Quasigroups and Related Systems **31** (2023), 275 – 286

https://doi.org/10.56415/qrs.v30.23

Topological S-act congruence

Sunil Kumar Maity and *Monika Paul*

Abstract. In this paper, we establish the necessary and sufficient condition for an equivalence relation ρ on an S-act A endowed with a topology such that A/ρ becomes a Hausdorff topological S-act. Also, we show that if A_1 and A_2 be two topological S-acts, then for any homomorphism $\varphi : A_1 \to A_2$, $A_1/\ker\varphi$ is a topological S-act if and only if φ is φ-saturated continuous. Moreover, we establish for any two congruences θ_1 and θ_2 on an S-act A endowed with a topology, $\theta_1 \cap \theta_2$ is a topological S-act congruence on A if and only if the mapping $\varphi : A \to A/\theta_1 \times A/\theta_2$, defined by $\varphi(a) = (a\theta_1, a\theta_2)$, for all $a \in A$, is φ-saturated continuous, where S is a topological semigroup.

1. Introduction and preliminaries

Analogous to topological group actions, topological semigroup actions plays an important role in the study of semigroup action theory. There are wide application of topological semigroup action in many fields like manifold, topological vector space etc. Properties of topological semigroup actions have been recently studied by many authors, for example, P. Normak, B. Khosravi and others (see [8], [4]). In [4], the author established a necessary and sufficient conditions for a congruence on a topological S-act to be topological S-act congruence.

Recall that a semigroup $(S, \cdot)$ is a nonempty set together with a binary operation on S satisfying the associative law, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in S$. Let S be a semigroup and A be a nonempty set. Then A is said to be a left S-act if there is an action $\lambda : S \times A \to A$ defined by $\lambda(s, a) = sa$ such that $(st)a = s(ta)$, for all $s, t \in S$ and $a \in A$. Throughout this paper, by an S-act, we always mean a left S-act. An equivalence relation θ on an S-act A is said to be a congruence on A if, for all $a, b \in A$ and $s \in S$, $(a, b) \in \theta$

2010 Mathematics Subject Classification: 22A20, 20M30

Keywords: Topological semigroup, S-act topological congruence, topological S-act congruence

implies $(sa, sb) \in A$. For any two S-acts A and B, a mapping $f : A \to B$ is said to be a homomorphism if $f(sa) = sf(a)$, for all $s \in S, a \in A$.

A semigroup S endowed with a topology τ is said to be a topological semigroup if the binary operation $\mu : \underset{(x,y)\mapsto xy}{S\times S\to S}$ is continuous. Let S be a semigroup endowed with a topology τ and A be a nonempty set endowed with a topology τ_A. Then A is said to be an S-act topological space if the action $\lambda : S \times A \to A$ is continuous. Now an S-act topological space A is said to be a topological S-act if S is a topological semigroup.

Let S be a semigroup endowed with a topology τ and A be an S-act endowed with a topology τ_A. Also, let θ be an equivalence relation on A. Consider the natural mapping $\pi : A \to A/\theta$ defined by $\pi(a) = a\theta$, for all $a \in A$. Define a topology on A/θ as follows : a subset U of A/θ is open in A/θ if and only if $\pi^{-1}(U)$ is open in A. With this topology, π is a quotient map and A/θ is called a quotient space. Two S-acts A and B endowed with topologies τ_A and τ_B respectively are said to be topologically isomorphic if there exists a homomorphism $\varphi : A \to B$ which is also a homeomorphism. Moreover, for a semigroup S endowed with a topology, a congruence θ on an S-act A endowed with a topology τ is said to be a S-act topological congruence if A/θ is an S-act topological space. In addition, for a topological semigroup S, a congruence θ on an S-act A endowed with a topology τ is said to be a topological S-act congruence if A/θ is a topological S-act.

2. Congruence on a topological S-act

Let S be a semigroup endowed with a topology τ, A be an S-act endowed with a topology τ_A and θ be an equivalence relation on A. Consider the natural map $\pi : A \to A/\theta$ defined by $\pi(a) = a\theta$ and the set $\tau_\theta = \{B \in \tau_A : \pi^{-1}\pi(B) = B\}$. It can easily be shown that τ_θ is a topology on A. We first state a very useful result from [4].

Theorem 2.1. *Let A be a topological S-act and θ be a congruence on A. Then A/θ is a topological S-act if and only if (A, τ_θ) is a topological S-act.*

From Theorem 2.1, we have the following two corollaries which will be very useful in our discussion.

Corollary 2.2. *Let S be a semigroup endowed with a topology τ and A be an S-act endowed with a topology τ_A. Then for any congruence θ on A,*

A/θ is an S-act topological space if and only if (A, τ_θ) is an S-act topological space.

Corollary 2.3. *Let S be a topological semigroup and A be an S-act endowed with a topology τ_A. Then for any congruence θ on A, A/θ is a topological S-act if and only if (A, τ_θ) is a topological S-act.*

We now present a necessary and sufficient condition for an equivalence relation ρ on an S-act A endowed with a topology to be an S-act topological congruence.

Theorem 2.4. *Let S be a semigroup endowed with a topology τ and A be an S-act endowed with a topology τ_A. Then for any equivalence relation θ on A such that A/θ is a Hausdorff space, A/θ is an S-act topological space if and only if (A, τ_θ) is an S-act topological space.*

Proof. Let A/θ be an S-act topological space. Then θ is a congruence on A. Hence by Corollary 2.2, (A, τ_θ) is an S-act topological space. Conversely, let (A, τ_θ) be an S-act topological space. Let $(a, b) \in \theta$ and $s \in S$. If possible, let $(sa, sb) \notin \theta$. Then $[sa] \neq [sb]$, where by $[x]$ we mean the θ-equivalence class containing the element $x \in X$. Since S/θ is Hausdorff, there exist disjoint open sets G and H containing $[sa]$ and $[sb]$ respectively in A/θ. Then $sa \in \pi^{-1}(G)$ and $sb \in \pi^{-1}(H)$. Since π is continuous, it follows that $\pi^{-1}(G)$ and $\pi^{-1}(H)$ are open in (A, τ_θ). Now (A, τ_θ) being S-act topological space, there exist open sets U_1 and V_1 in (S, τ) and (A, τ_θ) respectively such that $s \in U_1$, $a \in V_1$ and $U_1V_1 \subseteq \pi^{-1}(G)$. Similarly, there exist open sets U_2 and V_2 in (S, τ) and (A, τ_θ) respectively such that $s \in U_2$, $b \in V_2$ and $U_2V_2 \subseteq \pi^{-1}(H)$. Let $U = U_1 \cap U_2$. Then $s \in U$. As $(a, b) \in \theta$, we have $[a] = [b] \in \pi(V_1) \cap \pi(V_2)$ and thus $a, b \in V_1 \cap V_2 = V$ (say). This implies $sa, sb \in UV \subseteq \pi^{-1}(G) \cap \pi^{-1}(H)$ which implies that $G \cap H \neq \emptyset$, a contradiction. Therefore, $[sa] = [sb]$ and hence θ is a congruence on S. Consequently, by Corollary 2.2, it follows that A/θ is an S-act topological space. □

Using Theorem 2.4, we at once have the following corollary.

Corollary 2.5. *Let (S, τ) be a topological semigroup and A be an S-act endowed with a topology τ_A. Then for any equivalence relation θ such that A/θ is Hausdorff, A/θ is a topological S-act if and only if (A, τ_θ) is a topological S-act.*

It is well known that image of an S-act is an S-act. But continuous image of a topological S-act need not be a topological S-act. This follows from the following examples.

Example 2.6. Consider the topological semigroup $S = (\mathbb{Z}_6, \tau_{dis})$, where τ_{dis} is the discrete topology on $\mathbb{Z}_6$. Also, let $A = S$ and $B = (\mathbb{Z}_6, \tau_1)$, where $\tau_1 = \{\mathbb{Z}_6, \emptyset, \{\bar{4}\}\}$. Then A is a topological S-act. Consider the map $\varphi : A \to B$ defined by $\varphi(\bar{x}) = \overline{2x}$. Then $\varphi(A) = \{\bar{0}, \bar{2}, \bar{4}\}$ and the subspace topology on $\varphi(A)$ is given by $\tau_{\varphi(A)} = \{\varphi(A), \emptyset, \{\bar{4}\}\}$. It is easy to verify that φ is continuous and $\varphi(A)$ is an S-act. Now, for the open set $\{\bar{4}\}$ in $\varphi(A)$ with $s \cdot a \in \{\bar{4}\}$, where $s = \bar{2} \in S$ and $a = \bar{2} \in \varphi(A)$, there is no open set U containing a in $\varphi(A)$ such that $\{s\} \cdot U \subseteq \{\bar{4}\}$, where $\{s\}$ is open in S. Hence $\varphi(A)$ is not a topological S-act.

We now establish some sufficient conditions for which continuous image of a topological S-act will be a topological S-act. For this purpose, we first define φ-saturated continuity of between two S-acts A and B.

Definition 2.7. Let S be a semigroup endowed with a topology τ. Let A_1 and A_2 be S-acts endowed with topologies τ_1 and τ_2 respectively. Then a mapping $\varphi : A_1 \to A_2$ is said to be *φ-saturated continuous* if for any subset W of A_2 with $\varphi^{-1}(W)$ is open in A_1 and $s \in S$, $a \in A_1$ with $sa \in \varphi^{-1}(W)$, there exist open sets U and V in (S, τ) and $(A_1, \tau_{\ker\varphi})$ containing s and a respectively such that $UV \subseteq \varphi^{-1}(W)$, where $\ker\varphi = \{(a, b) \in A_1 \times A_1 : \varphi(a) = \varphi(b)\}$.

Now we characterize φ-saturated continuous map between two topological S-acts.

Proposition 2.8. *Let S be a topological semigroup. Any injective mapping $\varphi : A \to B$ between two topological S-acts A and B is always φ-saturated continuous.*

Proof. Let $\varphi : A \to B$ be an injective mapping between two topological S-acts A and B. Let W be a subset of B with $\varphi^{-1}(W)$ is open in A and $s \in S$, $a \in A$ with $sa \in \varphi^{-1}(W)$. Now A being a topological S-act, there exist open sets U and V in S and A containing s and a respectively such that $UV \subseteq \varphi^{-1}(W)$. Now φ being injective, $\varphi^{-1}(\varphi(V)) = V$ and this implies that $V \in \tau_{\ker\varphi}$. Hence the result. $\square$

By a counter-example below, we conclude that the converse of the Proposition 2.8 may not be true.

Example 2.9. Let $S = A = B = (\mathbb{Z}_6, \cdot_6, \tau)$, where $\tau = \{\mathbb{Z}_6, \emptyset, \{\bar{0}, \bar{3}\}, \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}\}$. Then S is a topological semigroup where A and B can be thought of as a topological S-acts. Consider the mapping $\varphi : A \to B$ defined by $\varphi(a) = 2a$, for all $a \in A$. Clearly, φ is not injective. But it can be easily verify that φ is φ-saturated continuous.

By the following example, we prove that any φ-saturated continuous map φ between two topological S-acts may not be continuous.

Example 2.10. Let $S = B = (\mathbb{Z}_6, \cdot_6, \tau_{discrete})$ and $A = (\mathbb{Z}_6, \cdot_6, \tau_{indiscrete})$. Then S is a topological semigroup where A and B are topological S-acts. Consider the identity map $id_A : A \to B$. Now id_A being injective, id_A is a id_A-saturated continuous map. But it can be easily check that id_A is not a continuous map.

Now we discuss the topological influence of a φ-saturated continuous image of a S-act endowed with a topology.

Theorem 2.11. *Let S be a topological semigroup. Let A and B be two S-acts endowed with topologies τ_A and τ_B respectively such that (A, τ_A) is a topological S-act. Also, assume that $\varphi : A \to B$ be a homomorphism which is a quotient map. Then B is a topological S-act if any one of the following two conditions holds:*

(i) φ is an open map.

(ii) φ is φ-saturated continuous.

Proof. (i). Let φ be open. Let W be open in B with $s\varphi(a) \in W$, where $s \in S$ and $a \in A$. Then $sa \in \varphi^{-1}(W)$. Because of the continuity of φ, $\varphi^{-1}(W)$ is open in A. Now, A being a topological S-act, there exit open sets U in S and V in A containing s and a respectively such that $UV \subseteq \varphi^{-1}(W)$. Since φ is open, we must have $\varphi(V)$ is open in B with $\varphi(a) \in \varphi(V)$. Moreover, $U\varphi(V) \subseteq W$ and thus B is a topological S-act.

(ii). Let φ be φ-saturated continuous. Let W_1 be open in B with $t\varphi(a) \in W_1$, where $t \in S$ and $a \in A$. Then $ta \in \varphi^{-1}(W_1)$. Because of the continuity of φ, $\varphi^{-1}(W_1)$ is open in A. Now, φ being φ-saturated continuous, there exist open sets U_1 and V_1 in S and $(A, \tau_{\ker\varphi})$ containing t and a respectively such that $U_1V_1 \subseteq \varphi^{-1}(W_1)$. One can easily verify that $\varphi^{-1}(\varphi(V_1)) = V_1$. Now φ being a quotient map, $\varphi(V_1)$ is open in B with $\varphi(a) \in \varphi(V_1)$. Also, $U_1\varphi(V_1) \subseteq W_1$. Hence B is a topological S-act. $\square$

We know that the kernel of a homomorphism between two S-acts is always a congruence. But the kernel of a homomorphism between two topological S-acts need not be a topological S-act congruence. Using Theorem 2.11, we have the following corollary which ensures that the kernel of a homomorphism between two topological S-acts is a topological S-act congruence.

Corollary 2.12. *Let S be a topological semigroup. Let A and B be two S-acts endowed with topologies τ_1 and τ_2 respectively such that (A, τ_1) is a topological S-act. Also, assume that $\varphi : A \to B$ is a homomorphism which is also a quotient map. If φ is φ-saturated continuous, then B is topologically isomorphic to $A/\ker\varphi$ and hence $A/\ker\varphi$ is a topological S-act.*

Remark 2.13. For any two S-acts A and B, any homomorphism $\varphi : A \to B$ induces an S-act $A/\ker\varphi$. But for any two topological S-acts A and B, any homomorphism $\varphi : A \to B$ need not induce topological S-act $A/\ker\varphi$, i.e., $A/\ker\varphi$ need not be a topological S-act. Now, we establish a necessary and sufficient condition on the homomorphism $\varphi : A \to B$ so that $A/\ker\varphi$ will be a topological S-act.

Theorem 2.14. *Let S be a semigroup endowed with a topology τ. Also, let A_1 and A_2 be S-acts endowed with topologies τ_1 and τ_2 respectively. Then for any mapping $\varphi : A_1 \to A_2$ such that $\ker\varphi$ is a congruence on A_1, $A_1/\ker\varphi$ is an S-act topological space if and only if φ is φ-saturated continuous.*

Proof. Let us define a map $f : A_1/\ker\varphi \to A_2$ by $f(a\ker\varphi) = \varphi(a)$, for all $a\ker\varphi \in A_1/\ker\varphi$. Then $f \circ \pi = \varphi$, where $\pi : A_1 \to A_1/\ker\varphi$ is defined by $\pi(a) = a\ker\varphi$, for all $a \in A_1$.

First suppose that $A_1/\ker\varphi$ is an S-act topological space. Let W be a subset of A_2 such that $\varphi^{-1}(W)$ is open in A_1. Let $s \in S$, $a \in A_1$ with $sa \in \varphi^{-1}(W)$. Then $sa \in \pi^{-1}(f^{-1}(W))$. Now, $A_1/\ker\varphi$ being an S-act topological space, by Corollary 2.2, $(A_1, \tau_{\ker\varphi})$ is an S-act topological space. So, there exist open sets U and V containing s and a in (S, τ) and $(A_1, \tau_{\ker\varphi})$ respectively such that $UV \subseteq \pi^{-1}(f^{-1}(W))$. This implies that $UV \subseteq \varphi^{-1}(W)$ and hence φ is φ-saturated continuous.

Conversely, let φ be φ-saturated continuous. Let $G \in \tau_{\ker\varphi}$ and $t \in S$, $a \in A_1$ with $ta \in G$. Then $ta \in G = \pi^{-1}(\pi(G))$. Since f is injective, $G = \pi^{-1}(\pi(G)) = \varphi^{-1}(f(f^{-1}(\varphi(G))))$. Now, φ being φ-saturated continuous, there exist open sets U_1 and V_1 containing t and a in S and $(A_1, \tau_{\ker\varphi})$

respectively such that $U_1V_1 \subseteq \varphi^{-1}(f(f^{-1}(\varphi(G)))) = G$. So, $(A_1, \tau_{\ker\varphi})$ is an S-act topological space and hence by Corollary 2.2, $A_1/\ker\varphi$ is an S-act topological space. □

Corollary 2.15. *Let S be a topological semigroup. Also, let A_1 and A_2 be two S-acts endowed with topologies τ_1 and τ_2 respectively. Then for any mapping $\varphi : A_1 \to A_2$ such that $\ker\varphi$ is a congruence on A_1, $A_1/\ker\varphi$ is a topological S-act if and only if φ is φ-saturated continuous.*

Corollary 2.16. *Let S be a semigroup endowed with a topology τ. Also, let A_1 and A_2 be two S-acts endowed with topologies τ_1 and τ_2 respectively. Then for any homomorphism $\varphi : A_1 \to A_2$, $A_1/\ker\varphi$ is an S-act topological space if and only if φ is φ-saturated continuous.*

Corollary 2.17. *Let S be a topological semigroup. Also, let A_1 and A_2 be two topological S-acts. Then for any homomorphism $\varphi : A_1 \to A_2$, $A_1/\ker\varphi$ is a topological S-act if and only if φ is φ-saturated continuous.*

Using Corollary 2.15, we will prove that for any mapping $\varphi : A_1 \to A_2$ between two topological S-acts, the continuity of φ does not imply the φ-saturated continuity of φ. By the following example, we will prove this fact.

Example 2.18. [6, Example 2.7] We consider the topological semigroup $S = \{(a, b) \in \mathbb{Q} \times \mathbb{R} : b \geqslant 0\}$ with respect to the binary operation $((x, y), (a, b)) \mapsto (x + a, min(y, b))$. Let $I = \{(a, b) \in S : b = 0\}$. Then from [6], it follows that S/ρ_I is not a topological semigroup, where ρ_I is the Rees congruence induced by the ideal I on the semigroup S. Let us define a mapping $\varphi : S \to S$ by for all $(a, b) \in S$,

$$\varphi((a, b)) = \begin{cases} (a, b) & b > 0 \\ (0, 0) & b = 0. \end{cases}$$

Then it can be easily shown that φ is a continuous mapping with $\ker\varphi = \rho_I$. We claim that φ is not φ-saturated continuous. Because if so, then by Corollary 2.15, it follows that $S/\ker\varphi = S/\rho_I$ is a topological semigroup which is not true. Hence φ is not φ-saturated continuous.

Theorem 2.19. *Let S be a semigroup endowed with a topology τ. Also, let A_1 and A_2 be two S-acts endowed with topologies τ_1 and τ_2 respectively. Then for any mapping $\varphi : A_1 \to A_2$, if $A_1/\ker\varphi$ is Hausdorff, then $A_1/\ker\varphi$ is an S-act topological space if and only if φ is φ-saturated continuous.*

Proof. Let $A_1/\ker\varphi$ is an S-act topological space. Then $\ker\varphi$ is a congruence on A and hence by Theorem 2.14, φ is φ-saturated continuous.

Conversely, let φ be φ-saturated continuous. First we show that $\ker\varphi$ is a congruence on A_1. For this, let $(a,b)\in\ker\varphi$ and $s\in S$. If possible, let $(sa,sb)\notin\ker\varphi$. Then $[sa]\neq[sb]$. Since $A_1/\ker\varphi$ is Hausdorff, there exist disjoint open sets U and V in $A_1/\ker\varphi$ containing $[sa]$ and $[sb]$ respectively. Then $sa\in\pi^{-1}(U)$ and $sb\in\pi^{-1}(V)$. Since π is continuous, we have $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open in (A_1,τ_1). Then $sa\in\pi^{-1}(U)=\varphi^{-1}(f(U))$ and $sb\in\pi^{-1}(V)=\varphi^{-1}(f(V))$, where the mapping $f: A_1/\ker\varphi\to A_2$, defined by $f(a\ker\varphi)=\varphi(a)$, is a continuous injective homomorphism. Now, φ being φ-saturated continuous, there exist open sets U_1, U_2 in (S,τ) and V_1, V_2 in $(A_1,\tau_{\ker\varphi})$ such that $U_1V_1\subseteq\varphi^{-1}(f(U))=\pi^{-1}(U)$, $U_2V_2\subseteq\varphi^{-1}(f(V))=\pi^{-1}(V)$, where $s\in U_1\cap U_2$ and $a\in V_1, b\in V_2$. Set $G=U_1\cap U_2$. Then $s\in G$. Moreover, $(a,b)\in\ker\varphi$ implies $[a]=[b]\in\pi(V_1)\cap\pi(V_2)$ and hence $a,b\in V_1\cap V_2=H$ (say). Therefore, $sa,sb\in GH\subseteq\pi^{-1}(U)\cap\pi^{-1}(V)$ and thus $U\cap V\neq\emptyset$, a contradiction. Therefore, $[sa]=[sb]$ and thus $\ker\varphi$ is a congruence on A_1. Consequently, by Theorem 2.14, it follows that $A_1/\ker\varphi$ is an S-act topological space. □

Corollary 2.20. *Let S be a topological semigroup. Also, let A_1 and A_2 be two S-acts endowed with topologies τ_1 and τ_2 respectively. Then for any mapping $\varphi: A_1\to A_2$, if $A_1/\ker\varphi$ is Hausdorff, then $A_1/\ker\varphi$ is a topological S-act if and only if φ is φ-saturated continuous.*

3. Intersection and join of S-act congruences

It is well known that the intersection of finite number of congruences on an S-act A is again a congruence on A. But for a topological semigroup S, intersection of two topological S-act congruences on a topological S-act A may not be a topological S-act congruence. In this section, we establish a necessary and sufficient condition so that intersection of two topological S-act congruences on a topological S-act A to be a topological S-act congruence.

Before going to this result, we give a counter example to show that intersection of any S-act topological congruences on an S-act endowed with a topology τ may not be a S-act topological congruence.

Example 3.1. Consider the semigroup $S=(\mathbb{Z}_6,\cdot_6,\tau)$, where $\tau=\{\mathbb{Z}_6,\emptyset,\{\bar0,\bar3\},\{\bar1,\bar2,\bar4,\bar5\},\{\bar4\}\}$. Let $S=A$. Then A is an S-act endowed with topology τ. It is clear that A is not a S-act topological space. Now let $\theta_1 = \{(x,x) : x \in A\} \cup \{(\bar{2},\bar{4}),(\bar{4},\bar{2}),(\bar{1},\bar{5}),(\bar{5},\bar{1})\}$ and $\theta_2 = \{(x,x) : x \in A\} \cup \{(\bar{2},\bar{5}),(\bar{5},\bar{2})(\bar{1},\bar{4}),(\bar{4},\bar{1}),(\bar{0},\bar{3}),(\bar{3},\bar{0})\}$. Then θ_1 and θ_2 are congruences on A. Clearly, $\theta_1 \cap \theta_2 = \{(x,x); x \in A\}$. Now $\tau_{\theta_1} = \tau_{\theta_2} = \{\mathbb{Z}_6, \emptyset, \{\bar{0},\bar{3}\}, \{\bar{1},\bar{2},\bar{4},\bar{5}\}\}$. It follows that (A,τ_{θ_1}) and (A,τ_{θ_2}) are S - act topological spaces. But $\tau_{\theta_1\cap\theta_2} = \tau$ and this implies that $\theta_1 \cap \theta_2$ is not an S-act topological congruence.

Now we establish some sufficient conditions for the intersection of two S-act topological congruences on an S-act A endowed with a topology τ to be a S-act topological congruence.

Theorem 3.2. *Let S be a semigroup endowed with a topology τ and A be an S-act endowed with a topology τ_A. Let θ_1 and θ_2 be two S-act topological congruences on A. If $\tau_{\theta_1} \cup \tau_{\theta_2}$ is a basis for $\tau_{\theta_1\cap\theta_2}$, then $\theta_1 \cap \theta_2$ is an S-act topological congruence on A.*

Proof. Let $\theta = \theta_1 \cap \theta_2$. We first show that $\tau_{\theta_1} \cup \tau_{\theta_2} \subseteq \tau_\theta$. Consider the natural epimorphisms $\pi : A \to A/\theta$, $\pi_1 : A \to A/\theta_1$ and $\pi_2 : A \to A/\theta_2$. Let $U \in \tau_{\theta_1}$. Then $\pi_1^{-1}(\pi_1(U)) = U$. Clearly, $U \subseteq \pi^{-1}(\pi(U))$. Let $x \in \pi^{-1}(\pi(U))$. Then $\pi(x) = \pi(u)$, for some $u \in U$. This implies that $(x,u) \in \theta \subseteq \theta_1$ and so $\pi_1(x) = \pi_1(u)$. From this, we have $x \in \pi_1^{-1}(\pi_1(U)) = U$ and thus $\pi^{-1}(\pi(U)) = U$. Therefore, $U \in \tau_\theta$ and hence $\tau_{\theta_1} \subseteq \tau_\theta$. Similarly, one can show that $\tau_{\theta_2} \subseteq \tau_\theta$. Therefore, $\tau_{\theta_1} \cup \tau_{\theta_2} \subseteq \tau_\theta$. Let $G \in \tau_\theta$ and $s \in S$, $a \in A$ with $sa \in G$. Then there exists a basic open set $W \in \tau_{\theta_1} \cup \tau_{\theta_2}$ such that $sa \in W \subseteq G$. Without any loss of generality, we assume that $W \in \tau_{\theta_1}$. Since θ_1 is an S-act topological congruence, there exist open sets U_1 and V_1 containing s and a in (S,τ) and (A,τ_{θ_1}) respectively such that $U_1V_1 \subseteq W$. As $\tau_{\theta_1} \subseteq \tau_\theta$, $V_1 \in \tau_\theta$. Hence $U_1V_1 \subseteq W \subseteq G$. Therefore, (A,τ_θ) is an S-act topological space and hence by Corollary 2.2, it follows that $\theta = \theta_1 \cap \theta_2$ is an S-act topological congruence on A. □

Corollary 3.3. *Let S be a topological semigroup and A be an S-act endowed with a topology τ_A. Let θ_1 and θ_2 be two topological S-act congruences on A. If $\tau_{\theta_1} \cup \tau_{\theta_2}$ is a basis for $\tau_{\theta_1\cap\theta_2}$, then $\theta_1 \cap \theta_2$ is a topological S-act congruence on A.*

Now, we come to the part where we talk about the join of two topological S-act congruences on a topological S-act A. In the following results, we establish a necessary and sufficient condition for the join of two topological

S-act congruences on a topological S-act to be again a topological S-act congruence.

Theorem 3.4. *Let S be a semigroup endowed with a topology τ and A be an S-act endowed with a topology τ_A. For any two S-act topological congruences θ_1 and θ_2 on A, $\theta_1 \vee \theta_2$ is an S-act topological congruence on A if and only if $(A, \tau_{\theta_1} \cap \tau_{\theta_2})$ is an S-act topological space.*

Proof. Let $\theta = \theta_1 \vee \theta_2$. Then by Corollary 2.2, θ is an S-act topological congruence on A if and only if (A, τ_θ) is an S-act topological space. For this, it suffices to show that $\tau_\theta = \tau_{\theta_1} \cap \tau_{\theta_2}$. Consider the natural epimorphisms $\pi : A \to A/\theta$, $\pi_1 : A \to A/\theta_1$ and $\pi_2 : A \to A/\theta_2$. Let $U \in \tau_\theta$. Then $\pi^{-1}(\pi(U)) = U$. Clearly, $U \subseteq \pi_1^{-1}(\pi_1(U))$. Let $x \in \pi_1^{-1}(\pi_1(U))$. Then $\pi_1(x) = \pi_1(u)$, for some $u \in U$. This implies that $(x, u) \in \theta_1 \subseteq \theta$ and so $\pi(x) = \pi(u)$. Then $x \in \pi^{-1}(\pi(U)) = U$. So, $\pi_1^{-1}(\pi_1(U)) = U$. This implies that $U \in \tau_{\theta_1}$. Thus, we have $\tau_\theta \subseteq \tau_{\theta_1}$. Similarly, $\tau_\theta \subseteq \tau_{\theta_2}$. Therefore, $\tau_\theta \subseteq \tau_{\theta_1} \cap \tau_{\theta_2}$. For the reverse inclusion let, $W \in \tau_{\theta_1} \cap \tau_{\theta_2}$. Then $\pi_1^{-1}(\pi_1(W)) = W$ and $\pi_2^{-1}(\pi_2(W)) = W$. Let $y \in \pi^{-1}(\pi(W))$. Then $\pi(y) = \pi(w)$, for some $w \in W$. This implies that $(y, w) \in \theta_1 \vee \theta_2$. Then by [2, Proposition 5.14], for some $n \in \mathbb{N}$, there exist elements $x_1, x_2, \ldots, x_{2n-1}$ in A such that $(y, x_1) \in \theta_1, (x_1, x_2) \in \theta_2, (x_2, x_3) \in \theta_1, \ldots, (x_{2n-1}, w) \in \theta_2$. Then $\pi_2(x_{2n-1}) = \pi_2(w)$ for some $w \in W$ and so $x_{2n-1} \in \pi_2^{-1}(\pi_2(W)) = W$. Now, $(x_{2n-2}, x_{2n-1}) \in \theta_1$ implies $x_{2n-2} \in \pi_1^{-1}(\pi_1(W)) = W$. Continuing in this way, we have $x_1 \in W$ and this implies that $y \in \pi_1^{-1}(\pi_1(W)) = W$. Therefore, $\pi^{-1}(\pi(W)) = W$ and so $W \in \tau_\theta$. Hence $\tau_\theta = \tau_{\theta_1} \cap \tau_{\theta_2}$. □

Corollary 3.5. *Let S be a topological semigroup and A be an S-act endowed with a topology τ_A. For any two topological S-act congruences θ_1 and θ_2 on A, $\theta_1 \vee \theta_2$ is a topological S-act congruence on A if and only if $(S, \tau_{\theta_1} \cap \tau_{\theta_2})$ is a topological S-act.*

We now establish necessary and sufficient conditions for the intersection of two topological S-act congruences on a topological S-act to be a topological S-act congruence.

Theorem 3.6. *Let S be a semigroup endowed with a topology τ and A be an S-act endowed with a topology τ_A. For any two congruences θ_1 and θ_2 on A, $\theta_1 \cap \theta_2$ is an S-act topological congruence on A if and only if the mapping $\varphi : A \to A/\theta_1 \times A/\theta_2$ defined by $\varphi(a) = (a\theta_1, a\theta_2)$, for all $a \in A$, is φ-saturated continuous.*

Proof. Since $\ker\varphi = \theta_1 \cap \theta_2$, the result follows from Theorem 2.14. □

Corollary 3.7. *Let S be a topological semigroup and A be an S-act endowed with a topology τ_A. Then for any two congruences θ_1 and θ_2 on A, $\theta_1 \cap \theta_2$ is a topological S-act congruence on A if and only if the mapping $\varphi : A \to A/\theta_1 \times A/\theta_2$ defined by $\varphi(a) = (a\theta_1, a\theta_2)$, for all $a \in A$, is φ-saturated continuous.*

Definition 3.8. *Let X and Y be topological spaces. A mapping $\varphi : X \to Y$ is said to be a weakly quotient mapping if for any subset A of Y, $\varphi^{-1}(A)$ is open in X implies A is open in Y.*

Theorem 3.9. *Let S be a semigroup endowed with a topology τ, A be an S-act endowed with a topology τ_A and θ_1, θ_2 be two S-act topological congruences on A. If the mapping $\varphi : A \to A/\theta_1 \times A/\theta_2$ defined by $\varphi(a) = (a\theta_1, a\theta_2)$, for all $a \in A$, is weakly quotient, then $\theta_1 \cap \theta_2$ is an S-act topological congruence on A.*

Proof. Clearly, $\ker\varphi = \theta_1 \cap \theta_2$. Consider the natural semigroup epimorphisms $\pi : A \to A/(\theta_1 \cap \theta_2)$, $\pi_1 : A \to A/\theta_1$ and $\pi_2 : A \to A/\theta_2$. To show $\theta_1 \cap \theta_2$ is an S-act topological congruence on A, it is suffices to show that φ is φ-saturated continuous. Let G be a subset of $A/\theta_1 \times A/\theta_2$ such that $\varphi^{-1}(G)$ is open in A and $s \in S$, $a \in A$ with $sa \in \varphi^{-1}(G)$. Since φ is weakly quotient, G is open in $A/\theta_1 \times A/\theta_2$. As $sa \in \varphi^{-1}(G)$, we have $(sa\theta_1, sa\theta_2) \in G$ and hence there exists a basic open set $U \times V$ in $A/\theta_1 \times A/\theta_2$ such that $(sa\theta_1, sa\theta_2) \in U \times V \subseteq G$, where U is open in A/θ_1 and V is open in A/θ_2. This implies that $sa \in \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$. Since A/θ_1 is an S-act topological space, (A, τ_{θ_1}) is an S-act topological space. So there exist open sets U_1 containing s and V_1 containing a in (S, τ) and (A, τ_{θ_1}) respectively such that $U_1V_1 \subseteq \pi_1^{-1}(U)$. Similarly, there exist open sets U_2 containing s and V_2 containing a in (S, τ) and (A, τ_{θ_2}) respectively such that $U_2V_2 \subseteq \pi_2^{-1}(V)$. Let $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Then U_3 and V_3 are open sets containing s and a in (S, τ) and (A, τ_A) respectively. We now show that $\pi^{-1}(\pi(V_3)) = V_3$. For this, let $z \in \pi^{-1}(\pi(V_3))$. Then $\pi(z) \in \pi(V_3)$ and so $\pi(z) = \pi(v)$, for some $v \in V_3$. Now, $v \in V_3$ implies $v \in V_1 \cap V_2$ and $\pi(z) = \pi(v)$ implies $(z, v) \in \theta_1 \cap \theta_2 \subseteq \theta_1$. This implies $\pi_1(z) = \pi_1(v)$ and $v \in V_1$. Therefore, $z \in \pi_1^{-1}(\pi_1(V_1)) = V_1$. Similarly, we can show that $z \in V_2$. Thus, $z \in V_1 \cap V_2 = V_3$ and hence $\pi^{-1}(\pi(V_3)) = V_3$. Therefore, $V_3 \in \tau_{\theta_1 \cap \theta_2}$ and $U_3V_3 \subseteq U_1V_1 \subseteq \pi_1^{-1}(U)$ and $U_3V_3 \subseteq U_2V_2 \subseteq \pi_2^{-1}(V)$. Let $t \in U_3$ and $c \in V_3$. Then $(tc\theta_1, tc\theta_2) \in U \times V \subseteq G$

implies $\varphi(tc) \in G$, i.e., $tc \in \varphi^{-1}(G)$. Therefore, $U_3V_3 \subseteq \varphi^{-1}(G)$ and hence φ is φ-saturated continuous. Consequently, $\theta_1 \cap \theta_2$ is an S-act topological congruence on A. $\square$

Corollary 3.10. *Let S be a topological semigroup, A be an S-act endowed with a topology τ_A and θ_1, θ_2 be two topological S-act congruences on A. If the mapping $\varphi : A \to A/\theta_1 \times A/\theta_2$ defined by $\varphi(a) = (a\theta_1, a\theta_2)$, for all $a \in A$, is weakly quotient, then $\theta_1 \cap \theta_2$ is a topological S-act congruence on A.*

Acknowledgement. The authors are thankful to the Referee for valuable suggestions which have definitely improved the presentation of this article. The presented research was supported by CSIR, India. CSIR Award no.: 09/028(1001)/2017-EMR-1.

References

[1] **A.H. Clifford and G.B. Preston**, The Algebraic Theory of Semigroup (Vol-I), Math. Surveys, 7, Amer. Math. Soc., (1961).

[2] **J.M. Howie**, *An Introduction to Semigroup Theory*, London, New York, 1976.

[3] **T. Husain**, *Introduction to Topological Groups*, W. B. Saunders company, (1966).

[4] **B. Khosravi**, *Congruence on topological S-acts and topological semigroups*, J. Appl. Sci., **10**(2010), no. 9, 766 – 771.

[5] **R.J. Koch, J.A. Hildebrant and J.H. Carruth**, *The Theory of Topological Semigroups*, Marcel Dekker, Inc., (1983).

[6] **J. Lawson and B. Madison**, *On congruences and cones*, Math. Z. **120**(1971), 18 – 24.

[7] **J.R. Munkress**, *Topology*, 2nd edition, PHI Learning Private Limited, Delhi, (2014).

[8] **P. Normak**, *Topological S-acts: preliminaries and problems*, Transformation Semigroups, Univ. Essex, Colchester, (1993), 60 – 69.

[9] **M. Petrich and N.R. Reilly**, *Completely Regular Semigroups* Wiley, New York, (1999).

Received June 16, 2022

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata - 700019, India
E-mail: skmpm@caluniv.ac.in (Maity), paulmonika007@gmail.com (Paul)