t - h_i^2 M_i \nu_2(t)/\nu_1''(0) - \\
&\quad - h_i^2 M_{i+1}(t - \nu_1(t) - \nu_2(t))/\nu_1''(1) \\
S'(x) &= \delta_i^{(1)} - h_i M_i \nu_2'(t)/\nu_1''(0) - \\
&\quad - h_i M_{i+1}(1 - \nu_1'(t) - \nu_2'(t))/\nu_1''(1) \\
S''(x, u_i) &= M_{i+1}(\nu_1''(t) + \nu_2''(t))/\nu_1''(1) - M_i \nu_2''(t)/\nu_1''(0)
\end{aligned}$$

The values of unknown coefficients of the spline in this case are computed as follows:

$$M_i = \nu_1''(1)\nu_1''(0)\delta_i^{(2)}/[\lambda_i \nu_1''(1) - \mu_i \nu_1''(0)], \quad i = 1, (1), n-1,$$

where $\delta_i^{(2)} = (\delta_i^{(1)} - \delta_{i-1}^{(1)})/(h_{i-1} + h_i)$.

There are no difficulties to get expressions for computation of M_0 and M_n, which remain unknown yet, using end conditions presented above.

3 Error analisys.

Problem of the interpolation accuracy using spline (1) is of great interest. Let $W_\infty^n[a,b]$ be the real Sobolev space

$$W_\infty^n[a,b] = \{f \in C^{n-1}[a,b] : f^{(n-1)} abs.cont.; f^{(n)} \in L_\infty[a,b]\}$$

The following theorem can be stated

Theorem 1 *If $f(x) \in W^2_\infty[a,b]$, then for the spline (1) generated by the functions $(\nu_1(t), \nu_2(t)) \in L$, where thr second derivative of the function $\nu_1(t)$ has different signes at the points 0 and 1, the following estimate*

$$\|S^{(k)} - f^{(k)}\|_C = O(\bar{h}^{2-k}), \quad k = 0, 1,$$

is valid, where $\bar{h} = \max_i h_i$.

Proof. Let's denote by $S_E(x)$ the corresponding Hermite spline (see [5]). Then the following identity

$$R(x) = S(x) - f(x) = S(x) - S_E(x) + S_E(x) - f(x)$$

can be obtained for the error term of interpolation. From the last relation we get

$$|R(x)| \le |S(x) - S_E(x)| + |S_E(x) - f(x)|. \tag{8}$$

It was shown (see [5]) that

$$\|S_E(x) - f(x)\|_C = O(\bar{h}^2), \tag{9}$$

therefore we have to estimate the first term of the right-hand side in (8). The next inequality

$$|S(x) - S_E(x)| \le \bar{h} \max_i(|m_i - f'_i|) \max_t(|\nu_2(t)| + |t - \nu_1(t) - \nu_2(t)|) \tag{10}$$

can be easy obtained. Taking into account (7) it can be proved that

$$|m_i - f'_i| \le \bar{h}\|f''\|_\infty/4.$$

As a result from (10) it follows

$$|S(x) - S_E(x)| = O(\bar{h}^2).$$

Thus, from (8)-(10) we get

$$\|S(x) - f(x)\|_C = O(\bar{h}^2).$$

In an analogous way the corresponding result for the derivative can be obtained. So, the proof of the theorem is ended.

4 Bidimensional interpolation.

Let us continue with the problem of bidimensional interpolation. Let's assume that the mesh $\Delta = \Delta_x \times \Delta_y$ is given on the domain $\Omega = [a, b] \times [c, d]$, where $\Delta_x : a = x_0 < x_1 < \ldots < x_n = b$ and $\Delta_y : c = y_0 < y_1 < \ldots < y_r = d$. Let us suppose that values $f(x_i, y_j) = f_{ij}, \quad i = 0, (1), n; \quad j = 0, (1), r$, are known at tne knots of mesh Δ. The interpolant $S(x, y)$ such that $S(x_i, y_j) = f_{ij}, \quad i = 0, (1), n; \quad j = 0, (1), r$, and $S(x, y) \in C^{2,2}(\Omega)$ is to be constructed. By $C^{2,2}(\Omega)$ the class of functions $g(x, y)$, which are continuous together with their derivatives $\partial^{k+l} g(x, y)/\partial^k x \partial^l y; \quad k, l = 0, 1, 2$, on the domain Ω is denoted.

Let us define splines as follows: on $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$

$$S(x, y) = \phi_i(t) F_{ij} \psi_j(\tau), \tag{11}$$

where

$$F_{ij} = \begin{pmatrix} f_{ij} & f_{ij+1} & m_{ij}^{(1,0)} & m_{ij+1}^{(0,1)} \\ f_{i+1j} & f_{i+1j+1} & m_{i+1j}^{(0,1)} & m_{i+1j+1}^{(0,1)} \\ m_{ij}^{(1,0)} & m_{ij+1}^{(1,0)} & m_{ij}^{(1,1)} & m_{ij+1}^{(1,1)} \\ m_{i+1j}^{(1,0)} & m_{i+1j+1}^{(1,0)} & m_{i+1j}^{(1,1)} & m_{i+1j+1}^{(1,1)} \end{pmatrix},$$

$\partial^{k+l} S(x_i, y_j)/\partial x^k \partial y^l = m_{ij}^{(k,l)}$; $k, l = 0, 1$;
$\phi_i(t)$ and $\psi_j(\tau)$ are vector functions of the following form

$$\phi_i(t) = (1 - \nu_1(t), \quad \nu_1(t), \quad h_i \nu_2(t), \quad h_i(t - \nu_1(t) - \nu_2(t))), \tag{12}$$

$$\psi_j(\tau) = (1 - \nu_1(\tau), \quad \nu_1(\tau), \quad l_j \nu_2(\tau), \quad l_j(\tau - \nu_1(\tau) - \nu_2(\tau)))^T, \tag{13}$$

$t = (x - x_i)/h_i, \quad h_i = x_{i+1} - x_i, \quad \tau = (y - y_j)/l_j, \quad l_j = y_{j+1} - y_j, \quad T$ denotes the operation of transposition and the pair of functions $(\nu_1(t, u), \nu_2(t, u))$ belongs to L.

The next natural question arises - are the interpolation conditions held? Taking into account the definition of the set of generating functions L we have $\psi^T(0) = \phi(0) = (1, 0, 0, 0)$ and $\psi^T(1) = \phi(1) = (0, 1, 0, 0)$. Then it is easy to obtain from (11), that the spline holds interpolation conditions in every domain Ω_{ij}, $i = 0,(1),n-1$, $j = 0,(1),r-1$. Let us introduce the following notations

$$\mu_j^* = l_{j-1}/(l_{j-1}+l_j), \quad \lambda_j^* = 1-\mu_j^*,$$

$$\delta_{ij}^{(1,0)} = (f_{i+1j} - f_{ij})/h_i, \quad i = 0,(1),n-1; \quad j = 0,(1),r;$$

$$\delta_{ij}^{(0,1)} = (f_{ij+1} - f_{ij}/l_j, \quad i = 0,(1),n; \quad j = 0,(1),r-1.$$

Then unknown coefficients of spline are determined using the next formulae

$$m_{ij}^{(1,0)} = (\lambda_i\nu_1''(1)\delta_{i-1j}^{(1,0)} - \mu_i\nu_1''(0)\delta_{ij}^{(1,0)})/(\lambda_i\nu_1''(1) - \mu_i\nu_1''(0)), \tag{14}$$

$$i = 1,(1),n-1; \quad j = 0,(1),r;$$

$$m_{ij}^{(0,1)} = (\lambda_j^*\nu_1''(1)\delta_{ij-1}^{(0,1)} - \mu_j^*\nu_1''(0)\delta_{ij}^{(0,1)})/(\lambda_j^*\nu_1''(1) - \mu_j^*\nu_1''(0)), \tag{15}$$

$$i = 0,(1),n; \quad j = 1,(1),r-1;$$

and

$$m_{ij}^{(1,1)} = [\lambda_i\nu_1''(1)(m_{ij}^{(0,1)} - m_{i-1j}^{(0,1)})/h_{i-1} -$$

$$-\mu_i\nu_1''(0)(m_{i+1j}^{(0,1)} - m_{ij}^{(0,1)})/h_i]/(\lambda_i\nu_1''(1) - \mu_i\nu_1''(0)), \tag{16}$$

$$i = 1,(1),n-1; \quad j = 1,(1),r-1;$$

or

$$m_{ij}^{(1,1)} = [\lambda_j^*\nu_1''(1)(m_{ij}^{(1,0)} - m_{ij-1}^{(1,0)})/l_{j-1} -$$

$$-\mu_j^*\nu_1''(0)(m_{ij+1}^{(1,0)} - m_{ij}^{(1,0)})/l_j^*]/(\lambda_i\nu_1''(1) - \mu_i^*\nu_1''(0)), \tag{17}$$

$$i = 1,(1),n-1; \quad j = 1,(1),r-1.$$

Substituting in (16) expressions for $m_{ij}^{(0,1)}$, given in (15), we get (17). So, formulae (16) and (17) are equivalent and values of $m_{ij}^{(1,1)}$ can be computed using one of them. As a result there are coefficients at the knots situated on the border of the domain Ω, which remain unknown, namely at the knots situated on the lines $x = x_i$, $i = 0, n$, and $y = y_j$, $j = 0, r$. Thus the interpolant is constructed on the domain $\Omega' = [x_1, x_{n-1}] \times [y_1, y_{r-1}]$ only. In this situation there are two possibilities - either to construct an interpolant on the domain Ω' only or to determine the values of the coefficients which remain unknown yet in an appropriate way. So, unknown values of coefficients can be determined using end conditions. We do not present here the corresponding formulae, which can be easy derived.

Now we have to prove that the second requirement of the problem is satisfied, namely the requirement of smoothness of the interpolant S. Taking into account (2) it is easy to determine that

$$\begin{aligned}
[\psi'(0)]^T = \phi'(0) &= (0,0,1,0);\\
[\psi'(1)]^T = \phi'(1) &= (0,0,0,1);\\
\phi_i''(0) &= (-\nu_1''(0)/h_i^2, \nu_1''(0)/h_i^2, \quad \nu_2''(0)/h_i, \quad 0);\\
\phi_i''(1) &= (-\nu_1''(1)/h_i^2, \quad \nu_1''(1)/h_i^2, \quad 0, \quad -\nu_1''(1)/h_i),\\
\psi_j''(0) &= (-\nu_1''(0)/l_j^2, \quad \nu_1''(0)/l_j^2, \quad \nu_2''(0)/l_j, \quad 0)^T;\\
\psi_j''(1) &= (-\nu_1''(1)/l_j^2, \quad \nu_1''(1)/l_j^2, \quad 0, \quad -\nu_1''(1)/l_j)^T.
\end{aligned}$$

Now the validity of the next equalities

$$\phi_i(0)F_{ij} = \phi_{i-1}(1)F_{i-1j}; \quad F_{ij}\psi_j(0) = F_{ij-1}\psi_{j-1}(1); \qquad (18)$$

$$\phi_i'(0)F_{ij} = \phi_{i-1}'(1)F_{i-1j}; \quad F_{ij}\psi_j'(0) = F_{ij-1}\psi_{j-1}'(1); \qquad (19)$$

$$\phi_i''(0)F_{ij} = \phi_{i-1}''(1)F_{i-1j}; \quad F_{ij}\psi_j''(0) = F_{ij-1}\psi_{j-1}''(1). \qquad (20)$$

can be proved. There are no difficulties when proving the first two groups. It is more comlpicate to do this for the last group but there is no special difficulties in this case too. Taking into account (18)-(20) the continuity of the spline and its derivatives along the lines $x = x_i$ and $y = y_j$ follows immediately. So, we have constructed the spline (11), which solves the problem of bidimensional interpolation and this spline is the local one.

5 Examples of generating functions.

In what follows we will give some examples of generating functions, since the natural question arises - is the set of generating functions L

nonempty? It is easy to prove that the following pairs of functions

$$\nu_1(t) = 3t^2 - 2t^3; \quad \nu_2(t) = t(1-t)^3 \tag{21}$$

$$\nu_1(t) = t^2/(2t^2 - 2t + 1); \quad \nu_2(t) = -2t^5 + 5t^4 - 3t^3 - t^2 + t. \tag{22}$$

belong to the set L. We should note that having a pair of generating functions new ones can be constructed as follows. Let us assume that $(\nu_1(t), \nu_2(t)) \in L$. Let us set now $\phi_1(t,u) = \nu_1(t)\psi(t,u)$ and $\phi_2(t,u) = \nu_2(t)\gamma(t,u)$, where u is vector of free parameters of the spline, $\gamma(1,u) = 1$, $\gamma'(1,u) = 1$, $\psi(1,u) = 1$, $\psi'(1,u) = 0$, $\psi(0,u) = \gamma(0,u)$ and functions $\gamma(t,u)$ and $\psi(t,u)$ are twice continuously differentiable on t. It is easy to prove that the pair of functions $(\phi_1(t,u), \phi_2(t,u))$ belong to L too. There is an example of functions γ and ψ given below, namely if $\psi(t,u) = 1$ then $\gamma(t,u) = 1 - t(1-t)\exp(-u(1-t))$, where u is free parameter of spline.

6 Numerical examples.

Some examples which illustrate the algorithms presented above are given below. The test functions were taken from [2], namely $f_1(x) = \exp(x)$, $f_2(x) = \exp(-10x)$, $f_3(x) = \sin(\pi x)$ and $f_4(x) = 1/(1 + 100(x-0.5)^2)$. The errors of interpolation using spline generated by functions (21) are given in the tables 1-3. Initial data were given on the uniform mesh with the step h on the segment $[-h, 1+h]$. Errors of interpolation were computed in the following way

$$E_r = \max_{x \in \Delta'} (|f^{(r)}(x) - S^{(r)}(x)|, \quad r = 0, 1, 2,$$

where Δ' is a uniform mesh with the step $h/10$ on the segment [0,1].

Table 1

E_0				
h	f_1	f_2	f_3	f_4
0.1	$1.614E-3$	$3.38E-2$	$6.192E-3$	$2.981E-2$
0.01	$1.69E-5$	$5.94E-4$	$6.17E-5$	$1.255E-3$
0.001	$1.7E-7$	$6.22E-6$	$6.17E-7$	$1.25E-5$

Table 2

E_1				
h	f_1	f_2	f_3	f_4
0.1	$4.98E-2$	1.75	$1.892E-1$	1
0.01	$5.2E-3$	$7.434E-1$	$1.895E-3$	$3.831E-2$
0.001	$5.22E-4$	$1.91E-2$	$1.895E-3$	$3.84E-2$

Table 3

E_2				
h	f_1	f_2	f_3	f_4
0.1	5.44	225.85	19.5	100
0.01	5.44	200.3	19.74	394.1
0.001	5.44	200	19.74	400

Errors of interpolation using spline generated by functions (22) are presented in the tables 4-5.

Table 4

E_0				
h	f_1	f_2	f_3	f_4
0.1	$4.39E-3$	$2.383E-2$	$6.192E-3$	$2.77E-2$
0.01	$1.69E-5$	$9.58E-5$	$6.17E-5$	$3.405E-4$
0.001	$1.7E-7$	$1.2422E-6$	$6.17E-7$	$2.57E-6$

Table 5

E_1				
h	f_1	f_2	f_3	f_4
0.1	$1.353E-1$	17.4	$1.892E-1$	2.1
0.01	$5.2E-3$	2.26	$1.895E-2$	$2,783-1$
0.001	$5.22E-4$	$2.363E-1$	$1.895E-3$	$2.51E-2$

Table 6

E_2				
h	f_1	f_2	f_3	f_4
0.1	14.8	81.8	19.5	139.5
0.01	5.44	81.21	19.74	197.2
0.001	5.44	93.1	19.74	189.2

7 Conclusions.

Numerical examples given above are in full accordance with the estimations of accuracy of interpolation using spline (1). It should be mentioned that splines from the family proposed in the present paper can be successfully used for solving problems of shape preserving interpolation.

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