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Menger hypercompositional algebras represented by medial n-ary hyperoperations

Thodsaporn Kumduang

Dedicated to the memory of Professor Valentin S. Trokhimenko

Abstract. The necessary and sufficient conditions under which a Menger algebra can be isomorphically represented by medial n-ary operations are proposed. Since a Menger hypercompositional algebra can be regarded as a generalization of a Menger algebra, for this reason, the situation for medial hyperoperations is further examined and a representation theorem of Menger hypercompositional algebras by such concepts is proved.

1. Introduction and preliminaries

It is widely accepted that Professor V.S. Trokhimenko, who is a Ukrainian mathematician, has a great contribution in the developments of Menger algebras and algebras of multiplace functions for a long time. Many papers concerning various classes of multiplace functions and their structural properties have been extensively studied in the past few decades, for instance, idempotent n-ary operations [10] and k-commutative n-place functions [11]. See [8, 9, 12, 13, 14, 15] for more related topics in this direction. It turned out that these works can be considered as nice connections between the study of algebra and the theory of functions. Unfortunately, V.S. Trokhimenko passed away in 2020 due to the pandamic of COVID-19. However, the paper that mentioned his personal life and scientific works was commemoratively collected by W.A. Dudek in [6].

Basically, for a fixed positive integer n, a Menger algebra of rank n is a pair of a nonempty set G and an $(n+1)$-ary operation on G which

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satisfies the superassociative law. Nowadays, Menger algebras were investigated in different aspects, for example, partial Menger algebras of terms [5], power Menger algebras of terms defined by order-decreasing transformations [28]. For other, see [4, 20, 23]. Fundamental properties of Menger algebras concerning quotient Menger algebras and isomorphism theorems for Menger algebras were recently examined in [18]. Now two elementary examples of Menger algebras are provided. The first one is the set $\mathbb{R}^+$ of all positive real numbers with the operation $\circ : (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$, defined by $\circ(x_0, \ldots, x_n) = x_0 \sqrt[n]{x_1 \cdots x_n}$. Another one is the set of all real numbers $\mathbb{R}$ with the following $(n+1)$-ary operation $\circ$, which is defined by $\circ(x, y_1 \ldots, y_n) = x + \frac{y_1 + \ldots + y_n}{n}$ for all $x, y_1 \ldots, y_n \in \mathbb{R}$. In a view of extensions, a Menger algebra of rank $n = 1$ is a semigroup. This means that a Menger algebra of rank n is a generalized structure of semigroups too.

Normally, semigroups and groups can be isomorphically represented by functions of one variables. Representations of other structures, for example, see [1, 22, 27]. Analogously, Menger algebras of some types are also studied in the same direction. It turned out that some types of Menger algebras of rank n can be represented by n-ary functions. In fact, let A^n be the n-th Cartesian product of a nonempty set A. Any mapping from A^n to A is called a *full n-ary function* or an *n-ary operation* if it is defined for all elements of A^n. The set of all such mappings is denoted by $T(A^n, A)$. One can consider the *Menger's superposition* on the set $T(A^n, A)$, i.e., an $(n+1)$-ary operation $\mathcal{O} : T(A^n, A)^{n+1} \to T(A^n, A)$ defined by

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)),$$

where $f, g_1, \ldots, g_n \in T(A^n, A), a_1, \ldots, a_n \in A$. A Menger algebra of all full n-ary functions, or a Menger algebra of all n-ary operations, is a pair of the set $T(A^n, A)$ of all full n-ary functions defined on A and the Menger composition of full n-ary functions satisfying the superassociative law. For an extensive information on functions, see [7, 21].

It is commonly seen that the study of hypercompositional algebra has become famous topics among mathematicians. One of outstanding classes of its is a semihypergroup, a hyperstructure that generalized semigroups but the composition of two elements is a nonempty set. There are several possibilities to construct generalizations of semihypergroups. Recall from [19] that a Menger hypercomposition algebra or a Menger hyperalgebra is a cuple $(G, \diamond)$ of a nonempty set G and one $(n+1)$-ary hyperoperation $\diamond$ on G satisfying the identity of the superassociativity. It can be noticed that a

Menger hyperalgebra can be reduced to a semihypergroup if we set $n = 1$. Furthermore, every Menger algebra is a Menger hypercomposition algebra. Normally, a representation is an essential part of the study of algebra, so representation theorems for Menger hyperalgebras are now recalled. Let A^n be the n-th Cartesian product of a nonempty set A. The symbol $P^*(A)$ stands for a power set of A without emptyset. On the set $T(A^n, P^*(A))$ of all *multivalued full n-ary functions* or *n-ary hyperoperations* $\alpha : A^n \to P^*(A)$, one can define the following $(n+1)$-ary operation $\bullet : T(A^n, P^*(A))^{n+1} \to T(A^n, P^*(A))$, called the *Menger superposition* $\bullet$, defined by

$$\bullet(f, g_1, \ldots, g_n)(x_1, \ldots, x_n) = \bigcup_{\substack{y_i \in g_i(x_1,\ldots,x_n) \\ i \in \{1,\ldots,n\}}} f(y_1, \ldots, y_n),$$

for all $i = 1, \ldots, n$ where $f, g_1, \ldots, g_n \in T(A^n, P^*(A))$, $x_1, \ldots, x_n \in A$. As a consequence, the set $T(A^n, P^*(A))$ of all multivalued full n-ary functions on A together with an $(n+1)$-ary operation $\bullet$ forms a Menger algebra.

This papar aims to apply a specific class of functions which are called medial operations (the formal definition will be recalled in the next section) into the study of Menger algebras and to describe properties of Menger hypercomposition algebras by such tools. In Section 2, the idea of medial operations is mainly presented and a representation theroem for Menger algebras via such concepts is mentioned. These lead us to generalized our study in Menger hypercompositional algebras. In addition, the conditions under which hyperstructure can be isomorphically represented by medial hyperoperations are found. Finally, some interesting remarks and some potential problems are given.

2. Results

This section begins with recalling some basic definitions of medial properties. An n-ary algebra (A, g) is said to be *medial* if it satisfies the identity

$$g(g(x_{11}, ..., x_{n1}), \ldots, g(x_{1n}, ..., x_{nn})) = g(g(x_{11}, ..., x_{1n}), \ldots, g(x_{n1}, ..., x_{nn})),$$

and an n-ary operation g on A is called medial. Furthermore, it has been studied by many authors under different names, such as Abelian, entropy, and bisymmetric algebras. On the other hand, if g satisfies the identity

$$g(g(x_{11}, ..., x_{n1}), \ldots, g(x_{1n}, ..., x_{nn})) = g(g(x_{nn}, ..., x_{n1}), \ldots, g(x_{1n}, ..., x_{11})),$$

then an algebra (A, g) is called *paramedial*. For more information about medial and paramedial properties can be found, for instance, in [2, 3, 16, 17, 24, 25, 26].

Example 2.1. Two interesting examples of mediality are collected.

(1) Every left (right) zero semigroup is a medial semigroup.

(2) Let (A, g) be an n-ary algebra. By an antiendomorphism on A we mean a mapping $\alpha : A \to A$, $\alpha(g(a_1, \dots, a_n)) = g(\alpha(a_n), \dots, \alpha(a_1))$. As a result if (A, g) is a paramedial n-ary algebra and $\gamma_1, \dots, \gamma_n$ are pair wise commuting antiendomorphisms of A, then an n-ary operation g^* on A, which is defined by

$$g^*(x_1, \dots, x_n) = g(\gamma_1(x_1), \dots, \gamma_n(x_n)),$$

is a medial n-ary operation.

By a_1^n, we mean the sequence $a_1, \dots, a_n$ for a positive integer n. However, it is not difficult to verify that the superassociativity does not valid for every medial n-ary operations. In order to state necessary and sufficient conditions representing an abstract Menger hypercompisitonal algebra by medial n-ary operations, we need a technical lemma.

Lemma 2.2. *For any medial n-ary operations f, g_{ij} on A, $i, j = 1, 2, \dots, n$, we have*

$$\mathcal{O}(f, \mathcal{O}(f, g_{11}, \dots, g_{n1}), \dots, \mathcal{O}(f, g_{1n}, \dots, g_{nn})) = \mathcal{O}(f, \mathcal{O}(f, g_{11}, \dots, g_{1n}), \dots, \mathcal{O}(f, g_{n1}, \dots, g_{nn})).$$

Proof. Let $a_1, \dots, a_n$ be elements in A. Then we obtain

$$\begin{aligned} & \mathcal{O}(f, \mathcal{O}(f, g_{11}, \dots, g_{n1}), \dots, \mathcal{O}(f, g_{1n}, \dots, g_{nn}))(a_1^n) \\ = \quad & f(\mathcal{O}(f, g_{11}, \dots, g_{n1})(a_1^n), \dots, \mathcal{O}(f, g_{1n}, \dots, g_{nn})(a_1^n)) \\ = \quad & f(f(g_{11}(a_1^n), \dots, g_{n1}(a_1^n)), \dots, f(g_{1n}(a_1^n), \dots, g_{nn}(a_1^n))) \\ = \quad & f(f(g_{11}(a_1^n), \dots, g_{1n}(a_1^n)), \dots, f(g_{n1}(a_1^n), \dots, g_{nn}(a_1^n))) \\ = \quad & f(\mathcal{O}(f, g_{11}, \dots, g_{1n})(a_1^n), \dots, \mathcal{O}(f, g_{n1}, \dots, g_{nn})(a_1^n)) \\ = \quad & \mathcal{O}(f, \mathcal{O}(f, g_{11}, \dots, g_{1n}), \dots, \mathcal{O}(f, g_{n1}, \dots, g_{nn}))(a_1^n). \end{aligned}$$

The proof is completed. □

As a consequence, we have

Theorem 2.3. *A Menger algebra $(G, \circ)$ of rank n is isomorphically represented by medial n-ary operations defined on some set if and only if $(G, \circ)$ satisfies the equation*

$$\circ(y, \circ(y, x_{11}^{n1}), \ldots, \circ(y, x_{1n}^{nn})) = \circ(y, \circ(y, x_{11}^{1n}), \ldots, \circ(y, x_{n1}^{nn}))$$

for all $y, x_{ij} \in G$ and $i, j \in \{1, \ldots, n\}$.

Proof. The necessity follows directly from the result of Lemma 2.2. Conversely, let $(G, \circ)$ be an arbitrary Menger algebra satisfying the equation

$$\circ(y, \circ(y, x_{11}^{n1}), \ldots, \circ(y, x_{1n}^{nn})) = \circ(y, \circ(y, x_{11}^{1n}), \ldots, \circ(y, x_{n1}^{nn})).$$

We now prove that there exists an n-ary operation induced by an element g of G. For this construction, consider the set $G' = G \cup \{e, c\}$ where e and c are different elements not containing in G. For every element $g \in G$, we assign an n-ary operation $\eta_g : (G')^n \to G'$ by setting

$$\eta_g(a_1^n) = \begin{cases} \circ(g, a_1^n) & \text{if } a_i \in G \text{ for all } 1 \leqslant i \leqslant n, \\ g & \text{if } a_i = e \text{ for all } 1 \leqslant i \leqslant n, \\ c & \text{otherwise.} \end{cases}$$

Firstly, we show that the n-ary operation η_g defined above is medial. For this, let $a_{ij} \in G'$ for $i, j = 1, \ldots, n$.

If all $a_{ij} \in G$, then according to the assumption, we have

$$\begin{aligned} \eta_g(\eta_g(a_{11}^{n1}), \ldots, \eta_g(a_{1n}^{nn})) &= \circ(g, \eta_g(a_{11}^{n1}), \ldots, \eta_g(a_{1n}^{nn})) \\ &= \circ(g, \circ(g, a_{11}^{n1}), \ldots, \circ(g, a_{1n}^{nn})) \\ &= \circ(g, \circ(g, a_{11}^{1n}), \ldots, \circ(g, a_{n1}^{nn})) \\ &= \eta_g(\eta_g(a_{11}^{1n}), \ldots, \eta_g(a_{n1}^{nn})). \end{aligned}$$

In the second case, if $a_{ij} = e$ for all $i, j \in \{1, \ldots, n\}$, then we obtain

$$\eta_g(\eta_g(a_{11}^{n1}), \ldots, \eta_g(a_{1n}^{nn})) = \eta_g(g, \ldots, g) = \circ(g, g, \ldots, g).$$

Moreover, $\eta_g(\eta_g(a_{11}^{1n}), \ldots, \eta_g(a_{n1}^{nn})) = \eta_g(g, \ldots, g) = \circ(g, g, \ldots, g)$.

In other case,

$$\eta_g(\eta_g(a_{11}^{n1}), \ldots, \eta_g(a_{1n}^{nn})) = \eta_g(c, \ldots, c) = c = \eta_g(\eta_g(a_{11}^{1n}), \ldots, \eta_g(a_{n1}^{nn})).$$

So, the n-ary operation η_g is medial.

Define a mapping $\phi : (G, \circ) \to (T(G^n, G), \mathcal{O})$ by $\phi(g) = \eta_g$ for all $g \in G$. To prove the injectivity of ϕ, let $g_1, g_2 \in G$. Suppose that $\phi(g_1) = \phi(g_2)$. Then for all $a_1, \ldots, a_n \in G$, we have $\eta_{g_1}(a_1, \ldots, a_n) = \eta_{g_2}(a_1, \ldots, a_n)$. In particular, $\eta_{g_1}(e,\ldots,e) = \eta_{g_2}(e,\ldots,e)$, which implies that $g_1 = g_2$. So, ϕ is injective. Finally, we show that the identity

$$\eta_{\circ(x,y_1,\ldots,y_n)} = \mathcal{O}(\eta_x, \eta_{y_1}, \ldots, \eta_{y_n})$$

holds for all $x, y_1, \ldots, y_n \in G$. For this, let $a_1, \ldots, a_n$ be arbitrary elements in G'. If $a_i \in G$ for all $1 \leqslant i \leqslant n$, then for $x, y_1, \ldots, y_n \in G$, applying the superassociativity of $(n+1)$-ary operation $\circ$ on G, we have

$$\begin{aligned}
\eta_{\circ(x,y_1,\ldots,y_n)}(a_1,\ldots,a_n) &= \circ(\circ(x,y_1,\ldots,y_n), a_1,\ldots,a_n) \\
&= \circ(x, \circ(y_1,a_1,\ldots,a_n), \ldots, \circ(y_n,a_1,\ldots,a_n)) \\
&= \eta_x(\eta_{y_1}(a_1,\ldots,a_n), \ldots, \eta_{y_n}(a_1,\ldots,a_n)) \\
&= \mathcal{O}(\eta_x, \eta_{y_1}, \ldots, \eta_{y_n})(a_1,\ldots,a_n).
\end{aligned}$$

If $(a_1,\ldots,a_n) = (e,\ldots,e)$, then $\eta_{\circ(x,y_1,\ldots,y_n)}(e,\ldots,e) = \circ(x,y_1,\ldots,y_n)$. On the other hand, we get $\eta_x(y_1,\ldots,y_n) = \eta_x(\eta_{y_1}(e,\ldots,e),\ldots,\eta_{y_n}(e,\ldots,e)) = \mathcal{O}(\eta_x,\eta_{y_1},\ldots,\eta_{y_n})(e,\ldots,e)$. Now, if $(a_1,\ldots,a_n) \in (G')^n \backslash (G^n \cup \{(e,\ldots,e)\})$, then we get $\eta_{\circ(x,y_1,\ldots,y_n)}(a_1,\ldots,a_n) = c$ and $\mathcal{O}(\eta_x,\eta_{y_1},\ldots,\eta_{y_n})(a_1,\ldots,a_n) = \eta_x(\eta_{y_1}(c,\ldots,c),\ldots,\eta_{y_n}(c,\ldots,c)) = \eta_x(c,\cdots,c) = c$, which implies

$$\eta_{\circ(x,y_1,\ldots,y_n)}(a_1,\ldots,a_n) = c = \mathcal{O}(\eta_x,\eta_{y_1},\ldots,\eta_{y_n})(a_1,\ldots,a_n).$$

This completes the proof of this theorem. □

Applyig the same construction of the n-ary operation η_g, we can prove a representation theorem of any Menger algebra by paramedial operations. So, we obtain the following corollary.

Corollary 2.4. *A Menger algebra $(G,\circ)$ of rank n is isomorphically represented by paramedial n-ary operations defined on some set if and only if $(G,\circ)$ satisfies the equation*

$$\circ(y, \circ(y, x_{11}^{n1}), \ldots, \circ(y, x_{1n}^{nn})) = \circ(y, \circ(y, x_{nn}, \ldots, x_{n1}), \ldots, \circ(y, x_{1n}, \ldots, x_{11}))$$

for all $y, x_{ij} \in G$ and $i, j \in \{1,\ldots,n\}$.

Now the investigation in Menger algebras is finished. We continue our study on Menger hypercompositional algebras. In our conjecture, the situation for Menger hypercompositional algebras is different. To attain this purpose, the concept of medial hyperoperations is now introduced. An n-ary hyperoperation f on A is said to be *medial* if

$$f(f(x_{11}^{n1}), \ldots, f(x_{1n}^{nn})) = f(f(x_{11}^{1n}), \ldots, f(x_{n1}^{nn})).$$

For convenience, we may rewrite the above identity in the following form:

$$\bigcup_{\substack{y_i \in f(x_{1i}^{ni}) \\ i \in \{1,\ldots,n\}}} f(y_1^n) = \bigcup_{\substack{y_i \in f(x_{i1}^{in}) \\ i \in \{1,\ldots,n\}}} f(y_1^n).$$

The following theorem presents a mediality of medial hyperoperations in a connection with permutations.

Theorem 2.5. *Let f be a medial n-ary hyperoperation on a nonempty set A and π be a permutation on $\{1,\ldots,n\}$. Then the n-ary hyperoperation $\widehat{f}$ on A, which is defined by $\widehat{f}(a_1,\ldots,a_n) = f(a_{\pi(1)},\ldots,a_{\pi(n)})$, is medial.*

Proof. For every $i,j \in \{1,\ldots,n\}$, let $a_{ij} \in A$. Then we obtain

$$\begin{aligned}
\bigcup_{\substack{b_i \in \widehat{f}(a_{1i},\ldots,a_{ni}) \\ i \in \{1,\ldots,n\}}} \widehat{f}(b_1,\ldots,b_n) &= \bigcup_{\substack{b_{\pi(i)} \in f(a_{\pi(1)\pi(i)},\ldots,a_{\pi(n)\pi(i)}) \\ i \in \{1,\ldots,n\}}} f(b_{\pi(1)},\ldots,b_{\pi(n)}) \\
&= \bigcup_{\substack{b_{\pi(i)} \in f(a_{\pi(i)\pi(1)},\ldots,a_{\pi(i)\pi(n)}) \\ i \in \{1,\ldots,n\}}} f(b_{\pi(1)},\ldots,b_{\pi(n)}) \\
&= \bigcup_{\substack{b_i \in \widehat{f}(a_{i1},\ldots,a_{in}) \\ i \in \{1,\ldots,n\}}} \widehat{f}(b_1,\ldots,b_n).
\end{aligned}$$

This shows that the n-ary hyperoperation $\widehat{f}$ is medial. □

Theorem 2.6. *A Menger hypercompositonal algebra $(G,\diamond)$ of rank n is isomorphically represented by medial n-ary hyperoperations defined on some set if and only if $(G,\diamond)$ satisfies the equation*

$$\bigcup_{\substack{y_i \in \diamond(y,x_{1i}^{ni}) \\ i \in \{1,\ldots,n\}}} \diamond(y,y_1^n) = \bigcup_{\substack{y_i \in \diamond(y,x_{i1}^{in}) \\ i \in \{1,\ldots,n\}}} \diamond(y,y_1^n)$$

for all $y, x_{ij} \in G$ and $i,j \in \{1,\ldots,n\}$.

Proof. Let $j = 1,\ldots,n$ and f, g_{1j}^{nj} be arbitrary medial n-ary hyperoperations. Then we have

$$\bullet(f,\bullet(f,g_{11}^{n1}),\ldots,\bullet(f,g_{1n}^{nn}))(a_1^n) = \bigcup_{\substack{y_i \in \bullet(f,g_{1i}^{ni})(a_1^n) \\ i \in \{1,\ldots,n\}}} f(y_1^n) = \bigcup_{\substack{y_i \in f(g_{1i}(a_1^n),\ldots,g_{ni}(a_1^n)) \\ i \in \{1,\ldots,n\}}} f(y_1^n)$$

$$= \bigcup_{\substack{y_i \in f(g_{i1}(a_1^n), \ldots, g_{in}(a_1^n)) \\ i \in \{1, \ldots, n\}}} f(y_1^n) = \bigcup_{\substack{y_i \in \bullet(f, g_{i1}, \ldots, g_{in})(a_1^n) \\ i \in \{1, \ldots, n\}}} f(y_1^n)$$
$$= \bullet(f, \bullet(f, g_{11}^{1n}), \ldots, \bullet(f, g_{n1}^{nn}))(a_1^n).$$

For the converse, let $G' = G \cup \{e, c\}$ where $e, c \notin G$ and $e \neq c$. Firstly, we now construct an n-ary hyperoperation G'. For each element $g \in G'$, an n-ary hyperoperation on G' can be defined by setting

$$\mu_g(a_1, \ldots, a_n) = \begin{cases} \diamond(g, a_1^n) & \text{if } a_1^n \in G; \\ \{g\} & \text{if } a_1 = \cdots = a_n = e; \\ \{c\} & \text{otherwise}. \end{cases}$$

Moreover, the extension of the multivalued full n-ary function is needed. For any nonempty subset A of G', the function μ_A is defined by

$$\mu_A(a_1^n) = \begin{cases} \diamond(A, a_1^n) & \text{if } a_1, \ldots, a_n \in G; \\ A & \text{if } a_1 = \cdots = a_n = e; \\ \{c\} & \text{otherwise}. \end{cases}$$

To show that μ_g is a medial n-ary hyperoperation, let $a_{ij} \in G'$ for every $i, j = 1, \ldots, n$. We first consider in the case when $a_{1j}^{nj} \in G$. Then we obtain

$$\bigcup_{\substack{b_i \in \mu_g(a_{1i}^{ni}) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \bigcup_{\substack{b_i \in \diamond(g, a_{1i}^{ni}) \\ i \in \{1, \ldots, n\}}} \diamond(g, b_1^n) = \bigcup_{\substack{b_i \in \diamond(g, a_{i1}^{in}) \\ i \in \{1, \ldots, n\}}} \diamond(g, b_1^n) = \bigcup_{\substack{b_i \in \mu_g(a_{i1}^{in}) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n).$$

In the second case, if $a_{1j} = \cdots = a_{nj} = e$ for all $j = 1, \ldots, n$, we have

$$\bigcup_{\substack{b_i \in \mu_g(a_{1i}^{ni}) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \bigcup_{\substack{b_i \in \mu_g(e, \ldots, e) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \bigcup_{\substack{b_i \in \{g\} \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \mu_g(g, \ldots, g)$$
$$= \diamond(g, g, \ldots, g) = \bigcup_{\substack{b_i \in \mu_g(e, \ldots, e) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \bigcup_{\substack{b_i \in \mu_g(a_{i1}^{in}) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n).$$

In other case, by the construction of μ_g, we have

$$\bigcup_{\substack{b_i \in \mu_g(a_{1i}^{ni}) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \bigcup_{\substack{b_i \in \{c\} \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n) = \{c\} = \bigcup_{\substack{b_i \in \mu_g(a_{i1}^{in}) \\ i \in \{1, \ldots, n\}}} \mu_g(b_1^n).$$

As a result, the hyperoperation μ_g with respect to each element g is medial.

Now we show that the mapping $\varphi : G \to \Lambda'$, which is defined by $\varphi(g) = \mu_g$ for all $g \in G$, is a strong isomorphism between $(G, \diamond)$ and

$(T(G^n, P^*(G)), \bullet)$ where $\Lambda' = \{\mu_g \mid g \in G\}$. In order to prove this property, we show

$$\mu_{\diamond(a,b_1,\dots,b_n)} = \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})$$

for any $a, b_1, \dots, b_n \in G'$.

Let $x_1, \dots, x_n \in G$. Then we first show that the equation

$$\mu_{\diamond(a,b_1,\dots,b_n)}(x_1^n) = \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(x_1^n).$$

holds. For this, let $a, b_1, \dots, b_n, x_1, \dots, x_n$ be arbitrary elements in G. Then

$$\begin{aligned}\mu_{\diamond(a,b_1,\dots,b_n)}(x_1^n) &= \diamond(\diamond(a, b_1^n), x_1^n) = \diamond(a, \diamond(b_1, x_1^n), \dots, \diamond(b_n, x_1^n)) \\ &= \diamond(a, \mu_{b_1}(x_1^n), \dots, \mu_{b_n}(x_1^n)) = \bigcup_{\substack{y_i \in \mu_{b_i}(x_1^n) \\ i \in \{1,\dots,n\}}} \diamond(a, y_1^n) \\ &= \bigcup_{\substack{y_i \in \mu_{b_i}(x_1^n) \\ i \in \{1,\dots,n\}}} \mu_a(y_1^n) = \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(x_1^n).\end{aligned}$$

Now let $x_1 = \dots = x_n = e$, then according to the definition of μ_A, we have

$$\begin{aligned}\mu_{\diamond(a,b_1,\dots,b_n)}(x_1^n) &= \mu_{\diamond(a,b_1^n)}(e, \dots, e) = \diamond(a, b_1^n) = \mu_a(b_1^n) = \bigcup_{\substack{y_i \in \{b_i\} \\ i \in \{1,\dots,n\}}} \mu_a(y_1^n) \\ &= \bigcup_{\substack{y_i \in \mu_{b_i}(e,\dots,e) \\ i \in \{1,\dots,n\}}} \mu_a(y_1^n) = \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(e, \dots, e) \\ &= \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(x_1^n),\end{aligned}$$

which implies $\mu_{\diamond(a,b_1,\dots,b_n)}(e, \dots, e) = \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(e, \dots, e)$.

Otherwise, we have $\mu_{\diamond(a,b_1,\dots,b_n)}(z_1^n) = \{c\}$ and

$$\begin{aligned}\bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(z_1^n) &= \bigcup_{\substack{y_i \in \mu_{b_i}(z_1^n) \\ i \in \{1,\dots,n\}}} \mu_a(y_1^n) = \bigcup_{\substack{y_i \in \{c\} \\ i \in \{1,\dots,n\}}} \mu_a(y_1, \dots, y_n) \\ &= \mu_a(c, \dots, c) = \{c\},\end{aligned}$$

which shows $\mu_{\diamond(a,b_1,\dots,b_n)}(z_1^n) = \bullet(\mu_a, \mu_{b_1}, \dots, \mu_{b_n})(z_1^n)$. This completes the proof of the homomorphism property.

In order to prove that μ_g is injective, suppose $\mu_a = \mu_b$. Since e is an element in the domian of μ_a and μ_b, then $\mu_a(e, \dots, e) = \mu_b(e, \dots, e)$, and $\{a\} = \{b\}$. Hence, $a = b$. So the mapping $\varphi : g \mapsto \mu_g$ is an isomorphism. □

Corollary 2.7. *A Menger hypercompositional algebra $(G, \diamond)$ of rank n is isomorphically represented by paramedial n-ary hyperoperations defined on some set if and only if $(G, \diamond)$ satisfies the equation*

$$\diamond(y, \diamond(y, x_{11}^{n1}), \ldots, \diamond(y, x_{1n}^{nn})) =$$
$$\diamond(y, \diamond(y, x_{nn}, \ldots, x_{n1}), \ldots, \diamond(y, x_{1n}, \ldots, x_{11}))$$

for all $y, x_{ij} \in G$ *and* $i, j \in \{1, \ldots, n\}$.

3. Concluding remarks

In the given paper, applying medial operations in the study of medial algebras, a representation theorem for Menger algebras via such operations was proved. Several results connecting Menger hypercompositional algebras and medial hyperoperations were developed. The main goals of these studies were to introduce a novel concept of operations and hyperoperations that generated by a certain classes of mediality and to generalize the investigation in Menger algebras to Menger hypercompositional algebras. To achieve these two aims, some technical tools that derived from the idea of W.A. Dudek and V.S. Trokhimenko were applied.

Finally, two problems for the future research in this area are collected.

(1) Describe algebraic properties of medial operations and medial hyperoperations.

(2) According to Chapter 6 in the monograph [7], systems of multiplace functions are described. It is possible to generalize Menger systems to Menger hypercompsitional systems and try to discuss a construction of a mapping λ_g with repect to each element g in a family of Menger hypercompositonal system $(G_n)_{n \in I}$. Find necessary and sufficient conditions under which a Menger hypercompositional system can be represented by medial hyperoperations.

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School of General Science, Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin, Thailand

E-mail: Kumduang01@gmail.com