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Rough set theory applied to BCI-algebras

Wiesław A. Dudek, Young Bae Jun and Hee Sik Kim

Abstract

As a generalization of subalgebras/ideals in BCI-algebras, the notion of rough subalgebras/ideals is introduced, and some of their properties are discussed.

1. Introduction

In 1982, Pawlak introduced the concept of a rough set (see [13]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [14]). An algebraic approach to rough sets has been given by Iwiński [7]. Rough set theory is applied to semigroups and groups (see [10, 11]). In 1994, Biswas and Nanda [2] introduced and discussed the concept of rough groups and rough subgroups. Recently, Jun [8] applied rough set theory to BCK-algebras. In this paper, we apply the rough set theory to BCI-algebras, and we introduce the notion of upper/lower rough subalgebras/ideals in BCI-algebras, and discuss some of their properties.

Note that BCI-algebras are an algebraic characterization of some types of non-classical logics. Moreover, BCI-algebras are also a generalization of BCK-algebras. On the other side, BCI-algebras are a generalization of T-quasigroups, too. Namely, as it is proved in [3] and [4], a BCI-algebra is a quasigroup if and only if it is medial. Such BCI-algebra is uniquely determined by some abelian group. In fact, such BCI-algebra is isotopic to this group. The class of associative BCI-algebras coincides with the class of Boolean groups.

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2. Preliminaries

Recall that a *BCI-algebra* is an algebra $(G, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in G$,

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCI-algebra G, the relation $\leqslant$ defined by $x \leqslant y$ if and only if $x * y = 0$ is a partial order on G. In any BCI-algebra the following two identities hold:

(P_1) $x * 0 = x$,
(P_2) $(x * y) * z = (x * z) * y$.

A non-empty subset S of a BCI-algebra G is said to be a *subalgebra* of G if $x * y \in S$ whenever $x, y \in S$. A non-empty subset A of a BCI-algebra G is called an *ideal* of G, denoted by $A \sqsubseteq G$, if

- $0 \in A$,
- $x * y \in A$ and $y \in A$ imply $x \in A$.

An ideal A of a BCI-algebra G is said to be *closed* if $0 * x \in A$ for all $x \in A$. Note that an ideal of a BCI-algebra may not be a subalgebra in general, but every closed ideal is closed with respect to a BCI-operation, i.e. it is a subalgebra (cf. [5]).

A non-empty subset A of a BCI-algebra G is called a *p-ideal* of G if it satisfies the following two conditions

- $0 \in A$,
- $(x * z) * (y * z) \in A$ and $y \in A$ imply $x \in A$.

Note that in BCI-algebras every p-ideal is an ideal, but not converse (see [15]).

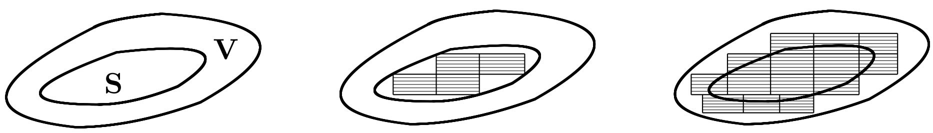
Let ρ be a congruence relation on G, that is, ρ is an equivalence relation on G such that $(x, y) \in \rho$ implies $(x * z, y * z) \in \rho$ and $(z * x, z * y) \in \rho$ for all $z \in G$. The set of all equivalence classes of G with respect to ρ will be denoted by G/ρ. On G/ρ we define an operation $*$ putting $[x]_\rho * [y]_\rho = [x * y]_\rho$ for all $[x]_\rho, [y]_\rho \in G/\rho$. It is clear that such operation is well-defined, but $(G/\rho, *, [0]_\rho)$ may not be a BCI-algebra, because G/ρ does not satisfy the fourth condition of a BCI-algebra.

For any non-empty subsets A and B of a BCI-algebra G we define the *complex multiplication* putting $A * B = \{x * y \mid x \in A,\ y \in B\}$.

3. Roughness of some ideals

Let V be a set and ρ an equivalence relation on V and let $\mathcal{P}(V)$ denote the power set of V. For all $x \in V$, let $[x]_\rho$ denote the equivalence class of G with respect to ρ. Define the functions ρ_- and ρ^- from $\mathcal{P}(V)$ to $\mathcal{P}(V)$ putting for every $S \in \mathcal{P}(V)$

$$\rho_-(S) = \{x \in V \mid [x]_\rho \subseteq S\},$$
$$\rho^-(S) = \{x \in V \mid [x]_\rho \cap S \neq \emptyset\}.$$



$S \subset V$ $\qquad$ $\rho_-(S) \subseteq S$ $\qquad$ $S \subseteq \rho^-(S)$

$\rho_-(S)$ is called the *lower approximation* of S while $\rho^-(S)$ is called the *upper approximation*. The set S is called *definable* if $\rho_-(S) = \rho^-(S)$ and *rough* otherwise. The pair (V, ρ) is called an *approximation space*.

Directly from the definition by simple calculations we can see that the following proposition holds.

Proposition 1. *Let A and B be non-empty subsets of a BCI-algebra G. If ρ is a congruence relation on G, then the following hold:*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$,
(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$,
(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,
(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$ *and* $\rho^-(A) \subseteq \rho^-(B)$,
(5) $\rho_-(A \cup B) \supseteq \rho_-(A) \cup \rho_-(B)$,
(6) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$,
(7) $\rho^-(A) * \rho^-(B) \subseteq \rho^-(A * B)$,
(8) $\rho_-(A) * \rho_-(B) \subseteq \rho_-(A * B)$ *whenever* $\rho_-(A) * \rho_-(B) \neq \emptyset$ *and* $\rho_-(A * B) \neq \emptyset$.

Proposition 2. *If* ρ *is a congruence relation on a* BCI*-algebra* G*, then the following are equivalent:*

(1) $x * y \in [0]_\rho$ *and* $y * x \in [0]_\rho$ *imply* $(x, y) \in \rho$,

(2) ρ *is regular, i.e.* $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$ *implies* $[x]_\rho = [y]_\rho$,

(3) $(G/\rho, *, [0]_\rho)$ *is a* BCI*-algebra.*

Proof. (1) ⇒ (2) Suppose $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$. Then $[x * y]_\rho = [0]_\rho = [y * x]_\rho$, and so $(x * y, 0) \in \rho$ and $(y * x, 0) \in \rho$. It follows from (1) that $(x, y) \in \rho$. Hence $[x]_\rho = [y]_\rho$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Let $x, y \in G$ be such that $x * y \in [0]_\rho$ and $y * x \in [0]_\rho$. Then

$$[x]_\rho * [y]_\rho = [x * y]_\rho = [0]_\rho = [y * x]_\rho = [y]_\rho * [x]_\rho \,.$$

It follows from the fourth condition of the definition of a BCI-algebra that $[x]_\rho = [y]_\rho$. Thus $(x, y) \in \rho$. This completes the proof. □

Theorem 3. *If* ρ *is a congruence relation on* G*, then* $[0]_\rho$ *is a closed ideal, and hence a subalgebra of* G*.*

Proof. Obviously, $0 \in [0]_\rho$. Let $x, y \in G$ be such that $x * y \in [0]_\rho$ and $y \in [0]_\rho$. Then $(x * y, 0) \in \rho$ and $(y, 0) \in \rho$. Since ρ is a congruence relation on G, it follows from (P_1) that $(x * y, x) = (x * y, x * 0) \in \rho$ so that $(x, 0) \in \rho$, that is, $x \in [0]_\rho$. If $x \in [0]_\rho$, then $(x, 0) \in \rho$ and hence $(0 * x, 0) = (0 * x, 0 * 0) \in \rho$, that is, $0 * x \in [0]_\rho$. Hence $[0]_\rho$ is a closed ideal of G. □

Definition 4. A non-empty subset S of a BCI-algebra G is called an *upper* (resp. a *lower*) *rough subalgebra* (or, (*closed*) *ideal*) of G if the upper (resp. nonempty lower) approximation of S is a subalgebra (or, (closed) ideal) of G. If S is both an upper and a lower rough subalgebra (or, (closed) ideal) of G, we say that S is a *rough subalgebra* (or (*closed*) *ideal*) of G.

Theorem 5. *Every subalgebra is a rough subalgebra.*

Proof. Let S be a subalgebra of a BCI-algebra G. Taking $A = B = S$ in Proposition 1(8), we have

$$\rho_-(S) * \rho_-(S) \subseteq \rho_-(S * S) \subseteq \rho_-(S)$$

because S is a subalgebra of G. Hence $\rho_-(S)$ is a subalgebra of G, that is, S is a lower rough subalgebra of G. We now show that $\rho^-(S)$ is a subalgebra of G. Let $x, y \in \rho^-(S)$. Then $[x]_\rho \cap S \neq \emptyset$ and $[y]_\rho \cap S \neq \emptyset$. Thus there exist $a_x, a_y \in S$ such that $a_x \in [x]_\rho$ and $a_y \in [y]_\rho$. It follows that $(a_x, x) \in \rho$ and $(a_y, y) \in \rho$ so that $(a_x * a_y, x * y) \in \rho$, that is, $a_x * a_y \in [x * y]_\rho$. On the other hand, since S is a subalgebra of G, we have $a_x * a_y \in S$. Hence $a_x * a_y \in [x * y]_\rho \cap S$, that is, $[x * y]_\rho \cap S \neq \emptyset$. This shows that $x * y \in \rho^-(S)$. Therefore S is an upper rough subalgebra of G. This completes the proof. □

For any subset I of a BCI-algebra G, define a relation $\rho(I)$ on G induced by I in the following way:

$$(x, y) \in \rho(I) \iff x * y,\ y * x \in I.$$

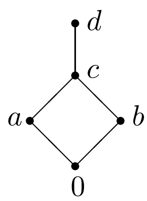
$\rho(I)_-(S)$ is called the *lower approximation* of S by I, while $\rho(I)^-(S)$ is called the *upper approximation* by I. In the case $\rho(I)_-(S) = \rho(I)^-(S)$ we say that S is called *definable* with respect to I. In otherwise S is *rough* with respect to I. Obviously $\rho(I)_-(G) = G = \rho(I)^-(G)$ for any $I \sqsubseteq G$. This means that any BCI-algebra is definable with respect to any its ideal.

If I is an ideal of G, then $\rho(I)$ is a regular congruence relation on G (see [9]). Note that in the case of BCI-quasigroups every subalgebra is an ideal. The converse is not true (see [4]), but a finite subset of such quasigroup is an ideal if and only if it is a subalgebra. Thus in BCI-algebras all relations $\rho(I)$ induced by a finite set I are regular congruences.

The following example shows that there exists non-empty subset S of G which is not an ideal, but for which S is an upper rough subalgebra of G. Hence we know that the notion of an upper rough subalgebra is an extended notion of a subalgebra.

Example 6. Let $G = \{0, a, b, c, d\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0



Then $I = \{0, a\} \sqsubseteq G$, and thus $[0]_{\rho(I)} = [a]_{\rho(I)} = I$, $[b]_{\rho(I)} = \{b\}$, $[c]_{\rho(I)} = \{c\}$, and $[d]_{\rho(I)} = \{d\}$. Consider a subset $S = \{a, b\}$ of G which is not a subalgebra of G. Then $\rho(I)^-(S) = \{0, a, b\}$ which is a subalgebra.

On the other hand, for $M = \{0, a, c\}$ which is a subalgebra but not an ideal, we have $\rho(I)^-(M) = \rho(I)_-(M) = M$. Hence M is definable with respect to I. It is not to difficult to see that M is not definable with respect to $J = \{0, b\} \sqsubseteq G$.

Proposition 7. *Every non-empty subset of a BCI-algebra is definable with respect to the trivial ideal* $\{0\}$.

Proof. If $a \in [x]_{\rho(\{0\})}$ then $(a, x) \in \rho(\{0\})$ and so $a * x \in \{0\}$ and $x * a \in \{0\}$. It follows that $a = x$ so that $[x]_{\rho(\{0\})} = \{x\}$ for all $x \in G$. Hence

$$\rho(\{0\})_-(S) = \big\{x \in G \mid [x]_{\rho(\{0\})} \subseteq S\big\} = S$$

and

$$\rho(\{0\})^-(S) = \big\{x \in G \mid [x]_{\rho(\{0\})} \cap S \neq \emptyset\big\} = S.$$

This completes the proof. □

Lemma 8. *If I and J are ideals of a BCI-algebra G such that $I \subseteq J$, then $\rho(I) \subseteq \rho(J)$.*

Proof. If $(x, y) \in \rho(I)$, then $x * y \in I \subseteq J$ and $y * x \in I \subseteq J$. Hence $(x, y) \in \rho(J)$, completing the proof. □

Remark 9. Let I and J be ideals of G such that $I \neq J$. Then $\rho(I)_-(J)$ is not an ideal of G in general. Indeed, it is easy to see that $I = \{0, a\}$ and $J = \{0, b\}$ are ideals of a *BCI*-algebra G defined in Example 6. But

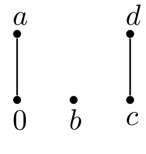
$$\rho(I)_-(J) = \{x \in G \mid [x]_{\rho(I)} \subseteq J\} = \{b\}$$

is not an ideal of G.

The following example shows that there exists a non-ideal J of G for which J is an upper rough ideal of G with respect to an ideal of G. Hence we know that the notion of an upper rough ideal is an extended notion of an ideal.

Example 10. Consider a *BCI*-algebra $G = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	c	b	c
a	a	0	c	b	b
b	b	b	0	c	c
c	c	c	b	0	0
d	d	c	b	a	0



Then for $I = \{0, a\} \sqsubseteq G$ we have $[0]_{\rho(I)} = [a]_{\rho(I)} = I$, $[b]_{\rho(I)} = \{b\}$, and $[c]_{\rho(I)} = [d]_{\rho(I)} = \{c, d\}$. Thus for $J = \{0, b, c\}$, which is not an ideal of G, we obtain

$$\rho(I)^{-}(J) = \{x \in G \mid [x]_{\rho(I)} \cap J \neq \emptyset\} = \{0, a, b, c, d\} \sqsubseteq G.$$

Theorem 11. *Let $I \subseteq J$ be two ideals of a BCI-algebra G. Then*

(1) *$\rho(I)_{-}(J)$ $(\neq \emptyset)$ is an ideal of G, that is, J is a lower rough ideal of G with respect to I.*

(2) *$\rho(I)^{-}(J)$ is an ideal of G, that is, J is an upper rough ideal of G with respect to I. Moreover if J is closed, then so is $\rho(I)^{-}(J)$.*

Proof. (1) Let $x \in [0]_{\rho(I)}$. Then $x = x * 0 \in I \subseteq J$ and so $[0]_{\rho(I)} \subseteq J$. Hence $0 \in \rho(I)_{-}(J)$. Let $x, y \in G$ be such that $x * y \in \rho(I)_{-}(J)$ and $y \in \rho(I)_{-}(J)$. Then $[y]_{\rho(I)} \subseteq J$ and

$$[x]_{\rho(I)} * [y]_{\rho(I)} = [x * y]_{\rho(I)} \subseteq J.$$

Let $a_x \in [x]_{\rho(I)}$ and $a_y \in [y]_{\rho(I)}$. Then $(a_x, x) \in \rho(I)$ and $(a_y, y) \in \rho(I)$, which imply $(a_x * a_y, x * y) \in \rho(I)$. Hence $a_x * a_y \in [x * y]_{\rho(I)} \subseteq J$. Since $a_y \in [y]_{\rho(I)} \subseteq J$, it follows that $a_x \in J$. Therefore $[x]_{\rho(I)} \subseteq J$, or equivalently, $x \in \rho(I)_{-}(J)$. This shows that $\rho(I)_{-}(J)$ is an ideal of G.

(2) Obviously, $0 \in \rho(I)^{-}(J)$. Let $x, y \in G$ be such that $y \in \rho(I)^{-}(J)$ and $x * y \in \rho(I)^{-}(J)$. Then $[y]_{\rho(I)} \cap J \neq \emptyset$ and $[x * y]_{\rho(I)} \cap J \neq \emptyset$, and so there exist $u, v \in J$ such that $u \in [y]_{\rho(I)}$ and $v \in [x * y]_{\rho(I)}$. Hence $(u, y) \in \rho(I)$ and $(v, x * y) \in \rho(I)$ which imply $y * u \in I \subseteq J$ and $(x * y) * v \in I \subseteq J$. Since $u, v \in J$ and J is an ideal, it follows that $y \in J$ and $x * y \in J$ so that $x \in J$. Note that $x \in [x]_{\rho(I)}$, thus $x \in [x]_{\rho(I)} \cap J$, that is, $[x]_{\rho(I)} \cap J \neq \emptyset$. Therefore $x \in \rho(I)^{-}(J)$, and consequently J is an upper rough ideal of G with respect to I. Now let $x \in \rho(I)^{-}(J)$. Then $[x]_{\rho(I)} \cap J \neq \emptyset$, and so there exists $a_x \in J$ such that $a_x \in [x]_{\rho(I)}$. Since J is closed, it follows that $0 * a_x \in J$ and hence

$$0 * a_x \in \left([0]_{\rho(I)} * [x]_{\rho(I)}\right) \cap J = [0 * x]_{\rho(I)} \cap J,$$

that is, $[0 * x]_{\rho(I)} \cap J \neq \emptyset$. Hence $0 * x \in \rho(I)^{-}(J)$. This completes the proof. □

Lemma 12. ([15, Theorem 4.1]) *An ideal I of a BCI-algebra G is a p-ideal if and only if for each $x, y, z \in G$,*

$$(x * z) * (y * z) \in I \text{ implies } x * y \in I.$$

It is not difficult to see that in the case of BCI-quasigroups every ideal is a p-ideal and conversely.

Theorem 13. *Let $I \sqsubseteq G$ and let J be a p-ideal of a BCI-algebra G containing I. Then $\rho(I)_-(J)$ $(\neq \emptyset)$ and $\rho(I)^-(J)$ are p-ideals of G.*

Proof. Let $x, y, z \in G$ be such that $(x * z) * (y * z) \in \rho(I)_-(J)$. Then

$$\big([x]_{\rho(I)} * [z]_{\rho(I)}\big) * \big([y]_{\rho(I)} * [z]_{\rho(I)}\big) = [(x * z) * (y * z)]_{\rho(I)} \subseteq J.$$

Let $w \in [x * y]_{\rho(I)} = [x]_{\rho(I)} * [y]_{\rho(I)}$. Then $w = a_x * a_y$ for some $a_x \in [x]_{\rho(I)}$ and $a_y \in [y]_{\rho(I)}$. From $a_x \in [x]_{\rho(I)}$ and $a_y \in [y]_{\rho(I)}$, we have $(a_x, x) \in \rho(I)$ and $(a_y, y) \in \rho(I)$. Taking $a_z \in [z]_{\rho(I)}$, then $(a_z, z) \in \rho(I)$. Since $\rho(I)$ is a congruence relation, we get $(a_x * a_z, x * z) \in \rho(I)$ and $(a_y * a_z, y * z) \in \rho(I)$, and thus

$$\big((a_x * a_z) * (a_y * a_z),\ (x * z) * (y * z)\big) \in \rho(I).$$

This means that

$$(a_x * a_z) * (a_y * a_z) \in [(x * z) * (y * z)]_{\rho(I)} \subseteq J.$$

Since J is a p-ideal, it follows from Lemma 12 that $w = a_x * a_y \in J$ so that $[x * y]_{\rho(I)} \subseteq J$, or equivalently, $x * y \in \rho(I)_-(J)$. Combining Theorem 11(1) and Lemma 12, $\rho(I)_-(J)$ is a p-ideal of G.

Now let $x, y, z \in G$ be such that $(x * z) * (y * z) \in \rho(I)^-(J)$ and $y \in \rho(I)^-(J)$. Then $[y]_{\rho(I)} \cap J \neq \emptyset$ and $[(x * z) * (y * z)]_{\rho(I)} \cap J \neq \emptyset$, and thus there are $a, b \in J$ such that $a \in [y]_{\rho(I)}$ and $b \in [(x * z) * (y * z)]_{\rho(I)}$. Hence $(a, y) \in \rho(I)$ and $\big(b, (x * z) * (y * z)\big) \in \rho(I)$, which imply that $y * a \in I \subseteq J$ and $\big((x * z) * (y * z)\big) * b \in I \subseteq J$. Since J is an ideal and since $a, b \in J$, we have $y \in J$ and $(x * z) * (y * z) \in J$. Since J is a p-ideal, it follows that $x \in J$. Note that $x \in [x]_{\rho(I)}$, and thus $x \in [x]_{\rho(I)} \cap J$, that is, $[x]_{\rho(I)} \cap J \neq \emptyset$. Therefore $x \in \rho(I)^-(J)$. This completes the proof. □

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W. A. Dudek
Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
50-370 Wrocław
Poland
e-mail: dudek@im.pwr.wroc.pl

Y. B. Jun
Department of Mathematics Education
Gyeongsang National University
Chinju (Jinju) 660-701
Korea
e-mail: ybjun@nongae.gsnu.ac.kr

H. S. Kim
Department of Mathematics
Hanyang University
Seoul 133-791
Korea
e-mail: heekim@hanyang.ac.kr