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Left twisted rings

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Abstract. We introduce the notion of a left-twisted ring, and we construct a left-zero ring which is not a ring. We show that such a left-twisted ring does not have an identity. Also, we show that every non-zero element of the left-twisted ring is a pseudo unit of it.

1. Introduction

The concept of several types of groupoids related to semigroups, viz., twisted semigroups for which twisted versions of the associative law hold was introduced by Allen et al. in [1]. Thus, if $(X, *)$ is a groupoid and if $\varphi : X^2 \to X^2$ is a function $\varphi(a, b) = (u, v)$, then $(X, *)$ is a *left-twisted semigroup* with respect to φ if for all $a, b, c \in X$, $a * (b * c) = (u * v) * c$. Moreover, right-twisted, middle-twisted and their duals, a dual left-twisted semigroup were also discussed. The class of groupoids defined over a field $(X, +, \cdot)$ via a formula $x * y = \lambda x + \mu y$, with $\lambda, \mu \in X$, fixed structure constants as twisted semigroups are discussed.

The basic idea came from the following observations. Let $X = \mathbf{R}$ be the set of all real numbers. We consider a binary operation $(\mathbf{R}, -)$ where "$-$" is the usual subtraction. Then $(x - y) - z \neq x - (y - z) = x - y + z$ in general, i.e., $(\mathbf{R}, -)$ is not a semigroup. Since $(x - y) - z = x - (y - (-z))$, if we define $u := x, v := -z$, then we have $(x - y) - z = u - (y - v)$, which looks like that "$-$" satisfies a version of the associative law in $\mathbf{R}$, i.e., there exists a map $\varphi : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\varphi(x, z) = (x, -z) = (u, v)$. Thus, we obtain a "twisted" associated law for $(\mathbf{R}, -)$, with the function φ defining the "nature" of the "twisted semigroup" of a particular type.

Kim and Neggers introduced in [2] the notion of $Bin(X)$, the collection of all groupoids defined on a non-empty set X. They showed that $(Bin(X), \Box)$ is a semigroup and the left zero semigroup on X acts as an identity in $(Bin(X), \Box)$. Let $(R, +, \cdot)$ be a commutative ring with identity

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and let $L(R)$ denote the collection of all groupoids $(R, *)$ such that, for all $x, y \in R$, $x * y := ax + by + c$, where $a, b, c \in R$ are fixed constants. Such a groupoid $(R, *)$ is said to be a *linear groupoid*. They showed that $(L(R), \Box)$ is a semigroup with identity. Neggers et al. introduced in [3] the notion of a Q-algebra, and showed that every quadratic Q-algebra $(X, *, e), e \in X$, has of the form $x * y = x - y + e$ when X is a field with $|X| \geqslant 3$.

In this paper, we construct a left-twisted ring which is not a ring on the basis of left-twisted semigroups on a field K, where $char(K) = p$, $K > p$, p is a prime by defining a binary operation $a * b := a^p b$ for all $a, b \in K$, and by defining an associator function φ, where $\varphi(a, b) := (a^{\frac{1}{p}}, b)$. We prove that such a left twisted ring $(K, +, \cdot, 0, 1)$ does not have an identity, but its non-zero element is a pseudo unit of it.

2. Preliminaries

Let $(X, *)$ be a groupoid for which there exists a function $\varphi : X^2 \to X^2$ such that, for all $a, b, c \in X$,

$$a * (b * c) = (u * v) * c, \tag{1}$$

where $\varphi(a, b) = (u, v)$, i.e., $u = u(a, b), v = v(a, b)$ are functions of two variables. We call $(X, *)$ a *left-twisted semigroup* with respect to the map φ. Such a map φ is called an *associator function* of the groupoid $(X, *)$.

Example 2.1. (cf. [1]) Let $\mathbf{R} = (\mathbf{R}, +, \cdot)$ be a real field and $\lambda \neq 0, \mu \in \mathbf{R}$. We define a binary operation "$*$" on $\mathbf{R}$ as follows: $x * y := \lambda x + \mu y$ for any $x, y \in \mathbf{R}$. If we define a map $\varphi(a, b) := (\frac{a}{\lambda}, b)$ and $\mu^2 = \mu$, then $(\mathbf{R}, *)$ is a left-twisted semigroup with respect to φ.

We may think of changing the equation (1) as follows:

$$(a * b) * c = a * (u * v), \tag{2}$$

where $\varphi(a, b) = (u, v)$, i.e., $u = u(a, b), v = v(a, b)$ are functions of two variables. We call $(X, *)$ a *right-twisted semigroup* with respect to φ.

Example 2.2. (cf. [1]) Consider $X := 2^A$ where $A \neq \emptyset$. If we define $a * b := a - b$ for any $a, b \in X$, then $(a * b) * c \neq a * (b * c)$. On the other hand, if we let $\varphi(b, c) := (b \cup c, \emptyset)$, then $(a * b) * c = (a - b) - c = a - (b \cup c)$, and $a * (u * v) = a - (b \cup c - \emptyset) = a - (b \cup c)$, proving that $(X, *)$ is a right-twisted semigroup with respect to φ.

Note that Example 2.2 is a typical example of a BCK-algebra which is also a right-twisted semigroup.

3. Left-twisted rings

An algebraic system $(X, +, *, 0, \varphi)$ is said to be a *left-twisted ring* if

(tr1) $(X, +, 0)$ is an abelian group,

(tr2) $(X, *, \varphi)$ is a left-twisted semigroup,

(tr3) for all $a, b, c \in X$,

$$a * (b + c) = a * b + a * c,$$
$$(a + b) * c = a * c + b * c.$$

Note that we can provide many examples of a left-twisted ring which are not a ring by applying Theorem 4.1 below using the change of prime number p.

Proposition 3.1. *Let* $(X, +, *, 0, \varphi)$ *be a left-twisted ring. Then*

(i) $a * 0 = a = 0 * a$ *for all* $a \in X$,

(ii) $a * (-b) = (-a) * b = -(a * b)$ *for all* $a, b \in X$.

Proof. (i). If $(X, +, *, 0, \varphi)$ is a left-twisted ring, then $a * 0 = a * (0 + 0) = a*0+a*0$ for all $a \in X$. Since $(X, +)$ is an abelian group, we have $a*0 = 0$ for all $a \in X$. Similarly, $0 * a = (0 + 0) * a = 0 * a + 0 * a$ implies $0 * a = 0$ for all $a \in X$.
(ii). By applying (i), we obtain

$$0 = a * 0 = a * (b + (-b)) = a * b + a * (-b).$$

It follows that $a*(-b) = -(a*b)$. Similarly, we obtain $(-a)*b = -(a*b)$. □

When we defined left-(resp., right-) twisted semigroup, we used the associator function $\varphi(a, b) = (u, v)$, i.e., $u = u(a, b), v = v(a, b)$ are functions of two variables. Since u and v are represented by a and b, we may define $u * v := \xi(a, b)$ for some $\xi : X^2 \to X^2$. We denote such a function ξ by $\widehat{\varphi}$, i.e., $u * v = \widehat{\varphi}(a, b)$.

Proposition 3.2. *Let $(X, +, *, 0, \varphi)$ be a left-twisted ring. Then, for all $a, b, c, d \in X$, we have*

$$\widehat{\varphi}(a+b, c) * d = \widehat{\varphi}(a, c) * d + \widehat{\varphi}(b, c) * d. \tag{3}$$

Proof. Given $a, b, c, d \in X$, since X is a left-twisted ring, there exist u, v in X such that $(a+b) * (c+d) = (u * v) * d$ where $\varphi(a+b, c) = (u, v)$. It follows that $u * v = \widehat{\varphi}(a+b, c)$, and hence we obtain

$$(a+b) * (c+d) = \widehat{\varphi}(a+b, c) * d. \tag{4}$$

Now, by applying (tr3), we obtain

$$\begin{aligned}(a+b) * (c * d) &= a * (c * d) + b * (c * d) \\ &= \widehat{\varphi}(a, c) * d + \widehat{\varphi}(b, c) * d. \end{aligned} \tag{5}$$

By (4) and (5), we prove the proposition. □

Corollary 3.3. *Let $(X, +, *, 0, \varphi)$ be a left-twisted ring. If $d (\in X)$ is right cancellative, then*

$$\widehat{\varphi}(a+b, c) = \widehat{\varphi}(a, c) + \widehat{\varphi}(b, c). \tag{6}$$

Proof. Straightforward. □

Corollary 3.4. *Let $(X, +, *, 0, \varphi)$ be a left-twisted ring. Then*

$$\widehat{\varphi}(0, c) * d = 0 \tag{7}$$

for all $c, d \in X$.

Proof. If we let $a = b = 0$ in Proposition 3.2, then

$$\widehat{\varphi}(0, c) * d = [\widehat{\varphi}(0, c) + \widehat{\varphi}(0, c)] * d = \widehat{\varphi}(0, c) * d + \widehat{\varphi}(0, c) * d.$$

This shows that $\widehat{\varphi}(0, c) * d = 0$. □

Let $(X, +, *, 0, \varphi)$ be a left-twisted ring. An element d in X is said to be a *right-non-zero-divisor* if $a * d = 0$ then $a = 0$.

Corollary 3.5. *Let $(X, +, *, 0, \varphi)$ be a left-twisted ring. If d in X is a right-non-zero-divisor, then $\widehat{\varphi}(0, c) = 0$ for all $c \in X$.*

Proof. It follows immediately from Corollary 3.4. □

Proposition 3.6. *Let $(X, +, *, 0, \varphi)$ be a left-twisted ring. If b in X is a right-non-zero-divisor, then $\widehat{\varphi}(a, 0) = 0$ for all $a \in X$.*

Proof. Given $a \in X$, we have $0 = a * 0 = a * (0 * b) = \widehat{\varphi}(a, 0) * b$. Since b is a right-non-zero-divisor, we obtain $\widehat{\varphi}(a, 0) = 0$ for all $a \in X$. □

4. Constructions of a left twisted ring

In this section, we construct a left twisted ring which is not a ring.

Theorem 4.1. *Let* $(K, +, \cdot, 0, 1)$ *be a field where* $char(K) = p$, $|K| > p$, p *is a prime. Define a binary operation* $a * b := a^p b$ *for all* $a, b \in K$, *and define a map* $\varphi(a, b) := (a^{\frac{1}{p}}, b)$. *Then* $(K, +, *, 0, \varphi)$ *is a left-twisted ring which is not a ring.*

Proof. We claim that $(K, *, \varphi)$ is a left-twisted semigroup. Given $a, b, c \in K$, we have $a * (b * c) = a^p(b * c) = a^p(b^p c) = (ab)^p c$. It follows that $(u * v) * c = \widehat{\varphi}(a, b) * c = (a^{\frac{1}{p}} * b) * c = (a^{\frac{1}{p}})^p b * c = ab * c = (ab)^p c = a * (b * c)$, proving the claim.

We claim that $(K, *)$ is not a semigroup. Let $a \notin GF(p)$. Then $a^p \neq a$ and hence $a \neq a^{\frac{1}{p}}$. Hence $a * (b * c) = (u * v) * c = ab * c = (a^{\frac{1}{p}} * b) * c$, which shows that $(K, *)$ is not a semigroup.

Finally, we show that (tr3) condition holds. Given $a, b, c \in K$, we have $a * (b + c) = a^p(b + c) = a^p b + a^p c = a * b + a * c$. Since $char(K) = p$, we obtain $(a + b) * c = (a + b)^p c = (a^p + b^p)c = a^p c + b^p c = a * c + b * c$. Hence $(K, +, *, 0, \varphi)$ is a left-twisted ring which is not a ring. □

Proposition 4.2. *Let* $(X, +, *, 0, \varphi)$ *be a left-twisted ring. Then*

(i) *if* $a * c \neq b * c$ *and* $c \neq 0$, *then* $a = b$,

(ii) *if* $a * c \neq a * d$ *and* $a \neq 0$, *then* $c = d$.

Proof. (i). Suppose $a * c = b * c$. Then $a^p c = b^p c$ and hence $(a - b)^p c = (a^p - b^p)c = 0$. Since $c \neq 0$ and K is a field, we obtain $(a - b)^p = 0$, proving that $a = b$.
(ii). Similar to (i), and we omit it. □

Theorem 4.3. *Let* $(K, +, \cdot, 0, 1)$ *be a field where* $|K| > p$, $char(K) = p$, *where* p *is a prime. Then a left-twisted ring* $(K, +, *, 0, \varphi)$ *does not have an identity.*

Proof. Assume that there exists $e \in K$ such that $a * e = a = e * a$ for all $a \in K$. It follows that $a^p e = a$. Since $a \neq 0$, we obtain $e = a^{1-p} = (\frac{1}{a})^{p-1} = \alpha^{p-1}$ where $\alpha = \frac{1}{a}$. This shows that $|K| = p$, i.e., $K = GF(p)$, a contradiction. Since $e * a = a$ and $a \neq 0$, we have $e^p a = a$, and hence $e^p = 1$. Hence e is a root of an equation $x^p - 1 = 0$. Since $x^p - 1 = 0$ has at most p such elements, $|K| = p$, a contradiction. □

Let $(K,+,*,0,\varphi)$ be a left-twisted ring described in Theorem 4.3. An element $u\in K$ is said to be a *pseudo unit* if $x\in X$, there exist $x_L, x_R\in K$ such that $x_L*u=x, u*x_R=x$, i.e., $(x_L)^pu=x, u^px_R=x$. It follows that $x_L=(\frac{x}{u})^{\frac{1}{p}}$ and $x_R=\frac{x}{u^p}$. Clearly, the identity 1 is a pseudo unit of K. For any $x\in K$, if we take $x_L:=x^{\frac{1}{p}}$ and $x_R:=x$, then 1 becomes a pseudo unit of K.

Proposition 4.4. *Let $(K,+,*,0,\varphi)$ be a left-twisted ring as in Theorem* 4.3. *Let $P(*):=\{u\in K\,|\,u: \text{a pseudo unit of } K\}$. Then $(P(*),*)$ is a subsemigroup of $(K,*)$ containing 1.*

Proof. Clearly, $1\in P(*)$. If $u,v\in X$, then $u*v=u^pv$. Given $x\in K$, we let $\alpha\in K$ such that $\alpha*(u^pv)=x$. It follows that $\alpha=(\frac{x}{u^pv})^{\frac{1}{p}}\in K$. Let $\beta\in K$ such that $(u*v)*\beta=x$. It follows that $(u^pv)^p\beta=x$, and hence $\beta=\frac{x}{(u^pv)^p}\in K$. If we take $x_L:=\alpha, x_R:=\beta$, then $u*v$ is a pseudo unit. □

Theorem 4.5. *Every non-zero element of K as in Theorem* 4.3 *is a pseudo unit of K.*

Proof. Let $u\notin P(*)$ with $u\neq 0$. Then there exists $x\in K$ such that $\alpha*u=x$ or $u*\beta=x$ is impossible for some $\alpha,\beta\in K$. It follows that $\alpha^pu=x$ or $u^p\beta=x$ is impossible. Since $u\neq 0$, we obtain $\alpha=(\frac{x}{u})^{\frac{1}{p}}$ or $\beta=\frac{x}{u^p}$ is impossible, a contradiction, since $\alpha,\beta\in K$. □

References

[1] **P.J. Allen, H.S. Kim and J. Neggers**, *Several types of groupoids induced by two-variable functions*, Springer Plus **5** (2016), 1715 – 1725.

[2] **H.S. Kim and J. Neggers**, *The semigroups of binary systems and some perspectives*, Bull. Korean Math. Soc. **45** (2008), 651 – 661.

[3] **J. Neggers, S.S. Ahn and H.S. Kim**, *On Q-algebras*, Int. J. Math & Math. Sci. **27** (2001), 749 – 757.

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