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On topological completely inverse AG^{**}-groupoids

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Abstract. We extend the classical theorem of R.Ellis to completely inverse AG^{**}-groupoids and we describe topologies on AG^{**}-groupoid induced by family of pseudometrics.

1. Introduction and prelimenaries

The way to create a topology on AG^{**}-groupoids using pseudometrics is not universal, because there are AG^{**}-groupoids with a topology that cannot be standardized. Nevertheless, the problem of describing thefamilies of pseudometrics that generate the topology on AG^{**}-groupoids, consistent with multiplication and inverse operations is interesting.

The topologies on semigroups and groups defined by the families of pseudometrics and norms are described in [1]. We now describe such topologies on AG^{**}-groupoids. First, we extend the Ellis result that each locally compact Hausdorff semitopological group is a topological group, on AG^{**}-groupoids, and describe topologies on completely inverse AG^{**}-groupoids defined by the families of left pseudometrics. Next we present the necessary and sufficient conditions under which such a topology is compatible with the AG^{**}-groupoid operation.

A groupoid $(X, \cdot)$ satisfying the identity $xy \cdot z = zy \cdot x$ is called an AG-groupoid. An AG-groupoid satisfying the identity $x \cdot yz = y \cdot xz$ is called an AG^{**}-groupoid. It is medial. Each AG-groupoid containing a left identity is an AG^{**}-groupoid (see [3] or [4]). Such AG-groupoid is paramedial, i.e. $xy \cdot zv = vy \cdot zx$ holds for all $x, y, z, v \in X$. An AG-groupoid with a left identity e in which the set of $x \in X$ for which there exists $x^{-1} \in X$ such that $x^{-1}x = xx^{-1} = e$ is called completely inverse.

The maps $\lambda_b : X \to X$, $x \mapsto bx$ and $\rho_b : X \to X$, $x \mapsto xb$ are called the left and right translation by b, respectively.

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An AG^{**}-groupoid with the topology τ defined on X is called a topological AG^{**}-groupoid if the multiplication and inversion are continuous in this topology.

2. Results

In the first we show that is possible to extend the remarkable R.Ellis theorem to the case of AG^{**}-groupoids.

Theorem 2.1. *Let τ be a locally compact topology defined on a completely inverse AG^{**}-groupoid with a left identity. Then $(X,\cdot,\tau)$ is a topological AG^{**}-groupoid if and only if λ_a is open and continuous for each $a \in X$.*

Proof. Let $x, y \in X$, and W be an neighborhood of the point xy^{-1}. As X satisfies the paramedial law for all $a \in X$ we obtain $xy^{-1} = (xx^{-1})x \cdot y^{-1} = ex \cdot y^{-1} = (a^{-1}a)x \cdot y^{-1} = y^{-1}x \cdot a^{-1}a = ax \cdot a^{-1}y^{-1} = ax \cdot (ay)^{-1}$. Since λ_a is open and continuous, then there exists U and V neighborhoods of x and y, respectively, such that $UV^{-1} \subset W$. Then the mapping $(x,y) \mapsto xy^{-1}$ of $X \times X$ into X is continuous, so $(X,\cdot,\tau)$ is a topological AG^{**}-groupoid.

The converse is obvious. □

A mapping $f : X \times X \to [0,+\infty)$ is called a pseudometric on X if $f(x,y) = f(y,x)$ and $f(x,y) \leqslant f(x,z) + f(z,y)$ for $x,y,z \in X$. The pseudometric f on an AG-groupoid $(X,\cdot)$ is said to be left (right) invariant if $f(zx,zy) = f(x,y)$ (resp. $f(xz,yz) = f(x,y)$) for $x,y,z \in X$.

Theorem 2.2. *If a topology on an AG-groupoid $(X,\cdot)$ is generated by a family of rihgt invariant pseudometrics, then all right translations ρ_a are continuous. If, in addition, X is completely inverse AG^{**}-groupoid with a right identity, then ρ_a are bijective.*

Proof. Let $\{f_i\}_{i\in I}$ be the family of right invariant pseudometrics. Then sets of the form $W = \{s \in X : f_i(s,xy) < \epsilon,\ f_1,\ldots,f_n \in \mathcal{F}\}$ form a pre-base of a topology τ on X, and $U = \{h \in X : f_i(h,x) < \epsilon, i \in I\}$ is a neighborhood of the point $x \in X$. If $h \in U$, then $f_i(ha,xa) = f_i(h,x) < \epsilon$. So, $ha \in W$. Consequently, ρ_a is continuous.

If $(X,\cdot)$ is a completely inverse AG^{**}-groupoid and e is its right identity e, then for each $y \in X$ there is $x \in X$ such that $\rho_a(x) = y$. Indeed, $xa = y$ means that $a^{-1} \cdot xa = a^{-1}y$, whence $x \cdot a^{-1}a = a^{-1}y$. So, $x = a^{-1}y \in X$. Therefore, ρ_a is bijective. □

Theorem 2.3. *For a completely inverse AG^{**}-groupoid $(X, \cdot)$ with a topology generated by a family $\mathcal{F}$ of left invariant pseudometrics the following conditions are equivalent.*

- (A) *for arbitrary $a, x \in X$, every $f \in \mathcal{F}$, and every $\alpha > 0$, there are $f_1, \ldots, f_n \in \{\mathcal{F}\}$ and $\beta > 0$ such that $f(sa, xa) < \alpha$ for all $s \in X$ for which $f_i(s, x) < \beta$ for $i = 1, \ldots, n$,*
- (B) *the multiplication $(x, y) \mapsto xy$ in (X, τ) is continuous,*
- (C) *the right translations ρ_a are continuous for each $a \in X$.*

If, in addition, X has a left identity, then each of the conditions (A), (B), (C) implies that $i : x \mapsto x^{-1}$ is continuous and injective.

Proof. $(A) \Rightarrow (B)$. Let $x, y \in X$. Consider the set

$$W = \{t \in X : g_i(t, xy) < \epsilon \text{ for all } g_1, \ldots, g_n \in \mathcal{F}\}.$$

Then there exist $f_1, \ldots, f_n \in \mathcal{F}$ and $\beta > 0$ such that $g_i(sy, xy) < \frac{\epsilon}{2}$ for every $i = 1, \ldots, n$ with all $s \in X$ satisfying $f_j(s, x) < \beta$ for all $j = 1, \ldots, n$. If s belong to the neighborhood $U = \{z \in X : f_j(z, x) < \beta,\ j = 1, \ldots, m\}$ of the point x, and h lie in the neighborhood $V = \{v \in X : g_i(v, y) < \frac{\epsilon}{2},\ i = 1, \ldots, n\}$ of the point y, then for $i = 1, \ldots, n$ we have $g_i(sh, xy) \leqslant g_i(sh, sy) + g_i(sy, xy) < g_i(h, y) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$. Consequently, $sh \in W$, and therefore the multiplication in (X, τ) is continuous.

$(B) \Rightarrow (C)$ obvious.

$(C) \Rightarrow (A)$. Let $x, a \in X$, $\alpha > 0$ and $f \in \mathcal{F}$. Then there exists a neighborhood U of the point x such that U is contained in the neighborhood $V = \{t \in X : f(t, xa) < \alpha\}$ of the point xa. Therefore, there are $f_1, \ldots, f_n \in \mathcal{F}$ and $\beta > 0$ such that the set $\{s \in X : f_i(s, x) < \beta,\ i = 1, \ldots, n\}$ is contained in U. Consequently, for such s we have $f(sa, xa) < \alpha$, i.e. (A) is satisfied.

Now, if a completely inverse AG^{**}-groupoid $(X, \cdot)$ with the left identity e satisfies (A), then for $U = \{z \in X : g_i(z, x^{-1}) < \epsilon,\ i = 1, \ldots, n\}$, where $g_1, \ldots, g_n \in \mathcal{F}$, there exist $f_1, \ldots, f_n \in \mathcal{F}$ and $\beta > 0$ such that $g_i(sx^{-1}, xx^{-1}) < \epsilon$ for $i = 1, \ldots, n$ and all s from the neighborhood $V = \{z \in X : f_j(z, x) < \beta,\ j = 1, \ldots, n\}$ of the point x. If $s \in V$, then for $i = 1, \ldots, n$ we have $g_i(s^{-1}, x^{-1}) = g_i(ss^{-1}, sx^{-1}) = g_i(e, sx^{-1}) = g_i(sx^{-1}, e) = g_i(sx^{-1}, xx^{-1}) < \epsilon$. Thus, $s^{-1} \in U$, so $i : x \mapsto x^{-1}$ is continuous. Moreover, if $i(x) = i(y)$, then $x^{-1} = y^{-1}$ and $e = yy^{-1} = yx^{-1}$. Consequently, $x = ex = yx^{-1} \cdot x = xx^{-1} \cdot y = ey = y$. Hence $i : x \mapsto x^{-1}$ is injective. $\square$

An AG-groupoid $(X, \cdot)$ with the left identity e is called an AG-group if for every $a \in X$ there exists $a^{-1} \in X$ such that $aa^{-1} = a^{-1}a = e$.

Theorem 2.4. *If a topology on an AG-group $(X, \cdot)$ is generated by a family of rihgt invariant pseudometrics, then all right translations ρ_a are continuous bijections.*

Proof. By Theorem 2.2 all right translations ρ_a are continuous. To prove that ρ_a are bijections consider the equation $\rho_a(x) = y$ where $a, y \in X$ are fixed. Then $xa = y$ and $xa \cdot a^{-1} = ya^{-1}$. So, $x = ex = a^{-1}a \cdot x = ya^{=1} \in X$, which shows that ρ_a is a bijection. □

Theorem 2.5. *For an AG^{**}-group $(X, \cdot)$ with a topology τ generated by a family $\mathcal{F}$ of left invariant pseudometrics the following conditions are equivalent.*

(A) *for arbitrary $a, x \in X$, every $f \in \mathcal{F}$, and every $\alpha > 0$, there are $f_1, \dots, f_n \in \{\mathcal{F}\}$ and $\beta > 0$ such that $f(sa, xa) < \alpha$ for all $s \in X$ for which $f_i(s, x) < \beta$ for $i = 1, \dots, n$,*

(B) *$(X, \cdot, \tau)$ is a topological AG-group,*

(C) *the right translations ρ_a are continuous for each $a \in X$.*

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