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Regularity of subsemigroups generated by ordered idempotents

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Abstract. An element e of an ordered semigroup S is called an ordered idempotent if $e \leqslant e^2$. If we consider a subsemigroup S' of an ordered semigroup S as an ordered semigroup then the set $Reg_{\leqslant}(S')$ of all ordered regular elements of S' is not identical with $Reg_{\leqslant}(S) \cap S'$ in general. Here we develop some equivalent conditions on the equality of these two sets for $S' = (Se]$, $(eS]$ and $(eSf]$, where e, f are ordered idempotents.

1. Introduction

The notion of regularity in a semigroup is derived from von Neumann's definition of a regular ring. As well as ring theory regularity plays important role in the study of semigroup theory. It received considerable attention in semigroup theory. In 1979, K. Nambooripad [13] published a influential paper on the structure of regular semigroups. The set $E(S)$ of all idempotents carries a certain structure. Subsemigroups generated by idempotents in a semigroup have another important feature in semigroup theory. It still remains a subject of higher interest to the researchers. T.E. Hall [4] proved that a regular semigroup is generated by its idempotents if and only if each principal factor is generated by its own idempotents. Due to T.E. Hall there is a very familiar question in the semigroup literature: from information about idempotents what information can be drawn about a semigroup?

The set $Reg(S)$ of all regular elements of S carries an important role in semigroup theory. For a subsemigroup T of a semigroup S we distinguish two regular subsets: $Reg(T)$ – regular elements of T, and $reg(T) = Reg(S) \cap T$ – elements of T regular in S. It is well known that, in general, $Reg(T) \subseteq reg(T)$. Mitrović [12] has characterized semigroup with $Reg(T) = reg(T)$, where T runs over one of the following families of subsemigroups: $\{Se : \ e \in E(S)\}$, $\{eS : \ e \in E(S)\}$, $\{eSf : \ e, f \in E(S)\}$. Moreover Mitrović ([11], Theorem 5.2.3) has proved that $Reg(T) = reg(T) \neq \phi$ if and only if S is hereditary uniformly π- regular semigroup. This paper is inspired by [12].

Bhuniya and Hansda, [1] have introduced the notion of ordered idempotents and have characterized an ordered semigroups in which every element is an ordered

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idempotent. The purpose of this paper is, starting from an ordered semigroup S to study regular parts of certain kinds of its subsemigroups generated by ordered idempotents.

2. Preliminaries

In this paper $\mathbb{N}$ is the set of all natural numbers. An *ordered semigroup* S is a partially ordered set $(S, \leqslant)$, and at the same time a semigroup $(S, \cdot)$ such that $(\forall a,\ b,\ x \in S)\ a \leqslant b \Rightarrow\ xa \leqslant xb$ and $ax \leqslant bx$. It is denoted by $(S, \cdot, \leqslant)$. For an ordered semigroup S and $H \subseteq S$, denote

$$(H] = \{t \in S : t \leqslant h, \text{ for some } h \in H\}.$$

H is called *downward closed* if $H = (H]$.

Let I be a nonempty subset of an ordered semigroup S. I is called a *left* (*right*) *ideal* of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. I is an *ideal* of S if it is both a left and a right ideal of S.

An element e of S is an *ordered idempotent* if $e \leqslant e^2$, [1]. The set of all ordered idempotents of S is denoted by $E_{\leqslant}(S)$. An element $a \in S$ is called *ordered regular* if there is $x \in S$ such that $a \leqslant axa$, i.e. $a \in (aSa]$. Clearly, if $a \leqslant axa$ then $ax,\ xa \in E_{\leqslant}(S)$. The set of all ordered regular elements of S is denoted by $Reg_{\leqslant}(S)$. An ordered semigroup S is *ordered regular* if $S = Reg_{\leqslant}(S)$. An ordered semigroup S is *right regular* if $a \in (a^2S]$ for every $a \in S$. Kehayopulu [7] defined an *ordered completely regular semigroup* as an ordered semigroup S such that $a \in (a^2Sa^2]$, for all $a \in S$. The set of all ordered completely regular elements is denoted by $Gr_{\leqslant}(S)$.

Before going to the main results we will state some preliminary results on ordered idempotents of an ordered semigroup.

Lemma 2.1. *Let S be an ordered semigroup and $Gr_{\leqslant}(S) \neq \phi$. Then for every $a \in Gr_{\leqslant}(S)$ there is $e \in E_{\leqslant}(S)$ such that $a \leqslant ea$ and $a \leqslant ae$.*

Proof. Consider $a \in Gr_{\leqslant}(S)$. Then there is $t \in S$ such that $a \leqslant a^2ta^2 \leqslant a(a^2ta^2ta^2) = ae$, where $e = a^2ta^2ta^2 \in E_{\leqslant}(S)$. Similarly $a \leqslant ea$. $\square$

Our next lemma is very much straight forward that follows similarly to the previous lemma.

Lemma 2.2. *Let S be an ordered semigroup and $e \in E_{\leqslant}(S)$. Then for every $a \in eSe$, $a \leqslant ea$ and $a \leqslant ae$.* $\square$

3. Subsemigroups generated by ordered idempotents

For a subsemigroup $T \subseteq S$, let $reg_{\leqslant}(T)$ denote the intersection $T \cap Reg_{\leqslant}(S)$. That is the set of all elements of T which are ordered regular in S.

Theorem 3.1. *Let S be an ordered semigroup and $E_{\leqslant}(S) \neq \phi$. Then for every $e \in E_{\leqslant}(S)$, $reg_{\leqslant}(eSe) = Reg_{\leqslant}(eSe)$.*

Proof. Suppose that $a \in reg_{\leqslant}(eSe)$. Since $a \in Reg_{\leqslant}(S)$ there is $x \in S$ such that $a \leqslant axa$. This implies that $a \leqslant aexea$, by Lemma 2.2. Since $exe \in eSe$ we have that $a \in Reg_{\leqslant}(eSe)$. Hence $reg_{\leqslant}(eSe) = Reg_{\leqslant}(eSe)$. □

Lemma 3.2. *Let S be an ordered semigroup and $e, f \in E_{\leqslant}(S)$. Then the following conditions hold on S:*

(1) $Reg_{\leqslant}((eSf]) = Reg_{\leqslant}((eS]) \cap Reg_{\leqslant}((Sf])$;

(2) $Gr_{\leqslant}((eSf]) = Gr_{\leqslant}((eS]) \cap Gr_{\leqslant}((Sf])$;

(3) $e \in E_{\leqslant}((eSf]) = E_{\leqslant}((eS]) \cap E_{\leqslant}((Sf])$;

(4) $reg_{\leqslant}((eSf]) = reg_{\leqslant}((eS]) \cap reg_{\leqslant}((Sf])$;

(5) $Gr_{\leqslant}((eSf]) = Gr_{\leqslant}((eSf])$.

Proof. (1). Let $b \in Reg_{\leqslant}((eS]) \cap Reg_{\leqslant}((Sf])$. Then there are $s_1, s_2 \in S$ such that $b \leqslant bes_1b$ and $b \leqslant bs_2fb$. So, we have $b \leqslant bes_1b \leqslant b(es_1bs_2f)b$. Thus, $b \in Reg_{\leqslant}((eSf])$. Also is obvious that $Reg_{\leqslant}((eSf])) \subseteq Reg_{\leqslant}((eS]) \cap Reg_{\leqslant}((Sf])$. Hence $Reg_{\leqslant}((eSf]) = Reg_{\leqslant}((eS]) \cap Reg_{\leqslant}((Sf])$.

(2). This proof is similar to (1).

(3). This is obvious.

(4). First suppose that $b \in reg_{\leqslant}((eS]) \cap reg_{\leqslant}((Sf])$. Since $b \in Reg_{\leqslant}(S)$, there is $y \in S$ such that $b \leqslant byb$. Also $b \in (eS] \cap (Sf]$. Therefore $b \leqslant byb$ implies that $b \in (eSf]$, and so $b \in Reg_{\leqslant}(S) \cap (eSf] = reg_{\leqslant}((eSf])$.

Also by Lemma 20 [5] we have that $(eSf] \subseteq (eS] \cap (Sf]$. So $reg_{\leqslant}((eSf]) = Reg_{\leqslant}(S) \cap (eSf] \subseteq Reg_{\leqslant}(S) \cap (eS] = reg_{\leqslant}((eS])$. Similarly $reg_{\leqslant}((eSf]) \subseteq reg_{\leqslant}((Sf])$. Hence $reg_{\leqslant}((eSf]) = reg_{\leqslant}((eS]) \cap reg_{\leqslant}((Sf])$.

(5). Let $a \in Gr_{\leqslant}((eSf]) = Gr_{\leqslant}(S) \cap (eSf]$. Since $a \in Gr_{\leqslant}(S)$ there is $t \in S$ such that $a \leqslant a^2ta^2$, which yields that $a \leqslant a^2(ata^2ta^2ta)a^2$. Now as $a \in (eSf]$ there is $s \in S$ such that $a \leqslant esf$. So from $a \leqslant a^2(ata^2ta^2ta)a^2$ we have that $a \leqslant a^2(esfta^2ta^2tesf)a^2$. Therefore $a \in Gr_{\leqslant}((eSf])$. Also it is evident that $Gr_{\leqslant}((eSf]) \subseteq gr_{\leqslant}((eSf])$. Therefore $Gr_{\leqslant}((eSf]) = gr_{\leqslant}((eSf])$. □

Lemma 3.3. *Let S be an ordered semigroup and $E_{\leqslant}(S) \neq \phi$. Then for every $e \in E_{\leqslant}(S)$, $Gr_{\leqslant}((Se]) = gr_{\leqslant}((Se])$.*

Proof. Let $a \in gr((Se]) = Gr_{\leqslant}(S) \cap (Se]$. Then there is $t \in S$ such that $a \leqslant a^2ta^2$. This implies that $a \leqslant a^2(ta^2ta)a^2$. Since $a \in (Se]$, there is $s \in S$ such that $a \leqslant se$. So $a \leqslant a^2(ta^2ta)a^2$ yields that $a \leqslant a^2(ta^2tse)a^2$. So $a \in Gr_{\leqslant}((Se])$, that is $Gr_{\leqslant}(S) \cap (Se] \subseteq Gr_{\leqslant}((Se])$. Also it is obvious that $Gr_{\leqslant}((Se]) \subseteq Gr_{\leqslant}(S) \cap (Se] = gr_{\leqslant}((Se])$. Hence $Gr_{\leqslant}((Se]) = gr_{\leqslant}((Se])$. □

Theorem 3.4. *Let S be a right regular ordered semigroup and $E_{\leqslant}(S) \neq \phi$. Then the following conditions are equivalent on S:*

(1) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((Se])) = Gr_{\leqslant}((Se])$;*

(2) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((Se])) = Reg_{\leqslant}((Se])$;*

(3) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((Se])) \subseteq LReg_{\leqslant}((Se])$;*

(4) *$Reg_{\leqslant}(S) \subseteq LReg_{\leqslant}(S)$;*

(5) *$Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$;*

(6) *for all $e, f \in E_{\leqslant}(S)$, $reg_{\leqslant}((eSf])) = Gr_{\leqslant}((eSf])$;*

(7) *for all $e \in E_{\leqslant}(S)$, $Reg_{\leqslant}((eSf])) = reg_{\leqslant}((eSf])$.*

Proof. (1) $\Rightarrow$ (2). Let $e \in E_{\leqslant}(S)$. Consider $x \in reg_{\leqslant}((Se])$. Then by (1) $x \in Gr_{\leqslant}((Se])$. So $x \leqslant x^2tx^2$ for some $t \in (Se]$. Since $x \in (Se]$ there is $s \in S$ such that $x \leqslant se$, which yields that $x \leqslant x(xtse)x$. Since $xtse \in (Se]$ we have that $reg_{\leqslant}((Se]) \subseteq Reg_{\leqslant}((Se])$ and so $reg_{\leqslant}((Se]) = Reg_{\leqslant}((Se])$.

(2) $\Rightarrow$ (3). Let $e \in E_{\leqslant}(S)$ choose $x \in reg_{\leqslant}((Se])$. By the given condition we have that $x \leqslant xzx$ for some $z \in (Se]$. Note that $zx \in E_{\leqslant}(S)$, so by (2) we have that $x \in (Szx] \cap Reg_{\leqslant}(S) = reg_{\leqslant}((Szx]) = Reg_{\leqslant}((Szx])$. This yields that $x \leqslant x(szx)x$ for some $s \in S$. Also $z \in (Se]$, then there is $s' \in S$ such that $z \leqslant s'e$, whence $x \leqslant x(ss'e)x^2$. Since $xss'e \in (Se]$ we have that $x \in LReg_{\leqslant}((Se])$. Therefore $reg_{\leqslant}((Se]) \subseteq LReg_{\leqslant}((Se])$.

(3) $\Rightarrow$ (4). Suppose that $x \in Reg_{\leqslant}(S)$. Then there is $y \in S$ such that $x \leqslant xyx$. Since $yx \in E_{\leqslant}(S)$ we have $x \in (Syx]$. Therefore $x \in (Syx] \cap Reg_{\leqslant}(S) = reg_{\leqslant}((Syx])$. So by the given condition $x \in LReg_{\leqslant}((Syx])$. So for some $z \in (Syx]$, $x \leqslant zx^2$. Hence $x \in LReg_{\leqslant}(S)$.

(4) $\Rightarrow$ (5). Let $a \in Reg_{\leqslant}(S)$. Then there is $x \in S$ such that $a \leqslant axa$. Now by right regularity of S we have $a \leqslant a^2sxa$. Also

$$asxa \leqslant asxaxa \leqslant asxa^2sxa = (asxa)(asxa).$$

Thus $asxa \in E_{\leqslant}(S)$. Say $f = asxa$. Then $a \in Reg_{\leqslant}(S) \cap (Sf]$, and so $a \in LReg_{\leqslant}(Sf]$, by condition (4). This yields that $a \leqslant ta^2$ for some $t \in (Sf]$. Now $a \leqslant a^2sxa \leqslant a^2sxta^2$, that is $a \in Gr_{\leqslant}(S)$. Hence $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$.

(5) $\Rightarrow$ (1). Consider $e \in E_{\leqslant}(S)$. By Lemma 3.3 it follows that $Gr_{\leqslant}((Se]) = Gr_{\leqslant}(S) \cap (Se]$. So by the given condition we have $Gr_{\leqslant}((Se]) = Reg_{\leqslant}(S) \cap (Se] = reg_{\leqslant}(Se]$.

(5) $\Leftrightarrow$ (6). First suppose that $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$. Note that $Gr_{\leqslant}((eSf]) \subseteq Reg_{\leqslant}((eSf]) \subseteq reg_{\leqslant}((eSf])$. Also $reg_{\leqslant}((eSf]) = Reg_{\leqslant}(S) \cap (eSf]$. Then by (5) and Lemma 3.2, $reg_{\leqslant}((eSf]) \subseteq Gr_{\leqslant}(S) \cap (eSf] = gr((eSf]) = Gr_{\leqslant}((eSf])$. Hence $reg_{\leqslant}((eSf]) = Gr_{\leqslant}((eSf])$.

For the converse part we first note that $Gr_{\leqslant}(S) \subseteq Reg_{\leqslant}(S)$. To show $Reg_{\leqslant}(S) \subseteq Gr_{\leqslant}(S)$, choose $a \in Reg_{\leqslant}(S)$. Then there is $t \in S$ such that $a \leqslant ata$, which yields that $a \leqslant atatata$. It is clear to see that $ta, at \in E_{\leqslant}(S)$, which gives that $a \in Reg_{\leqslant}((taSat]) \subseteq reg_{\leqslant}((taSat])$. Then by (6), $a \in Gr_{\leqslant}(taSat])$, that is $a \in Gr_{\leqslant}(S)$. Thus $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$.

(6) $\Leftrightarrow$ (7). First suppose that $reg_{\leqslant}((eSf]) = Gr_{\leqslant}((eSf])$. Now obviously $Reg_{\leqslant}((eSf]) \subseteq reg_{\leqslant}((eSf])$. Also, by (6), we have $reg_{\leqslant}((eSf]) = Gr_{\leqslant}((eSf]) \subseteq Reg_{\leqslant}((eSf])$. Thus $Reg_{\leqslant}((eSf]) = reg_{\leqslant}((eSf])$.

Conversely assume that $x \in reg_{\leqslant}((eSf])$. Then by condition (7), $y \in (eSf]$ such that $x \leqslant xyx$, which implies that $x \leqslant xyxyx$. Clearly $xy, yx \in E_{\leqslant}(S)$. So by condition (7) we have that $x \in Reg_{\leqslant}(S) \cap ((xySyx]) = reg_{\leqslant}((xySyx]) = Reg_{\leqslant}((xySyx])$. Thus $x \leqslant xtx$ for some $t \in (xySyx]$ so that $x \leqslant x(xysyx)x$ for some $s \in S$. Since $y \in (eSf]$ it follows that $x \in Gr_{\leqslant}((eSf])$. Hence $reg_{\leqslant}((eSf]) = Gr_{\leqslant}((eSf])$. □

In rest of this section we wish to characterize the equality of the regularity of the subsemigroups $(Se]$ and $(eS]$ for an ordered idempotent e.

Theorem 3.5. *Let S be a right regular ordered semigroup and $E_{\leqslant}(S) \neq \phi$. Then the following conditions are equivalent on S:*

(1) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((eS]) \subseteq reg_{\leqslant}((Se])$;*

(2) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((eS]) = reg_{\leqslant}((eSe])$;*

(3) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((eS]) = Reg_{\leqslant}((eSe])$ and $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$;*

(4) *for all $e \in E_{\leqslant}(S)$, $reg_{\leqslant}((eS]) \subseteq Reg_{\leqslant}((Se])$;*

(5) *for all $e \in E_{\leqslant}(S)$, $Gr_{\leqslant}((eSe]) = Gr_{\leqslant}((eS])$ and $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$;*

(6) *for all $e \in E_{\leqslant}(S)$, $Gr_{\leqslant}((eS]) \subseteq Gr_{\leqslant}((Se])$ and $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$.*

Proof. (1) $\Rightarrow$ (2). Let $e \in E_{\leqslant}(S)$. By Lemma 3.2 we have $reg_{\leqslant}((eSe) = reg_{\leqslant}((eS]) \cap reg_{\leqslant}((Se])$. Then by (1) it follows that $reg_{\leqslant}((eSe]) = reg_{\leqslant}((eS])$.

(2) $\Rightarrow$ (3). Let $e \in E_{\leqslant}(S)$. By Lemma 3.2 we have that $reg_{\leqslant}((eSe]) = reg_{\leqslant}((eS]) \cap reg_{\leqslant}((Se])$. Then by (2) it follows that $reg_{\leqslant}((eSe]) = reg_{\leqslant}((eS])$.

(3) $\Rightarrow$ (4). Let $e \in E_{\leqslant}(S)$. Note that $reg((eS]) = Reg((eSe]) \subseteq reg((eSe])$. Then by Theorem 3.2, $reg_{\leqslant}((eSe]) = reg_{\leqslant}((eS]) \cap reg_{\leqslant}((Se])$. This implies that $reg((eS]) \subseteq reg((Se])$, by condition (3).

(4) $\Rightarrow$ (5). Let $a \in Reg_{\leqslant}(S)$. Then for some $x \in S$, $a \leqslant axa$. Now $xax \leqslant xaxax$ and $xaxax \in (xaS] \cap Reg_{\leqslant}(S) = reg(xaS]$. Since $xa \in E_{\leqslant}(S)$, by condition (4) we have that $xax \in Reg_{\leqslant}((Sxa])$. So $xax \leqslant zxa$ for some $z \in S$. Therefore $a \leqslant axaxa \leqslant azxaa$, so $a \in (Sa^2]$, that is, $a \in LReg_{\leqslant}(S)$. So $Reg_{\leqslant}(S) \subseteq LReg_{\leqslant}(S)$. Thus $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$ by Theorem 3.4.

Now to show $Gr_{\leqslant}((eSe]) = Gr_{\leqslant}((eS])$ for some $e \in E_{\leqslant}(S)$, it is require only to proof $Gr_{\leqslant}((eS]) \subseteq Gr_{\leqslant}((eSe])$. For this let us assume $b \in Gr_{\leqslant}((eS])$. Then

$b \in Reg((eS]) \subseteq reg((eS]) \subseteq Reg((Se])$ follows from condition (4). Further, since $b \in Gr_{\leqslant}((eS])$ there is $s \in S$ such that $b \leqslant b^2esb^2$. Also as $b \in Reg_{\leqslant}((Se])$. So from some $s_1 \in S$ we have that $b \leqslant bs_1eb$. Therefore $b \leqslant b^2esb^2 \leqslant b^2(esbs_1e)b^2$. Hence $b \in Gr_{\leq}((eS])$. That is $Gr_{\leqslant}((eSe]) = Gr_{\leqslant}((eS])$.

(5) $\Rightarrow$ (6). This follows immediately.

(6) $\Rightarrow$ (1). Let $e \in E_{\leqslant}(S)$. Choose $a \in reg((eS]) = Reg(S) \cap (eS]$. Then by condition (6), $a \in Gr_{\leqslant}(S)$. Thus $a \in Gr_{\leqslant}(S) \cap (eS]$. So $a \in Gr_{\leqslant}(S)$ follows from condition (6). Hence $reg_{\leqslant}((eS]) \subseteq reg_{\leqslant}((Se])$. □

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