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Some linear conditions and their application to describing group isotopes

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Abstract

The uniqueness of a canonical decomposition of a group isotope is proved in [1]. Now we characterize components of a canonical decomposition of a group isotope from the main classes of quasigroups.

1. Some known results and notions

A groupoid $(A, \circ)$ is called an *isotope* of a groupoid $(B, \cdot)$, if there are bijections α, β, γ from A to B such that the equality

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y)$$

holds for all $x, y \in A$. The triple (α, β, γ) is called an *isotopy* between $(A, \circ)$ and $(B, \cdot)$. Bijections α, β, γ are called *left, right* and *middle components* of this isotopy. A groupoid isotopic to a group $(G, +)$ is called a *group isotope*. $(G, +)$ is called a *decomposition group*. It is easy to see that a group isotope is a quasigroup.

A transformation α of a group $(Q, +)$ is called: *unitary* if $\alpha(0) = 0$; *linear* (*alinear*) if there exist $a, b \in Q$ and an automorphism (antiautomorphism) θ of the group $(Q, +)$ such that $\alpha(x) = a + \theta(x) + b$ for all $x \in Q$; *left* and *right monoregular* if it satisfies the identity

$$\alpha(x + x) = \alpha(x) + x \quad \text{and} \quad \alpha(x + x) = x + \alpha(x),$$

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respectively. A linear unitary transformation is an automorphism.

If the left (right) and middle components of an isotopy are linear transformations of a decomposition group, then the isotopy is called *left* (*right*) *linear*. If the left (right) component is alinear but the middle component is linear then the corresponding isotope is called *left* (*rigdt*) *alinear*. A left and right linear (alinear) group isotope is called *linear* (*alinear*). A quasigroup linearly isotopic to a group is called a *linear quasigroup*. If, in addition, the group is abelian then the quasigroup is said to be *abelian*.

The right side of

$$x \cdot y = \alpha x + a + \beta y\,, \tag{1}$$

is called a (*middle*) *canonical decomposition* determined by an element $0 \in Q$ of a group isotope $(Q,\cdot)$, if $(Q,+)$ is a group (with 0 as its neutral element) and α, β are unitary permutations of $(Q,+)$. α and β are called *coefficients* of the canonical decomposition, a – the *free member*, $(Q;+)$ – the *canonical decomposition group*.

Left and *right* canonical decompositions are determined by:

$$x \cdot y = a + \alpha x + \beta y, \qquad x \cdot y = \alpha x + \beta y + a,$$

respectively. These three canonical decompositions are uniquely determined by an arbitrary element 0 from the set Q (cf. [1]).

In [1] the following two lemmas are proved.

Lemma 1. *If for permutations α, β, γ, δ, μ of a group $(Q,+)$ the identity $\alpha(\beta(x)+\gamma(y)) = \delta(x)+\mu(y)$ holds, then α is a linear transformation of $(Q,+)$. If in addition $\alpha 0 = 0$, then α is an automorphism of $(Q,+)$.*

Lemma 2. *If* (1) *is a canonical decomposition of a group isotope $(Q,\cdot)$ and α is an automorphism of its decomposition group $(Q,+)$, then in $(Q,\cdot)$ we have*

$$x/y = \alpha^{-1}x - \alpha^{-1}\beta y - \alpha^{-1}a = \alpha^{-1}x + \alpha^{-1}I_a^{-1}Ia + \alpha^{-1}I_a^{-1}I\beta y, \tag{2}$$

$$x \oslash y = \alpha^{-1}y - \alpha^{-1}\beta x - \alpha^{-1}a = \alpha^{-1}I_a^{\oplus}I\beta x \oplus \alpha^{-1}I_a^{\oplus}Ia \oplus \alpha^{-1}y. \tag{3}$$

In the sequel will be used the following result from [2].

Theorem 3. *Let* $(Q,\cdot,\Omega)$ *be a quasigroup algebra, where* $(Q,\cdot)$ *is a group isotope. If in the words* υ_1, υ_2, υ_3, υ_4, υ *of the signature* $\{\cdot\}\cup\Omega$ *a variable* x *(a variable* y*) appears only in the words* υ_1, υ_3 *(respectively,* υ_2, υ_4*) and, in addition, exactly one time in at least one of them, then the group isotope is:*

1) *left linear, if the identity* $(\upsilon_1(x)\cdot\upsilon_2(y))\cdot\upsilon=\upsilon_3(x)\cdot\upsilon_4(y)$ *holds in* $(Q,\cdot,\Omega)$,
2) *right linear, if the identity* $\upsilon\cdot(\upsilon_1(x)\cdot\upsilon_2(y))=\upsilon_3(x)\cdot\upsilon_4(y)$ *holds in* $(Q,\cdot,\Omega)$,
3) *left alinear, if the identity* $(\upsilon_1(x)\cdot\upsilon_2(y))\cdot\upsilon=\upsilon_4(y)\cdot\upsilon_3(x)$ *holds in* $(Q,\cdot,\Omega)$,
4) *right alinear, if the identity* $\upsilon\cdot(\upsilon_1(x)\cdot\upsilon_2(y))=\upsilon_4(y)\cdot\upsilon_3(x)$ *holds in* $(Q,\cdot,\Omega)$.

It is easy to see that the following lemma is true.

Lemma 4. *If a group isotope* $(Q,\cdot)$ *has the canonical decomposition* (1), *then*

$$e_x=x\backslash x=\beta^{-1}(-a-\alpha x+x), \tag{4}$$

$$1_x=x/x=\alpha^{-1}(x-\beta x-a), \tag{5}$$

$$R_{e_x}^{-1}(u)=\alpha^{-1}(u-x+\alpha x), \tag{6}$$

$$L_{1_x}^{-1}(u)=\beta^{-1}(\beta x-x+u),$$

where e_x *and* 1_x *are defined by the identities* $xe_x=1_xx=x$.

Also the following two results are proved in [2].

Theorem 5. *Let* $\{x_0,\dots,x_n\}$ *be the set of all variables in the words* w, v *of the signature* $(\cdot,/,\backslash)$ *and let* 0 *be a fixed element of* Q. *If a quasigroup* $(Q,\cdot)$ *is abelian or linear and in the words* w,v *every appearance of every variable is not contained between two appearances of another variable, then the following conditions are equivalent:*

1) *the identity* $w=v$ *holds in* $(Q,\cdot,/,\backslash)$,
2) $w(0,...,0,x_i,0,...,0)=v(0,...,0,x_i,0,...,0)$ *holds in* $(Q,\cdot,/,\backslash)$ *for every* $i=0,1,\dots,n$,

3) $w(0,\ldots,0) = v(0,\ldots,0)$ *and for the middle* 0*-canonical decomposition sums of all coefficients of every variable in* w *and* v *are identical.*

Theorem 6. *Let* $(Q,\cdot,\Omega)$ *be a quasigroup algebra, where* $(Q,\cdot)$ *is a group isotope. If the identity* $w_1(x)\cdot w_2(y) = w_3(y)\cdot w_4(x)$ *holds and two pairs of its subwords* (w_1,w_4) *and* (w_2,w_3) *contain all appearances of variables* x *and* y *(respectively) and there exists only one appearance of* x *in* w_1 *or* w_4 *(respectively,* y *in* w_2 *or* w_3*), then* $(Q,\cdot)$ *is isotopic to a commutative group.*

2. Some linear conditions

The aim of this section is description of positions of variables in some identities implying relations between the coefficients of the group isotope in the canonical decomposition.

Lemma 7. *Let* ω *be a word in a quasigroup algebra* $(Q,\cdot,\Omega)$*, where* $(Q,\cdot)$ *is a group isotope. Then the left bracketting*

$$\omega = (\ldots((\omega_n \underset{n}{\circ} \upsilon_{n-1}) \underset{n-1}{\circ} \upsilon_{n-2}) \underset{n-2}{\circ} \ldots) \underset{1}{\circ} \upsilon_0,$$

where $\underset{i}{\circ} \in \{\cdot, /\}$ *and* υ_i *is a subword of the word* ω*, can be represented in the additive form*

$$\alpha^{k_n}\omega_n + \alpha^{k_{n-1}}\rho_{n-1}a + \alpha^{k_{n-1}}\rho_{n-1}\beta\upsilon_{n-1} + ... + \alpha^{k_0}\rho_0 a + \alpha^{k_0}\rho_0\beta\upsilon_0,$$

where (1) *denotes the canonical decomposition of* $(Q,\cdot)$*,* k_i *denotes the difference between the numbers of operations* $(\cdot)$ *and* $(/)$ *in the sequence* $(\underset{1}{\circ}, \underset{2}{\circ}, ..., \underset{i}{\circ})$ *and*

$$\rho_i \ := \begin{cases} \varepsilon, & \text{if} \quad (\underset{i+1}{\circ}) = (\cdot), \\ \alpha^{-1}I_a^{-1}I, & \text{if} \quad (\underset{i+1}{\circ}) = (/), \end{cases}$$

for $i = 0,1,\ldots,n-1$.

Proof. We use the induction by n. For $n=1$ we have

$$\begin{aligned} \omega &= \alpha\omega_1 + a + \beta\upsilon_0, && \text{if} \quad (\underset{1}{\circ}) = (\cdot), \\ \omega &\overset{(3)}{=} \alpha\omega_1 + \alpha^{-1}I_a^{-1}Ia + \alpha^{-1}I_a^{-1}I\beta\upsilon, && \text{if} \quad (\underset{1}{\circ}) = (/). \end{aligned}$$

These decompositions coincide with the additive form, since $k_0 = 0$, $k_1 = 1 - 0 = 1$, $\rho_0 = \varepsilon$ when $(\underset{1}{\circ}) = (\cdot)$, and $k_1 = 0 - 1 = -1$, $k_0 = 0$, $\rho_0 = \alpha^{-1} I_a^{-1} I$ when $(\underset{1}{\circ}) = (/)$.

Assume, now that the lemma is true for $n-1$. If in the left bracketting of ω we denote $\omega_n \underset{n}{\circ} v_{n-1}$ by ω_{n-1}, then, by the assumption on $n-1$, we obtain

$$\omega = (\ldots(\omega_{n-1} \underset{n-1}{\circ} v_{n-2}) \underset{n-3}{\circ} \ldots) \underset{1}{\circ} v_0$$
$$= \alpha^{k_{n-1}}(\omega_n \underset{n}{\circ} v_{n-1}) + \alpha^{k_{n-2}}\rho_{n-2}a + \alpha^{k_{n-2}}\rho_{n-2}\beta v_{n-2} + ...$$
$$... + \alpha^{k_0}\rho_0 a + \alpha^{k_0}\rho_0\beta v_0,$$

which in the case $(\underset{n}{\circ}) = (\cdot)$ gives $\omega_{n-1} = \alpha\omega_n + a + \beta v_{n-1}$. But $k_n = k_{n-1} + 1$ and $\rho_{n-1} = \varepsilon$, therefore

$$\omega = \alpha^{k_{n-1}}(\alpha\omega_n + a + \beta v_{n-1}) + \alpha^{k_{n-1}}\rho_{n-1}a + \alpha^{k_{n-2}}\rho_{n-2}\beta v_{n-2} + \ldots$$
$$\ldots + \alpha^{k_0}\rho_0 a + \alpha^{k_0}\rho_0\omega_0$$
$$= \alpha^{k_{n-1}+1}\omega_n + \alpha^{k_{n-1}}a + \alpha^{k_{n-1}}\beta v_{n-1} + \alpha^{k_{n-1}}\rho_{n-1}a + \alpha^{k_{n-2}}\rho_{n-2}\beta v_{n-2} +$$
$$\ldots + \alpha^{k_0}\rho_0 a + \alpha^{k_0}\rho_0\omega_0,$$

which coincides with the additive form of ω.

In the case $(\underset{n}{\circ}) = (/)$ we have $k_n = k_{n-1} - 1$, $\rho_{n-1} = I\alpha^{-1}I_a^{-1}$ and

$$\omega_{n-1} \overset{(2)}{=} \alpha^{-1}\omega_n + \rho_{n-1}a + \rho_{n-1}\beta v_{n-1}.$$

Therefore

$$\omega = \alpha^{k_{n-1}}(\alpha^{-1}\omega_n + \rho_{n-1}a + \rho_{n-1}\beta v_{n-1}) + \alpha^{k_{n-2}}\rho_{n-2}a$$
$$+\alpha^{k_{n-2}}\rho_{n-2}\beta v_{n-2} + \ldots + \alpha^{k_0}\rho_0 a + \alpha^{k_0}\rho_0\omega_0,$$

which also gives the additive form of ω. □

Corollary 8. *A left bracketting* $\omega = (\ldots((v_n \cdot v_{n-1}) \cdot v_{n-2}) \cdot \ldots) \cdot v_0)$ *of the word* ω *in a left linear group isotope* $(Q, \cdot)$ *can be written in the form*

$$\omega = \alpha^n v_n + \alpha^{n-1}a + \alpha^{n-1}\beta v_{n-1} + \alpha^{n-2}a + \alpha^{n-2}\beta v_{n-2} + \ldots + a + \beta v_0.$$

Proof. Putting $(\underset{1}{\circ}) = ... = (\underset{n}{\circ}) = (\cdot)$ in Lemma 7 we obtain the above corollary, since in this case $\rho_i = \varepsilon$ for all $i = 0, \dots, n$. □

Theorem 9. *Assume that the identity $\omega = \upsilon$ holds in a quasigroup algebra $(Q, \cdot, /, \backslash, \Omega)$, where $(Q, \cdot)$ is a left linear group isotope, and the first variables in ω and υ are identical and appear in these words only once. If all nodal operations of the overwords of the first variable belong to the set $\{\cdot, /\}$, then the left coefficient α of the canonical decomposition of $(Q, \cdot)$ satisfies the condition $\alpha^{k_1-k_2-k_3+k_4} = \varepsilon$, where k_1, k_3 are the numbers of all nodal operations of the first variable overwords of ω and υ respectively, coinciding with $(\cdot)$, and k_2, k_4 are those coinciding with $(/)$.*

Proof. Let (1) be the canonical decomposition of $(Q, \cdot)$ and let x be the first variable in ω and υ. Applying Lemma 7 to the full left bracketting we see that these words begin with the variable x and that the left and right side of the identity $\omega = \upsilon$ may be written in the form given in Corollary 8. This means that the subword υ_0 contains only one variable x. Since this variable does not appear in other subwords, then replacing of all other variables by elements of Q we obtain

$$\alpha^{k_1-k_2}(x) + b = \alpha^{k_3-k_4}(x) + c,$$

where b, c are some fixed elements from Q. Since for $x = 0$ we have $b = c$, therefore $\alpha^{k_1-k_2} = \alpha^{k_3-k_4}$, which completes the proof. □

Lemma 10. *Let ω be a word in a quasigroup algebra $(Q, \cdot, \Omega)$, where $(Q, \cdot)$ is a group isotope. Then the right bracketting*

$$\omega = \upsilon_0 \underset{1}{\circ} (\upsilon_1 \underset{2}{\circ} \dots \underset{n-1}{\circ} (\upsilon_{n-1} \underset{n}{\circ} \omega_n) \dots),$$

where $\underset{i}{\circ} \in \{\cdot, \backslash\}$ and υ_i are subwords of the word ω, can be represented in the additive form

$$\omega = \beta^{k_0}\nu_0\upsilon_0 + \beta^{k_0}\nu_0 a + \beta^{k_1}\nu_1\alpha\upsilon_1 + \beta^{k_1}\nu_1 a + ...$$
$$... + \beta^{k_{n-1}}\nu_{n-1}\alpha\upsilon_{n-1} + \beta^{k_{n-1}}\nu_0\beta\upsilon_{n-1}a + \beta^{k_n}\omega_n,$$

where (1) denotes the canonical decomposition of $(Q, \cdot)$, k_i denotes the difference between the numbers of operations $(\cdot)$ and $(\backslash)$ in the *sequence* $(\underset{1}{\circ}, \underset{2}{\circ}, ..., \underset{i}{\circ})$ *and*

$$\nu_i \ := \ \begin{cases} \varepsilon, & if \quad (\underset{i+1}{\circ}) = (\cdot), \\ \beta^{-1} I_a I, & if \quad (\underset{i+1}{\circ}) = (\backslash), \end{cases}$$

for $i = 0, 1, \ldots, n-1$.

Proof. The proof is analogous to the proof of Lemma 7. □

Corollary 11. *A right bracketting* $\omega = \upsilon_0 \cdot (\upsilon_1 \cdot \ldots \cdot (\upsilon_{n-1} \cdot \upsilon_n) \ldots)$ *of the word* ω *of a right linear group isotope* $(Q, \cdot)$ *can be written in the form*

$$\omega = \alpha\upsilon_0 + a + \beta\alpha\upsilon_1 + \beta a + \beta^2\alpha\upsilon_2 + \beta^2 a + \cdots + \beta^{n-1} a + \beta^n \upsilon_n.$$

Proof. The proof is analogous to the proof of Corollary 8. □

Theorem 12. *Assume that the identity* $\omega = \upsilon$ *hold in a quasigroup algebra* $(Q, \cdot, /, \backslash, \Omega)$, *where* $(Q, \cdot)$ *is a right linear group isotope, and the last variables in* ω *and* υ *are identical and appear in these words only once. If all nodal operations of the overwords of the last variable belong to the set* $\{\cdot, \backslash\}$, *then the right coefficient* β *of the canonical decomposition of* $(Q, \cdot)$ *satisfies the condition* $\beta^{k_1 - k_2 - k_3 + k_4} = \varepsilon$, *where* k_1, k_3 *are the numbers of all nodal operations of the last variable overwords of* ω *and* υ *respectively, coinciding with* $(\cdot)$, *and* k_2, k_4 *are those coinciding with* $(\backslash)$.

Proof. The proof is analogous to the proof of Theorem 9. □

3. Axiomatics of some classes of isotopes

In this section we find criteria for a group isotope to belong to the main classes of quasigroups.

3.1. Moufang, Bol and IP-quasigroups

As it is well-known, a quasigroup $(Q, \cdot)$ is called

left IP-quasigroup, if there exists a transformation λ such that

$$\lambda x \cdot (x \cdot y) = y,$$

right IP-quasigroup, if there exists a transformation ρ such that

$$(x \cdot y) \cdot \rho(y) = x,$$

Moufang quasigroup, if:

$$(xy \cdot z)y = x \cdot y(e_y z \cdot y),$$
$$y(x \cdot yz) = (y \cdot x1_y)y \cdot z,$$

left Bol quasigroup, if:

$$z(x \cdot zy) = R_{e_z}^{-1}(z \cdot xz) \cdot y,$$

right Bol quasigroup, if:

$$(yz \cdot x)z = y \cdot L_{1_z}^{-1}(zx \cdot z).$$

Theorem 13. *For a group isotope $(Q, \cdot)$ the following statements are equivalent:*

1) *$(Q, \cdot)$ is a left IP-quasigroup,*
2) *$(Q, \cdot)$ is a left Bol quasigroup,*
3) *the right coefficient of the canonical decomposition of $(Q, \cdot)$ is involutive automorphism of the decomposition group.*

Proof. 1) $\Longrightarrow$ 3). Assume that the group isotope $(Q; \cdot)$ is a left IP-quasigroup. Then, by the canonical decomposition (1) of $(Q, \cdot)$, the equation defining a left IP-quasigroup may be written in the form

$$\alpha\lambda(x) + a + \beta(\alpha(x) + a + \beta(y)) = y,$$

where λ is as in the definition of a left IP-quasigroup.

This means that

$$\beta(R_a\alpha(x) + \beta(y)) = IR_a\alpha\lambda(x) + y,$$

where $I(x) = -x$, holds for all $x, y \in Q$. Thus, according to Theorem 1, β is a linear transformation of the group $(Q, +)$. Moreover, β (as a component of the canonical decomposition) is a unitary permutation of $(Q, +)$. Hence, β is an automorphism of $(Q, +)$.

Applying this fact and Theorem 12 to the equality defining a left IP-quasigroup we obtain the relation $\beta^{2-0+0-0} = \varepsilon$, which shows that β is an involutive automorphism of $(Q,+)$.

3) $\Longrightarrow$ 1). Let $(Q,\cdot)$ be an isotope of a group $(Q,+)$, (1) its canonical decomposition and β an involutive automorphism of $(Q,+)$. Putting

$$\lambda = \alpha^{-1}R_a^{-1}I\beta R_a\alpha \tag{7}$$

we obtain a transformation λ of Q such that

$$\begin{aligned}\lambda(x)\cdot(x\cdot y) &= R_a\alpha\lambda(x) + \beta(R_a\alpha(x) + \beta(y))\\ &= R_a\alpha\alpha^{-1}R_a^{-1}I\beta R_a\alpha(x) + \beta R_a\alpha(x) + \beta^2(y)\\ &= -\beta R_a\alpha(x) + \beta R_a\alpha(x) + y = y.\end{aligned}$$

Hence $(Q,\cdot)$ is a left IP-quasigroup.

2) $\Longrightarrow$ 3). Let a group isotope $(Q,\cdot)$ be a left Bol quasigroup. Fixing z in the identity defining a left Bol loop and applying Theorem 3 we obtain the right linearity of $(Q,\cdot)$. Because this identity is balanced with respect to y, then Theorem 12 implies $\beta^{3-0+0-1} = \varepsilon$, where β is a right coefficient of the canonical decomposition of $(Q,\cdot)$. Thus β is an involutive automorphism.

3) $\Longrightarrow$ 2). If β in the canonical decomposition (1) of $(Q,\cdot)$ is an involutive automorphism of $(Q,+)$, then

$$\begin{aligned}R_{e_z}^{-1}(z\cdot xz)\cdot y &\overset{(1)}{=} \alpha R_{e_z}^{-1}(z\cdot xz) + a + \beta y\\ &\overset{(6)}{=} (z\cdot xz) - z + \alpha z + a + \beta y\\ &\overset{(1)}{=} \alpha z + a + \beta(\alpha x + a + \beta z) - z + \alpha z + a + \beta y\\ &= \alpha z + a + \beta\alpha x + \beta a + z - z + \alpha z + a + \beta y\\ &= \alpha z + a + \beta\alpha x + \beta a + \alpha z + a + \beta y.\end{aligned}$$

Similarly

$$\begin{aligned}z(x\cdot zy) &\overset{(1)}{=} \alpha z + a + \beta(\alpha x + a + \beta(\alpha z + a + \beta y))\\ &= \alpha z + a + \beta\alpha x + \beta a + \alpha z + a + \beta y,\end{aligned}$$

which proves that $(Q,\cdot)$ is a left Bol quasigroup. $\square$

Theorem 14. *For a group isotope $(Q,\cdot)$ the following statements are equivalent:*

1) *$(Q,\cdot)$ is a right IP-quasigroup,*
2) *$(Q,\cdot)$ is a right Bol quasigroup,*
3) *the left coefficient of the canonical decomposition of $(Q,\cdot)$ is an involutive automorphism of the decomposition group.*

Proof. The proof is analogous to the proof of Theorem 13. □

Theorem 15. *For a group isotope $(Q,\cdot)$ the following statements are equivalent:*

1) *$(Q,\cdot)$ is an IP-quasigroup,*
2) *$(Q,\cdot)$ is a Moufang quasigroup,*
3) *$(Q,\cdot)$ is a Bol quasigroup,*
4) *all coefficients of the canonical decomposition of $(Q,\cdot)$ are involutive automorphisms of the decomposition group.*

Proof. The equivalence of 1), 3) and 4) follows from Theorems 13 and 14.

2) $\Longleftrightarrow$ 4). Let $(Q,\cdot)$ be a Moufang quasigroup. Putting

$$\upsilon_1 = xy, \quad \upsilon_2 = z, \quad \upsilon = y, \quad \upsilon_3 = x, \quad \upsilon_4 = y(e_y z \cdot y)$$

in the first identity defining this quasigroup and applying Theorem 3 we obtain the right linearity of $(Q,\cdot)$. In the analogous way, the second identity from the definition of a Moufang quasigroup gives the left linearity of $(Q,\cdot)$. Thus $(Q,\cdot)$ is a linear group isotope. But for linear group isotopes this equivalence is proved in [4]. □

A *left* (*right*) symmetri c quasigroup is defined as a quasigroup satisfying the identity $x\cdot(x\cdot y) = y$ (respectively, $(x\cdot y)\cdot y = x$). A quasigroup which is left and right symmetric is called *symmetric* or a *TS-quasigroup.*

Corollary 16. *A group isotope $(Q,\cdot)$ is a left (right) symmetric quasigroup iff the decomposition group $(Q,+)$ is commutative and the right (left) coefficient β of its canonical decomposition is an automorphism of $(Q,+)$ such that $\beta(x) = -x$ for all $x \in Q$.*

Proof. Every left symmetric quasigroup is a left IP-quasigroup, where $\lambda = \varepsilon$. From the proof of Theorem 13 follows $\beta = I$, i.e. $\beta(x) = -x$ for all $x \in Q$. But such defined β is an automorphism only in commutative groups. The converse is obvious.

In the case of a right symmetric quasigroup the proof is analogous. □

3.2. F-quasigroups

Note that a *left* (*right*) F-quasigroup is defined as a quasigroup $(Q, \cdot)$ satisfying the identity

$$x \cdot yz = xy \cdot e_x z, \tag{8}$$

(respectively, $xy \cdot z = x1_z \cdot yz$).

Theorem 17. *A group isotope $(Q, \cdot)$ with a canonical decomposition (1) is a left F-quasigroup iff β is an automorphism of the group $(Q, +)$, β commutes with α and α satisfies the identity*

$$\alpha(x + y) = x + \alpha y - x + \alpha x. \tag{9}$$

Proof. Let $(Q, \cdot)$ be a group isotope satisfying (8). If (1) is a canonical decomposition of $(Q, \cdot)$, then (8) together with Theorem 3 imply that β is an automorphism of $(Q, +)$.

Moreover, (8) for $z = \beta^{-1}(-a)$ and $x = \alpha^{-1}(t - a)$ gives

$$t + \beta\alpha y = \alpha(t + \beta y) + \gamma t, \tag{10}$$

where γ is a some permutation of Q.

This identity $y = 0$ implies $\gamma t = -\alpha t + t$. Hence (10) may be written in the form

$$t + \beta\alpha y = \alpha(t + \beta y) - \alpha t + t,$$

which for $t = 0$ gives $\alpha\beta = \beta\alpha$. This fact together with the transposition of βy and y in (10) implies

$$t + \alpha y = \alpha(t + y) - \alpha t + t,$$

which proves (9).

Conversely, let $(Q,\cdot)$ be a group isotope with the canonical decomposition described in Theorem.

Putting $y=-x$ in (9) we obtain $0=x+\alpha(-x)-x+\alpha(x)$, i.e.

$$x+\alpha(-x)=-\alpha x+x. \tag{11}$$

Hence

$$\begin{aligned}
xy\cdot e_x z &\overset{(1)}{=} \alpha(\alpha x+a+\beta y)+a+\beta(\alpha e_x+a+\beta z)\\
&= \alpha((\alpha x+a)+\beta y)+a+\beta(\alpha e_x)+\beta a+\beta^2 z\\
&\overset{(9)}{=} \alpha x+a+\alpha\beta y-(\alpha x+a)+\alpha(\alpha x+a)+a+\alpha\beta e_x+\beta a+\beta^2 z\\
&\overset{(4)}{=} \alpha x+a+\alpha\beta y-(\alpha x+a)+\alpha(\alpha x+a)+a+\\
&\qquad +\alpha(-(\alpha x+a)+x)+\beta a+\beta^2 z\\
&\overset{(9)}{=} \alpha x+a+\alpha\beta y-(\alpha x+a)+\alpha(\alpha x+a)+a-(\alpha x+a)+\\
&\qquad +\alpha x+\alpha x+a+\alpha(-(\alpha x+a))+\beta a+\beta^2 z\\
&= \alpha x+a+\alpha\beta y-(\alpha x+a)+\alpha(\alpha x+a)+(\alpha x+a+\\
&\qquad +\alpha(-(\alpha x+a)))+\beta a+\beta^2 z\\
&\overset{(11)}{=} \alpha x+a+\alpha\beta y-(\alpha x+a)+\alpha(\alpha x+a)-\alpha(\alpha x+a)+\\
&\qquad +(\alpha x+a)+\beta a+\beta^2 z\\
&= \alpha x+a+\alpha\beta y+\beta a+\beta^2 z\\
&= \alpha x+a+\beta\alpha y+\beta a+\beta^2 z=\alpha x+a+\beta(\alpha y+a+\beta z)\\
&= x\cdot(y\cdot z),
\end{aligned}$$

which proves that $(Q,\cdot)$ is a left F-quasigroup. □

Corollary 18. *If a group isotope is a left F-quasigroup, then it is right linear. It is linear iff the left coefficient of its canonical decomposition commutes with every inner automorphism of the decomposition group.*

Proof. The first part follows from Theorem 17. If a linear group isotope is a left F-quasigroup, then, as it is proved in [4], the left coefficient of its canonical decomposition commutes with every inner automorphism of the decomposition group.

Conversely, if α commutes with every inner automorphism of the group $(Q,+)$, then (9) may be rewritten in the form:

$$\alpha(x+y)=\alpha(x+y-x)+\alpha x,$$

which for $u=x+y-x$ implies $\alpha(u+x)=\alpha u+\alpha x$. Hence α is an automorphism of the group $(Q;+)$. □

Corollary 19. *If a group isotope is a left F-quasigroup, then it is left alinear iff its decomposition group is commutative.*

Proof. Theorem 17 implies (9), which may be rewritten in the form $\alpha y+\alpha x=x+\alpha y-x+\alpha x$, because α is an antiautomorphism of $(Q,+)$. This implies the commutativity of the group $(Q,+)$.

The converse is obvious. □

Theorem 20. *A group isotope $(Q,\cdot)$ with a canonical decomposition (1) is a right F-quasigroup iff α is an automorphism of the group $(Q,+)$, α commutes with β and β satisfies the identity*

$$\beta(y+z)=\beta z-z+\beta y+z.$$

Proof. The proof is analogous to the proof of Theorem 17. □

3.3. Alternative quasigroups

A quasigroup $(Q,\cdot)$ is called *left* (*right*) alternative if it satisfies the identity $x\cdot(x\cdot z)=(x\cdot x)\cdot z$ (respectively, $(x\cdot y)\cdot y=x\cdot(y\cdot y)$).

Theorem 21. *A group isotope $(Q,\cdot)$ with the canonical decomposition (1) is left alternative iff $\beta=\varepsilon$ and $\alpha=R_a^{-1}\theta^{-1}$, where θ is a right monoregular permutation of the group $(Q,+)$.*

Proof. If a group isotope $(Q,\cdot)$ with the canonical decomposition (1) is left alternative, then the identity $x\cdot(x\cdot z)=(x\cdot x)\cdot z$ may be rewritten in the form

$$\alpha x+a+\beta(\alpha x+a+\beta z)=\alpha(\alpha x+a+\beta x)+a+\beta z.$$

Replacing in this identity $a+\beta z$ by z and αx by x we obtain

$$x+a+\beta(x+z)=\alpha(x+a+\beta\alpha^{-1}x)+z,$$

which for $z = 0$ gives

$$x + a + \beta x = \alpha(x + a + \beta\alpha^{-1}x). \tag{12}$$

Therefore the previous identity may be written in the form

$$x + a + \beta(x + z) = x + a + \beta x + z.$$

Hence $\beta(x + z) = \beta x + z$, and in the consequence $\beta = \varepsilon$. Thus (12) implies

$$\alpha^{-1}(x + a + x) = x + a + \alpha^{-1}x.$$

Replacing x by $x - a$ we see that $\theta = R_a^{-1}\alpha^{-1}$ is a right monoregular permutation.

Conversely, let the relations $\beta = \varepsilon$ and θ be a right monoregular permutation of the group $(Q; +)$, then

$$\begin{aligned} x \cdot (x \cdot z) &\overset{(1)}{=} \alpha x + a + \beta(\alpha x + a + \beta z) = \alpha x + a + \alpha x + a + z \\ &= (\alpha x + a + \alpha x) + a + z = \alpha(\alpha x + a + x) + a + z \\ &\overset{(1)}{=} (x \cdot x) \cdot z \end{aligned}$$

completes the proof. □

Corollary 22. *A left alternative group isotope is a left loop.*

Proof. Indeed, $\beta = \varepsilon$ implies

$$(\alpha^{-1}(-a)) \cdot y \overset{(1)}{=} \alpha(\alpha^{-1}(-a)) + a + y = -a + a + y = y$$

for every $y \in Q$. Thus $\alpha^{-1}(-a)$ is a left unit of $(Q, \cdot)$. □

In the similar way as Theorem 21 we can prove

Theorem 23. *A group isotope $(Q, \cdot)$ with the canonical decomposition (1) is a right alternative quasigroup iff $\alpha = \varepsilon$, and $\beta = R_a^{-1}\theta^{-1}$, where θ is a left monoregular permutation of the group $(Q, +)$.*

Corollary 24. *A right alternative group isotope is a right loop.*

3.4. Semimedial quasigroups

A quasigroup $(Q,\cdot)$ is called *left semimedial* if it satisfies the identity

$$xx \cdot yz = xy \cdot xz,$$

and *right semimedial* if it satisfies the identity $xy \cdot zz = xz \cdot yz$. A quasigroup which is left and right semimedial is called *semimedial*. It is a special case of so-called *medial* quasigroups, i.e. quasigroups satisfying the identity $xy \cdot uv = xu \cdot yv$.

Theorem 25. *A group isotope $(Q,\cdot)$ is left semimedial iff there exists a group $(Q,+)$, an element $a \in Q$, a permutation α of Q and an automorphism β of $(Q,+)$ such that*

$$L_{\alpha a}\beta\alpha = \alpha R_a \beta, \tag{13}$$

$$x \cdot y = \alpha x + \beta y + a, \tag{14}$$

$$\alpha(x+y) = \alpha x + \beta x + \alpha y - \beta x \tag{15}$$

for all $x, y \in Q$.

Proof. By Theorem 3, a left semimedial group isotope $(Q,\cdot)$ is right linear and has the decomposition (14), where β is an automorphism of the group $(Q,+)$.

Thus from (14) and $00 \cdot yz = 0y \cdot 0z$, where $\beta z = -a$, we obtain $\alpha a + \beta\alpha y = \alpha(\beta y + a)$, which gives (13) and

$$\beta\alpha y = -\alpha a + \alpha(\beta y + a).$$

This together with (14) and $xx \cdot yz = xy \cdot xz$ for $\beta z + a = 0$, $\beta y + a = u$ and $\alpha x = v$ implies

$$\alpha(v + \beta x + a) - \alpha a + \alpha u = \alpha(v+u) + \beta v,$$

which for $u = 0$ gives $\alpha(v + \beta x + a) - \alpha a = \alpha v + \beta v$.

Applying this identity to the previous we obtain (15).

Conversely, if a group isotope $(Q,\cdot)$ has the canonical decomposition (14) such that (13) and (15) are satisfied, then

$$\begin{aligned} xx\cdot yz &\overset{(14)}{=} \alpha(xx)+\beta(yz)+a\\ &\overset{(14)}{=} \alpha(\alpha x+\beta x+a)+\beta(\alpha y+\beta z+a)+a\\ &\overset{(15)}{=} \alpha^2x+\beta\alpha x+\alpha(\beta x+a)-\beta\alpha x+\beta\alpha y+\beta^2z+\beta a+a\\ &\overset{(13)}{=} \alpha^2x+\beta\alpha x+\alpha a+\beta\alpha x-\beta\alpha x+\beta\alpha y+\beta^2z+\beta a+a\\ &= \alpha^2x+\beta\alpha x+\alpha a+\beta\alpha y+\beta^2z+\beta a+a. \end{aligned}$$

and

$$\begin{aligned} xy\cdot xz &\overset{(14)}{=} \alpha(xy)+\beta(xz)+a\\ &\overset{(14)}{=} \alpha(\alpha x+\beta y+a)+\beta(\alpha x+\beta z+a)+a\\ &\overset{(15)}{=} \alpha^2x+\beta\alpha x+\alpha(\beta y+a)-\beta\alpha x+\beta\alpha x+\beta^2z+\beta a+a\\ &\overset{(13)}{=} \alpha^2x+\beta\alpha x+\alpha a+\beta\alpha y+\beta^2z+\beta a+a. \end{aligned}$$

This proves that $(Q,\cdot)$ is left semimedial. □

Corollary 26. *A left semimedial group isotope is right linear. It is left linear iff it is medial.*

Proof. The first part of the statement follows from Theorem 25. By Toyoda-Bruck's Theorem a medial group isotope is linear, and by [4] a semimedial linear group isotope is medial. □

Theorem 27. *A group isotope $(Q,\cdot)$ is right semimedial iff there exists a group $(Q,+)$, an element $a\in Q$, an automorphism α of $(Q,\cdot)$ and a permutation β of Q such that $\beta(x+y)=-\alpha y+\alpha x+\alpha y+\beta y$, $\beta L_a\alpha=R_{\beta a}\alpha\beta$ and $x\cdot y=a+\alpha x+\beta y$ for all $x,y\in Q$.*

Proof. The proof is analogous to the proof of Theorem 25. □

Corollary 28. *A group isotope is medial iff it is semimedial.*

Corollary 29. *A group isotope $(Q,\cdot)$ is commutative iff its decomposition group is commutative and $\alpha=\beta$.*

Corollary 30. *A group isotope $(Q,\cdot 0$ is unipotent iff it has the decomposition $x\cdot y=\alpha x-\alpha y+a$ or $x\cdot y=a+\beta x-\beta y$.*

Corollary 31. *The canonical decomposition group of a commutative unipotent group isotope is a Boolean group.*

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