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On the weight of finite groups

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Abstract. For a finite group G, let $W(G)$ denotes the set of the orders of the elements of G. In this paper we study $|W(G)|$ and show that the cyclic group of order n has the maximum value of $|W(G)|$ among all groups of the same order. Furthermore we study this notion in nilpotent and non-nilpotent groups and state some inequality for it. Among the result we show that the minimum value of $|W(G)|$ is power of 2 or it pertains to a non-nilpotent group.

1. Introduction

Let G be a finite group. The connection between structure and the set of the orders of the elements of G, has been studied in several works. In 1932, Levi and Waerden [4] showed that under some conditions the groups with weight 2 are nilpotent of class at most 3. Later in 1937, Neumann [6] proved that if $W(G) = \{1, 2, 3\}$, then G is an elementary abelian-by-prime order group. Sanov [9] showed that, when $W(G) \subseteq \{1, 2, 3, 4\}$ G is a locally finite group. Novikov and Adjan [7] in 1968 answered negatively to the following question. Does the finiteness of $W(G)$ imply G to be locally finite? In the same line of research Gupta et. al, [3] proved if $W(G) \subseteq \{1, 2, 3, 4, 5\}$ and $W(G) \neq \{1, 5\}$, then G is locally finite. In 2007, D. V. Lytkina [5] showed that for the group G, with $W(G) = \{1, 2, 3, 4\}$, either G is an extension of an elementary abelian 3-group by a cyclic or a quaternion group, or it is an extension of a nilpotent 2-group of class 2 by a subgroup of S_3. The sum of element orders in finite groups is studied by Amiri, Jafarian Amiri and Isaacs [1]. We denote by $|W(G)|$, the number of element orders of G. The group G is m-weight group, if $|W(G)| = m$. It is easy to see that if G is trivial, then $|W(G)| = 1$. If G be a non-trivial group then, the weight of G is at least 2. In the following lemma, we state a result about 2-weight group.

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Lemma 1.1. *Let G be a group, then G is a 2-weight group if and only if* $\exp(\mathrm{G}) = p$.

Proof. First assumethat, G is a 2-weight group. If $\exp(\mathrm{G}) = p$ has two distinct prime divisors p and q, then $\{1, p, q\} \subseteq W(G)$, so $exp(G)$ must be a p-number for some prime p. Now, if $exp(G) = p^n$, for some $n \geqslant 2$, then, $\{1, p, p^2\} \subseteq W(G)$. The converse is trivial. □

2. Preliminary results

This section contains some basic properties on the weight of a finite group. The following proposition shows the relation of the weight of a direct product of a finite number of finite groups with the weights of its factors.

Proposition 2.1. *Let H and K be two arbitrary finite groups, then*

$$|W(H \times K)| \leqslant |W(H)| \times |W(K)|,$$

and the equality holds if $(exp(H), exp(K)) = 1$.

Proof. Let $m \in W(H \times K)$ then, there exists $(h, k) \in H \times K$, such that $m = o(h, k) = [o(h), o(k)] = \dfrac{o(h)}{g_1} \times \dfrac{o(k)}{g_2} = rs$. Since $[o(h), o(k)]$ is the least common multiple of $o(h)$ and $o(k)$ and $g_1 g_2 = gcd(o(h), o(k))$, on the other hand $r = \dfrac{o(h)}{g_1}, s = \dfrac{o(k)}{g_2}$. So we have $r \in W(H)$ and $s \in W(K)$. Hence $|W(H \times K)| \leqslant |W(H)| \times |W(K)|$. Now, if $(exp(H), exp(K)) = 1$ and $(r, s) \in W(H) \times W(K)$, then there exsit $h \in H$ and $k \in K$ of orders r and s, respectively. Therefore, (h, k) is an element of $H \times K$ of order rs, so the result holds. □

Now, using induction in order to prove the following corollary.

Corollary 2.2. *Let ${G_i}_{i=1}^n$ be a family of finite groups. Then,* $|W(\prod_{i=1}^n G_i)| \leqslant \prod_{i=1}^n |W(G_i)|$. *Furthermore, the equality holds if the exponent of distinct direct factors are mutully coprime.*

It is easy to see that the cyclic group of order p^{m-1}, $C_{p^{m-1}}$ is an m-weight group, in which p is an arbitrary prime number, so for every natural number n, there exists a finite group (in fact a finite p-group) of weight m.

The following theorem gives an upper bound for the weight of a finite group in terms of its order.

Theorem 2.3. *Let G be a finite group of order n, then $|W(G)| \leqslant |W(C_n)|$ and the equality holds if and only if $G \cong C_n$.*

Proof. Since the order of each element of G is a divisor of n and $|W(C_n)| = d(n)$, in which $d(n)$ is the number of natural divisors of n, it is trivial, such that $|W(G)| \leqslant |W(C_n)|$. Now, if $|W(G)| = |W(C_n)|$, then $n \in W(G)$ and hence $G \cong C_n$. □

3. Nilpotent groups

In this section, we state some facts on $W(G)$, when G is a nilpotent group. The following proposition gives the upper and lower bound for $W(G)$, when G is a finite nilpotent group.

Proposition 3.1. *Let $\aleph$ be class of nilpotent groups of order n, then for each $G \in \aleph$ we have*

$$2^{|\pi(n)|} \leqslant |W(G)| \leqslant d(n),$$

and equality in the first inequality holds if and only if all Sylow subgroups of G has prime exponent.

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then $d(n) = (\alpha_1 + 1) \cdots (\alpha_k + 1)$. Let G be a nilpotent group of order n, so $G \cong \prod_{i=1}^{k} S_i$, in which S_i is the Sylow p_i-subgroup of G of order $p_i^{\alpha_i}$ $(1 \leqslant i \leqslant k)$. Now, by Proposition 2.1, we have $|W(G)| = \prod_{i=1}^{k} |W(S_i)|$. Applying, Theorem 2.3, thus $2 \leqslant |W(S_i)| \leqslant \alpha_i+1$, for all i, $1 \leqslant i \leqslant k$. So $2^{|\pi(n)|} \leqslant |W(G)| \leqslant \prod_{i=1}^{k}(\alpha_i + 1) = d(n)$. Hence, $|W(G)| = 2^{|\pi(n)|}$ if and only if $\alpha_i = 1$, for all i, $1 \leqslant i \leqslant k$ which is equal to $\exp(S_i) = p_i$, for all i, $1 \leqslant i \leqslant k$. □

As an immediate result we have.

Corollary 3.2. *Let G be a finite group of order n, if $|W(G)| < 2^{|\pi(n)|}$ then G is non-nilpotent.*

Theorem 3.3. *Let G be a group of prime weight then G is nilpotent if and only if G is a p-group.*

Proof. Since G is a nilpotent group we have $G = P_1 \times \cdots \times P_k$ so $W(G) = W(P_1) \cdots W(P_k)$ this implies $k = 1$ hence G is a p-group □

Immediate consequence of Theorem 3.3, we get the following corollary.

Corollary 3.4. *In the class of all finite groups of prime weight, each group is either a p-group or non-nilpotent.*

Proposition 3.5. (See [8, Theorem 1]) *Suppose that* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$*, in which* p_i*'s are distinct prime numbers. Then, every finite group of order* n *is a nilpotent group if and only if* $p_i \nmid p_j^{\beta_j} - 1$*, for each* j*,* $0 < \beta_j \leqslant \alpha_j$ *and* $i \neq j$*.*

In above proposition such these numbers are called nilpotent numbers. Now in order to prove our main result, we need the following results.

Lemma 3.6. *Every finite nilpotent group of order* n *is cyclic if and only if* n *is square free.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be decomposition of n into prime factors and G be a nilpotent group of order n. By Proposition 3.1, we have $2^k \leqslant |W(G)| \leqslant |W(C_n)|$, since every nilpotent group of order n is cyclic, so both inequalities are in fact equality and hence $\alpha_i = 1$, for all i, $1 \leqslant i \leqslant k$. Conversely, let G be a nilpotent group of order $n = p_1 \cdots p_k$. Applying, Proposition 3.1 again, so we have $|W(G)| = 2^k = d(n) = |W(C_n)|$, it implies that $G \cong C_n$. □

Using, the above lemma we can prove the following theorem.

Theorem 3.7. *Every finite group of order* n *is cyclic if and only if* $n = p_1 \cdots p_k$*, in which* $p_1 < \ldots < p_k$ *and* $p_i \nmid p_{i+s} - 1$*, where* $1 \leqslant i \leqslant k-1$ *and* $1 \leqslant s \leqslant k-i$*.*

Proof. If every finite group of order n is cyclic, then by Lemma 3.6 and Proposition 3.5, the result holds. If $n = p_1 \cdots p_k$, in which $p_1 < \ldots < p_k$ and $p_i \nmid p_{i+s} - 1$, where $1 \leqslant i \leqslant k-1$ and $1 \leqslant s \leqslant k-i$, then every group of order n is nilpotent, so we have $|W(G)| = 2^k = d(n) = |W(C_n)|$ and hence $G \cong C_n$. □

4. Non-nilpotent groups

This section is devoted to some results on non-nilpotent groups.

Let $\mathcal{K}_{(n)}$ denote the class of all groups of order n.

Definition 4.1. *We say that* $\mathcal{K}_{(n)}$ *has non-nilpotency property if there exists a non-nilpotent group* T *in* $\mathcal{K}_{(n)}$*, such that* $\min\{|W(G)| \mid G \in \mathcal{K}_{(n)}\} = |W(T)|$.

Theorem 4.2. *If $\mathcal{K}_{(n)}$ has non-nilpotency property, then $\mathcal{K}_{(nm)}$, has also non-nilpotency property, for any natural number m, such that $(n, m) = 1$.*

Proof. Let H be a nilpotent group of order nl, since $(n, l) = 1$ and H is nilpotent, there exist normal subgroups N and L of H, such that $|L| = l$, $|N| = n$ and $H = N \times L$. Now, as $N \in \mathcal{K}_{(n)}$ and $\mathcal{K}_{(n)}$ has non-nilpotency property, so there is a non-nilpotent group T in $\mathcal{K}_{(n)}$ such that

$$|W(T)| = \min\{|W(G)| \mid G \in \mathcal{K}_{(n)}\}$$

so

$$|W(T)| \leqslant |W(N)|.$$

If $E = T \times L$, then E is also a non-nilpotent group, and clearly $|T| = |N| = n$ and $|L| = l$. Now, we have

$$|W(E)| = |W(T \times L)| = |W(T)||W(L)| \leqslant W(N)||W(L)| = |W(N \times L)| = |W(H)|.$$

So, as E is a non-nilpotent group, and H is nilpotent group in$\mathcal{K}_{(nl)}$ and $|W(E)| \leqslant W(H)|$, then $\mathcal{K}_{(nl)}$ has non-nilpotency property. $\square$

Example 4.3. It is easy to see that $\mathcal{K}_{(6)}$ has the non-nilpotency property, so $\mathcal{K}_{(30)}$ has the non-nilpotency property, we know that

$$\mathcal{K}_{(30)} = \{C_{30}, C_3 \times D_{10}, C_5 \times D_6, D_{30}\}$$

and

$$\omega(C_{30}) = 8, \omega(C_3 \times D_{10}) = 6, \omega(C_6 \times D_6) = 6 \text{ and } \omega(D_{30}) = 5.$$

Therefore, the minimum weight occurs at the non-nilpotent group D_{30} .

In the following lemma, we construct non-nilpotent groups with small enough weights.

Lemma 4.4. *Let p and q be two distinct prime numbers and $\alpha \in Aut(C_q^r)$ be of order p. If $\{a_1, \ldots, a_m\}$ be the standard generating set for C_p^m, then the semidirect product C_p^m and C_q^r, by the homomorphism $\mu : C_p^m \to Aut(C_q^r)$, such that $\mu(a_i) = \alpha$, for each i, $i = 1, \ldots, m$, is a non-nilpotent group with weight at most 4.*

Proof. Let $b \neq 0$ and $(0, b) \in C_p^m \ltimes C_q^r$. Clearly $(0, b)^q = (0, b^q) = (0, 0)$ and hence $o(0, b) = q$. So, if $a \neq 0$ and $(a, 0) \in C_p^m \ltimes C_q^r$, we have $(a, 0)^p = (a^p, 0) = (0, 0)$, it implies that $o(a, 0) = p$

Now, assume that $a \neq 0$ and $b \neq 0$, as $(a,b)^{pq} = (0,0)$ and $o(a,b) \leqslant pq$, it follows that

$$W(C_p^m \ltimes C_q^r) \subseteq \{1, p, q, pq\},$$

therefore $C_p^m \ltimes C_q^r$ is a non-nilpotent group with maximum weight 4.

□

We use the following useful result in the next theorem.

Proposition 4.5. (See [2]) *For a finite p-group G,* $\mathrm{Aut}(G) \cong Gl(n,p)$ *if and only if G is an elementary abelian p-group of order p^n.*

Theorem 4.6. *The class of $\mathcal{K}_{(n)}$ has non-nilpotency property, for any non-nilpotent natural number n.*

Proof. As n is not a nilpotent number according to Proposition 3.5, there exist distinct and prime divisors p and q of n such that

$$p \mid q^i - 1$$

Now, we consider $n = p^m q^r k$ that $(pq, k) = 1$. By Proposition 4.5, we have

$$\left|Aut(C_q^r)\right| = (q^r - 1)(q^r - q)(\dots(q^r - q^{r-1})$$

As

$$p \mid q^i - 1,$$

thus

$$p \mid (q^i - 1)q^{r-i} = q^r - q^{r-i}.$$

Therefore, $p \mid \left|Aut(C_q^r)\right|$ and hence there exists $\alpha \in Aut(C_q^r)$ with $o(\alpha) = p$. Now, if $\{a_1, \dots, a_m\}$ is standard generator set of C_p^m, we consider homomorphism μ, such that

$$\mu : C_p^m \to Aut(C_q^r)$$

given by $\mu(a_i) = \alpha$ for $i = 1, \dots, m$. We get semidirect product C_p^m and C_q^r, by homomorphism μ. Then, $C_p^m \ltimes C_q^r$ is a non-nilpotent group of order $p^m q^r$. On the other hand by Lemma 4.4, we have

$$\left|W(C_p^m \ltimes C_q^r)\right| \leqslant 4$$

So, if G is a nilpotent group of order $p^m q^r$, then we have

$$|W(G)| \geqslant 2^2 = 4$$

Thus, we conclude that $\mathcal{K}_{(p^m q^r)}$ has nonnilpotency property. Since $(pq, k) = 1$ and $p^m q^r k = n$, by Theorem 4.2, $\mathcal{K}_{(n)}$ has non-nilpotency property. □

Theorem 4.7. *Let n be an even number, such that n is not a power of 2, then $\mathcal{K}_{(n)}$ has the non-nilpotency property.*

Proof. Suppose that $n = 2^{\alpha_1} p^{\alpha_2} q_3^{\alpha_3} \cdots q_r^{\alpha_r}$, for some $r \geq 2$. Since 2 is a divisor of $\left|Aut\left(Z_p^{\alpha_2}\right)\right|$, we have $\omega\left(\mathbb{Z}_2^{\alpha_1} \ltimes \mathbb{Z}_p^{\alpha_2}\right) \subseteq \{1, 2, p, 2p\}$. Now, let G be a nilpotent group of order n, thus $\omega(G) \geqslant 2^r$, also we have

$$\omega\left(\left(\mathbb{Z}_2^{\alpha_1} \ltimes \mathbb{Z}_p^{\alpha_2}\right) \times \mathbb{Z}_{q_3}^{\alpha_3} \times \ldots \times \mathbb{Z}_{q_r}^{\alpha_r}\right) \leqslant 4(2^{r-2}) = 2^r$$

Therefore

$$\omega\left(\left(\mathbb{Z}_2^{\alpha_1} \ltimes \mathbb{Z}_p^{\alpha_2}\right) \times \mathbb{Z}_{q_3}^{\alpha_3} \times \ldots \times \mathbb{Z}_{q_r}^{\alpha_r}\right) \leqslant \omega(G)$$

and the results hold. □

Example 4.8. $\mathcal{K}_{(12)}$, $\mathcal{K}_{(22)}$ and $\mathcal{K}_{(30)}$ has the non-nilpotency property. We know that $\mathcal{K}_{(12)} = \{A_4, D_{12}, T, C_{12}, C_3 \times C_2 \times C_2\}$ in which

$$T =< a, b \mid a^4 = b^3 = 1; a^{-1}ba = b^{-1} >.$$

We have $\omega(T) = \omega(D_{12}) = \omega(C_2 \times C_2 \times C_3) = 4$ also $\omega(A_4) = 3$ and $\omega(C_{12}) = 6$.

$\mathcal{K}_{(22)} = \{C_{22}, D_{22}\}$, $\omega(C_{22}) = 4$ and $\omega(D_{22}) = 3$.

$\mathcal{K}_{(30)} = \{C_{30}, C_3 \times D_{10}, C_5 \times D_6, D_{30}\}$(see Theorem 4.2).

Here, we can prove the main theorem.

Theorem 4.9. *Let G be a finite group of order n, then $|W(G)| \leqslant |W(C_n)|$. If $\min\{|W(G)| \mid |G| = n\} = m$, then $m = 2^{|\pi(n)|}$ or there is a nonnilpotent group T that $|T| = n$ and $|W(T)| = m$. In other words, the class of groups of order n, cyclic group C_n has the most weight and if the least weight on the above groups equals m, then m is a power of 2, such that the power equals to numbers of distinct prime factors of n. Therefore m is the weight of a non-nilpotent group.*

Proof. Let C_n be a cyclic group of order n. If m is a divisor of n, then $m \in W(G)$ and it follows that

$$\{m \in \mathbb{Z} \mid m > 0, m \mid n\} \subseteq W(C_n).$$

Now, if G is a group of order n and $m \in W(G)$, then $m \mid n$ and hence

$$W(G) \subseteq \{m \in \mathbb{Z} \mid m > 0, m \mid n\}.$$

Thus, $W(G) \subseteq W(C_n)$, and so we have

$$|W(G)| \leqslant |W(C_n)|.$$

For the finite group G if n is a nilpotent number, then

$$|W(G)| \geqslant 2^{|\pi(n)|},$$

If n is not a nilpotent number, then $\mathcal{K}_{(n)}$ has nonnilpotency property. So, there exists a nonnilpotent group T in $\mathcal{K}_{(n)}$, such that for every group G in $\mathcal{K}_{(n)}$, we have

$$|W(T)| \leqslant |W(G)| .$$

Hence

$$|W(T)| = \min \left\{ |W(G)| \mid G \in \mathcal{K}_{(n)} \right\} ,$$

Therefore, the proof is completed □

References

[1] **H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs**, *Sums of element orders in finite groups*, Commun. Algebra, **37** (2009), no. 9, 2978 – 2980.

[2] **S. Fouladi, R. Orfi**, *General linear groups as automorphism groups*, Intern. J. Algebra, **5** (2011), no. 27, 1327 – 1335.

[3] **N.D. Gupta and V.B. Mazurov**, *On groups with small orders of elements*, Bull. Australian Math. Soc., **6** (1999), 197 – 205.

[4] **F. Levi, B.L. Waerden**, *Uber eine besondere Klasse von Gruppen*, Abhandlungen Math. Seminar Univ. Hamburg, **9** (1932), 154 – 158.

[5] **D.V. Lytkina**, *Structure of a group with elements of order at most*, Siberian Math. J., **48** (2007), 283 – 287.

[6] **B.H. Neumann**, *Groups whose elements have bounded orders*, J. London Math. Soc., **12** (1937), 195 – 198.

[7] **P.S. Novikov, S.I. Adjan**, *Infinite periodic groups I*, Mathematics of the USSR-Izvestiya., **2** (1968), no. 1, 209 – 236.

[8] **J. Pakianathan, K. Shankar**, *Nilpotent numbers*, Amer. Math. Monthly, **107** (2000), no. 7, 631 – 634.

[9] **I.N. Sanov**, *Solution of Burnside's problem for exponent* 4, Leningrad Math. J., **10** (1940), 166 – 170.

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