Quasigroups and Related Systems **30** (2022), 317 – 327
https://doi.org/10.56415/qrs.v30.27

Normal filter in quasi-ordered residuated systems

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Abstract. The concept of quasi-ordered residuated systems was introduced by Bonzio and Chajda in 2018. The author introduced the concept of filters in such systems as well as some types of filters in them such as implicative and comparative filters. This article introduces the concept of a normal filter in a quasi-ordered residuated system and relates it to some other filters in such algebraic systems.

1. Introduction

Residuated lattices were introduced by Ward and Dilworth in [15]. The filter theory of residuated lattice has been widely studied, and some important results have been obtained. Normal filters in BL-algebra were defined in paper [13]. Borzooei and Paad studied the normal filter in BL-algebras (cf. [4]) by comparing it with other types of filters in residuated lattices. Ahadpanah and Torkzadeh are studied normal filters in residuated lattices (cf. [1]). Wang et al. also dealt with normal filters in some logical algebras (cf. [14]).

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRS), was introduced in 2018 by S. Bonzio and I. Chajda (cf. [3]). Previously, this concept was discussed in [2]. It should be noted that this algebraic system differs from the commutative residuated lattice ordered under a quasi-order (see Example 2.13):

- QRS does not have to be limited from below: and
- QRS, in the general case, does not have to be a lattice.

The author introduced and developed the concepts of filters (cf. [7]) in this algebraic structure as well as several types of filters such as implicative (cf.

2010 Mathematics Subject Classification: 08A02, 06A11
Keywords: Quasi-ordered residuated system, filter in quasi-ordered residuared system, implicative filter, comparative filter, normal filter.

[9]), associated (cf. [8]) and comparative filters (cf. [10]). In [9], it is shown that every comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter of $\mathfrak{A}$ and the reverse it need not be valid.

In the following, some preliminary claims and terms about quasi-ordered residuated systems are taken from the literature [2, 3, 7, 9, 10]. In Section 3, we define the concept of normal filters in a quasi-ordered residuated system and we prove some theorems that accurately describe the relationship between this notion and the other types of filters in such an algebraic structure.

2. Preliminaries

2.1. Concept of quasi-ordered residuated systems

In article [3], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1. [[3], Definition 2.1] A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:
(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.
We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following

Theorem 2.2 ([3], Proposition 2.1)**.** *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ be a residuated relational system. Then*
(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$*;*
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$*;*
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$*;*
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$*;*
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $' \preccurlyeq '$ on a set A is a binary relation which is reflexive and transitive.

Definition 2.3. [[3], Definition 3.1] A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \to, 1, \preccurlyeq \rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

Example 2.4. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (d,1), (b,b), (a,a), (c,c), (d,d), (c,a), (d,a), (d,b), (d,c)\}$.

Example 2.5. For a commutative monoid A, let $\mathfrak{P}(A)$ denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \to, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \to X := \{z \in A : Yz \subseteq X\}).$$

Example 2.6. Let $\mathbb{R}$ be a field of real numbers. Define a binary operations '$\cdot$' and '$\to$' on $A = [0, 1] \subset \mathbb{R}$ by

$$(\forall x, y \in [0,1])(x \cdot y := max\{0, x+y-1\}) \text{ and } x \to y := min\{1, 1-x+y\}).$$

Then, A is a commutative monoid with the identity 1 and $\langle A, \cdot, \to, \leqslant, 1 \rangle$ is a quasi-ordered residuated system.

Example 2.7. Any commutative residuated lattice $\langle A, \cdot, \to, 0, 1, \sqcap, \sqcup, R \rangle$ where R is a lattice quasi-order is a quasi-ordered residuated system.

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.8 ([3], Proposition 3.1). *Let A be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;
(10) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;
(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates an equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (9) and (10), this equivalence is compatible with the operations in $\mathfrak{A}$. Thus, $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$. The concept of a strong quasi-ordered residuated system is given by the following definition:

Definition 2.9. [[11], Definition 6] For a quasi-ordered residuated system $\mathfrak{A}$ it is said to be a *strong quasi-ordered residuated system* if the following holds

$$(\forall u, v \in A)((u \rightarrow v) \rightarrow v \equiv_{\preccurlyeq} (v \rightarrow u) \rightarrow u).$$

Example 2.10. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

and

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	b	b	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (b,b), (c,c), (a,b), (a,c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system.

2.2. Concept of filters

In this subsection we give some notions that will be used in this article.

Definition 2.11. [[7], Definition 3.1] For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies conditions

$(F2)$ $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, and
$(F3)$ $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

It is shown (cf. [7], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the conditions

(F-0) $1 \in F$ and

(F-1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice (cf. [7], Theorem 3.1).

Remark 2.12. In implicative algebras, the term 'implicative filter' is used instead of the term 'filter' we use (see, for example [5, 12]) because in the structure we study the concept of filter is determined more complexly than requirements (F3). It is obvious that our filter concept is also a filter in the sense of [5, 6, 12]. The term 'special implicative filter' is also used in the aforementioned sources if the implicative filter in the sense of [12] satisfies some additional condition.

Example 2.13. Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\to$' as follows, $(\forall y, v \in A)(u \cdot v := min\{u, v\})$ and $u \to v := 1$ if $u \leqslant v$ and $u \to v := v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} := \langle A, \cdot, \to, 1, \leqslant \rangle$ is a quasi-ordered residuated system. All filters in $\mathfrak{A}$ are in the form of $\langle x, 1]$, for $x \in \langle -\infty, 1]$.

Terms covering some of the requirements used herein to identify various types of filters in the observed algebraic structure are mostly taken from papers on UP-algebras. In some other algebraic systems, different terms are used to cover the concepts of implicative and comparative filters mentioned herein.

Definition 2.14. [[9], Definition 3.1] For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that an *implicative filter* of $\mathfrak{A}$ if (F2) and the following condition

(IF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \to v \in F) \implies u \to z \in F)$

are valid.

Definition 2.15. [[10], Definition 3.1] For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that a *comparative filter* of $\mathfrak{A}$ if (F2) and the following condition

(CF) $(\forall u, v, z \in A)((u \to ((v \to z) \to v) \in F \wedge u \in F) \implies v \in F)$

are valid.

Example 2.16. *Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.4. Then the set $F := \{1, a, b\}$ is a comparative filter in $\mathfrak{A}$.*

In [9] (Theorem 3.4), it is shown that every comparative filter in a quasi-ordered residuated system is an implicative filter and that the reverse does not have to be.

Example 2.17. (cf. [9], Example 3.3) Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.4. Then the subset $F := \{1, b\}$ is an implicative filter but it is not a comparative filter.

Notions and notations that are used but not previously determined in this paper can be found in ([2, 3, 7, 8, 9, 10]).

3. Normal filters in QRS

In this section, which is the main part of this paper, the concept of a normal filter in a quasi-ordered residuated system is presented and some of its important features are shown. Some of the assertions used in this article, although shown in our paper [9], will be shown again due to the consistency of the material presented in this paper.

Definition 3.1. A filter F of a quasi-ordered residuated system $\mathfrak{A}$ is called *normal* if the following holds

(NF) $(\forall x, y, z \in A)((z \to ((y \to x) \to x) \in F \wedge z \in F) \Rightarrow (x \to y) \to y \in F)$.

In the general case, the filter in a quasi-ordered residuated system does not have to be a normal filter.

Theorem 3.2. *If $\mathfrak{A}$ is a strong quasi-ordered residuated system, then any filter of $\mathfrak{A}$ is a normal filter of $\mathfrak{A}$.*

Proof. Let F be a filter of a strong quasi-ordered residuated system $\mathfrak{A}$. Suppose that $x, y, z \in A$ are elements such that $z \in F$ and $z \to ((y \to x) \to x) \in F$. Then $(y \to x) \to x \in F$ by (F-3). Since $(y \to x) \to x = (x \to y) \to y$ because $\mathfrak{A}$ is a strong quasi-ordered residuated system, we conclude that $(x \to y) \to y \in F$ is valid. So, F is a normal filter. □

Example 3.3. Let $\mathfrak{A}$ be as in Example 2.13. We will show that $\mathfrak{A}$ does not have any proper normal filter. Let $u, v \in A$ be such that $u < v < x$ and let $F := \langle x, 1]$ be a filter in $\mathfrak{A}$ where $x < 1$. Then $(v \to u) \to u = 1 \in F$ and $(u \to v) \to v = u \to v = u \notin F$. So, F is not a normal filter in $\mathfrak{A}$. Thus, F is not a normal filter in A according to Theorem 3.2.

In what follows, we need the following lemma

Lemma 3.4 ([9], Lemma 3.1). *Let a subset F of a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition (F2). Then the following holds*

$$(\forall u \in A)(u \in F \iff 1 \to u \in F).$$

Proof. Since $(\forall x \in A)(1 \to x \preccurlyeq x)$ and $(\forall x \in A)(x \preccurlyeq 1 \to x)$, by Proposition 2.3 (d) in [3], the proof of this lemma follows from (F2). □

Theorem 3.5. *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$. F is a normal filter of $\mathfrak{A}$ if and only if the followings holds*

(12) $(\forall x, y, z \in A)((y \to x) \to x \in F \implies (x \to y) \to y \in F)$.

Proof. Let F be a normal filter of a quasi-ordered residuated system $\mathfrak{A}$ and let $x, y, z \in A$ be elements such that $(y \to x) \to x \in F$. Since $(y \to x) \to x \in F$ is equivalent with $1 \to ((y \to x) \to x) \in F$ by Lemma 3.4 and since $1 \in F$, thus $(x \to y) \to y \in F$ because F is a normal filter of $\mathfrak{A}$.

Suppose that the filter F of a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition (12). Let $z \to ((y \to x) \to x) \in F$ and $z \in F$ be holds for all $x, y, z \in A$. Since F is a filter of $\mathfrak{A}$, then $(y \to x) \to x \in F$ by (F-3). Thus we get that $(x \to y) \to y \in F$ by the hypothesis (12). Hence, F is a normal filter of $\mathfrak{A}$. □

In proving the following result that connects comparative and normal filters in a quasi-ordered residuated system we need the following lemma

Lemma 3.6 ([9], Proposition 3.1). *For any comparative filter F in a quasi-ordered residuated system $\mathfrak{A}$ holds*

(13) $(\forall v, z \in A)((v \to z) \to v \in F \implies v \in F)$.

Theorem 3.7. *Any comparative filter in a quasi-ordered residuated sustem $\mathfrak{A}$ is a normal filter in $\mathfrak{A}$.*

Proof. Let F be a filter in $\mathfrak{A}$ and let $x, y, z \in A$ be such $(x \to y) \to y \in F$. By the clain (11) of Proposition 2.8, we conclude

$$x \preccurlyeq (y \to x) \to x.$$

Then

$$((y \to x) \to x) \to y \preccurlyeq x \to y$$

by (10) and by repeated procedure, we have

$$(x \to y) \to y \preccurlyeq (((y \to x) \to x) \to y) \to y.$$

Thus $(((y \to x) \to x) \to y) \to y \in F$ by (F2).

On the other hand, since $y \to x \preccurlyeq y \to x$ is equivalent to $(y \to x) \cdot y \preccurlyeq x$ according to (3) and $y \cdot (y \to x) \preccurlyeq x$, respectively, because the multiplication in A is commutative operation, we have

$$y \preccurlyeq (y \to x) \to x.$$

If we treat this inequality with $((y \to x) \to x) \to y$ using procedure (10) on the right, we get

$$(((y \to x) \to x) \to y) \to y \preccurlyeq (((y \to x) \to x) \to y) \to ((y \to x) \to x).$$

Since $(((y \to x) \to x) \to y) \to y \in F$, we obtain

$$(((y \to x) \to x) \to y) \to ((y \to x) \to x) \in F$$

according to (F2). If we denote $v =: (y \to x) \to x$, $z =: y$, then we recognize the previous condition as the hypothesis in Lemma 3.6. Therefore $(y \to x) \to x \in F$, since F is a comparative filter in $\mathfrak{A}$. So, F is a normal filter in $\mathfrak{A}$. □

Example 3.8. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

·	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	a	a	1

and

→	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	b
c	1	b	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (a,b), (a,c), (b,b), (c,b), (c,c)\}.$$

Subset $\{1\}$ is a normal filter of $\mathfrak{A}$ but it is not a comparative filter in $\mathfrak{A}$ because it does not satisfy the condition (13) since, for example, for $v = b$ and $u = a$, we have $(b \to a) \to b = 1 \in F$ while $b \notin F$.

Example 3.9. Let $\mathfrak{A}$ be as in Example 3.8. $F = \{1\}$ is a normal filter in $\mathfrak{A}$ but it is not an implicative filter in $\mathfrak{A}$ since it does not satisfy the condition (14). For example, for $u = b$ and $v = c$ we have $u \to (u \to v) = b \to (b \to c) = b \to b = 1 \in F$ but $u \to v = b \to c = b \notin F$.

We first express the following theorem:

Theorem 3.10. *Every comparative filter in a quasi-ordered residuated system is an implicative and a normal filter in it.*

Proof. The proof is obtained by combining Theorem 3.4 in [9] and Theorem 3.7. □

To demonstrate the following statement we need the following lemma

Lemma 3.11 ([9], Proposition 3.1)**.** *Let F be an implicative filter of a quasi-ordered residuated system A. Then the following holds*

(14) $(\forall u, v \in A)(u \to (u \to v) \in F \implies u \to v \in F)$.

Proof. If we put $v = u$ in (IF), we immediately obtain the claim of this proposition, since for every $u \in A$ always $u \to u \in F$ holds for every non-empty set F satisfying condition (F2). Indeed, $u \to u \in F$ follows from $u \preccurlyeq u$; whence $1 \preccurlyeq (u \to u)$ by $1 \in F$ and (F2). □

Theorem 3.12. *If F is an implicative and normal filter in a quasi-ordered residuated system $\mathfrak{A}$, then F is a comparative filter in $\mathfrak{A}$.*

Proof. Let F be an implicative and normal filter of a quasi-ordered residuated system $\mathfrak{A}$. Let us prove that F is a comparative filter in $\mathfrak{A}$. For this purpose, according to Theorem 3.2 in [9], it suffices to prove that (13) holds. Assume $(x \to y) \to x \in F$. Since

$$(x \to y) \to x \preccurlyeq (x \to y) \to ((x \to y) \to y)$$

is valid according to (10), from this inequality and from $(x \to y) \to x \in F$ we get $(x \to y) \to ((x \to y) \to y) \in F$ by (F2). Hence it follows $(x \to y) \to y \in F$ according to Lemma 3.11 because F is an implicative filter in A. So, $(y \to x) \to x \in F$, since F is a normal filter in $\mathfrak{A}$.

On the other hand, by the claim (11) of Proposition 2.8 we have $y \preccurlyeq x \to y$. Thus $(x \to y) \to x \preccurlyeq y \to x$ by (10). From this and from the hypothesis $(x \to y) \to x \in F$, it follows that $y \to x \in F$ in accordance with (F2). Finally, $(y \to x) \to x \in F$ and $y \to x \in F$ implies that $x \in F$ by (F3). We have shown that condition (13) is a valid formula and, therefore, F is a comparative filter in $\mathfrak{A}$. □

The notion of MV-filters in residuated lattices was introduced in [1] as follows: A subset F of a residuated lattice L is called an MV-filter of L if it is a filter of L that satisfies in the condition

(MVF) $(\forall u, v \in L)(((u \to v) \to v) \to ((v \to u) \to u) \in F)$.

Also, in [1] it is shown (Theorem 3.10) that every MV-filter of L is a normal filter of L. It has been shown there that the reverse need not be true (Example 3.11). In our case, the relationships between conditions (MVF) and (NF) are similar.

Theorem 3.13. *Any filter in a quasi-ordered residuated system $\mathfrak{A}$ which satisfies the condition (MVF), is a normal filter in $\mathfrak{A}$.*

Proof. Let F be a filter in $\mathfrak{A}$ and let $x, y \in A$ be such $(x \to y) \to y \in F$. By the hypothesis (MVF) we have $((x \to y) \to y) \to ((y \to x) \to x) \in F$. Since F is a filter in $\mathfrak{A}$, we conclude that $(y \to x) \to x \in F$. So, F is a normal filter in $\mathfrak{A}$ by Theorem 3.5. □

Example 3.14. Consider the quasi-ordered residuated system $\mathfrak{A}$ as in Example 3.8. The filter $F = \{1\}$ in $\mathfrak{A}$ is a normal filter in $\mathfrak{A}$ while it does not satisfy the condition (MVF), since $((c \to a) \to a) \to ((a \to c) \to x) = a \notin F$ holds.

4. Conclusion and further work

The condition (NF), taken from the theory of residual lattices and BL-algebras, is placed here in the context of a specific principle-logical environment. Notwithstanding these specificities, it is shown that the substructure of the normal filtera in a quasi-ordered residuated system, determined by the requirement (NF), have similar properties as that substructure in the mentioned algebraic structures. It is quite reasonable to assume that the requirement (NF) by which it is determined substructure of the normal filter and its properties do not depend much on the environment in which they are observed. It seems that a deeper understanding of requirements (NF) in different principle-logical environments could offer some answer to the aforementioned dilemma.

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Received July 19, 2021
Revised November 13, 2021

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