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A note on 2-prime and n-weakly 2-prime ideals of semirings

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Abstract. We introduce and study the concepts of 2-prime and n-weakly 2-prime (resp. weakly 2-prime) ideals in a commutative semiring. We prove that an integral semidomain S is a valuation semiring if and only if every proper ideal of S is 2-prime and in a principal ideal semidomain the concepts of primary, quasi-primary and 2-prime ideals coincide. We characterize semirings where 2-prime ideals are prime and also characterize semirings where every proper ideal is n-weakly 2-prime (resp. weakly 2-prime).

1. Introduction

A commutative semiring is a commutative semigroup $(S, \cdot)$ and a commutaive monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S.x = x.0_S = 0_S$ for all $x \in S$, both are connected by ring like distribuitivity. We say S is a semiring with identity if the multiplicative semigroup $(S, \cdot)$ has identity element. Throughout this paper, unless otherwise mentioned, all semirings are commutative with identity element $1 \neq 0$, in particular S will denote such a semiring.

A nonempty subset I of S is called an ideal of S if a, $b \in I$ and $r \in S$, then $a+b \in I$ and $ra \in I$. We define radical of an ideal I as $\sqrt{I} = \{x \in S : x^n \in I\}$ and residual of I by $a \in S$ as $(I : a) = \{s \in S : sa \in I\}$. Annihilator of an element a in a semiring S is defined as $\mathrm{Ann}(a) = \{x \in S : ax = 0\}$. For an element x of S, $(x) = Sx$ is the principal ideal of S generated by x. An ideal I of a semiring S is said to be subtructive (or k-ideal) if a, $a+b \in I$, $b \in S$ then $b \in I$. A nonzero element a of S is said to be a zero divisor if $ab = 0$ for some nonzero $b \in S$. For an ideal I of S, $Zd_S(I) = \{s \in S : sr \in I$ for some $r \notin I\}$ and $\sqrt[2]{I} = \{x \in S : x^2 \in I\}$. An ideal I of a semiring S is said to be proper if $I \neq S$ and an ideal generated by nth powers of

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elements of I is denoted as $I_n = (\{x^n : x \in I\})$ [?]. A semiring S is called a semidomain if $ab = ac$ implies $b = c$ for any $b, c \in S$ and for all nonzero $a \in S$. Similarly to the concept of field of fractions in ring theory, one can define the semifield of fractions $F(S)$ of a semidomain S ([5], p. 22). Let A be a multiplicatively closed subset of a semiring S. The relation is defined on the set $S \times A$ by $(s, a) \sim (t, b) \Leftrightarrow xsb = xat$ for some $a \in A$ is an equivalence relation and the equivalence class of $(s, a) \in S \times A$ denoted by s/a. The set of all equivalence classes of $S \times A$ under "$\sim$" denoted by $A^{-1}S$. The addition and multiplication are defined $s/a + t/b = (sb + ta)/ab$ and $(s/a)(t/b) = st/ab$. The semiring $A^{-1}S$ is called quotient semiring S by A. Suppose that S is a commutative semiring, A be a multiplicatively closed subset and I be an ideal. The set $A^{-1}I = \{a/b : a \in I, b \in A\}$ is an ideal of $A^{-1}S$. A proper ideal I of a semiring is said to be prime (resp. weakly prime) if for a, $b \in S$ such that $ab \in I$ (resp. $0 \neq ab \in I$) implies either $a \in I$ or $b \in I$. An ideal I of S is said to be primary if $ab \in I$ for some a, $b \in S$ implies $a \in I$ or $b \in \sqrt{I}$ and quasiprimary if $\sqrt{I}$ is a prime ideal of S. The notion of 2-prime (resp. weakly 2-prime ideal) as a generalisations of prime (resp. weakly prime) ideals in a commutative ring was introduced in [2, 7] and in a commutative semigroup in [6]. Moreover, rings in which concept of 2-prime, primary ideals coincide and rings in which 2-prime ideals are prime has been studied in [13]. These observations tempted us to study 2-prime (resp. weakly 2-prime) ideals in a commutative semiring.

In this article, firstly we define 2-prime ideals in a commutative semiring and state its relations with prime and quasi-primary ideals. Then we prove that every maximal ideal of a semiring without unity is 2-absorbing (Theorem 2.6). We define valuation ideal in a semiring and prove that a semidomain is a valuation semiring if and only if every proper ideal of the semidomain is 2-prime (Theorem 2.11). Also we prove that in a principal ideal semidomain the concepts of 2-prime, primary, quasi-primary ideals coincide (Theorem 2.15). In section 3, we characterize semirings in which 2-prime ideals are prime, defined as 2-P-semiring. In section 4, we define n-weakly 2-prime (resp. weakly 2-prime) ideals in a semiring. Then we characterize semirings in which every proper ideal is weakly 2-prime (Theorem 4.5) (resp. n-weakly 2-prime) (Theorem 4.6) and also studied some further properties of these ideals.

Before going to main work, we discuss some necessary preliminaries.

Theorem 1.1. (cf. [8]) *Let $I \subseteq P$ be ideals of a semiring S, where P is prime. Then the following statements are equivalent:*

(1) *P is a minimal prime ideal of I.*
(2) *For each $x \in P$, there is a $y \notin P$ and a nonnegative integer i such that $yx^i \in I$.*

2. 2-prime ideals

Definition 2.1. A proper ideal I of a semiring S is said to be a 2-*prime ideal* if $xy \in I$ for some $x, y \in S$ implies either $x^2 \in I$ or $y^2 \in I$.

The following lemmas are obvious, hence we omit the proof.

Lemma 2.2.
(1) *Every prime ideal of S is a 2-prime ideal of S.*
(2) *Every 2-prime ideal of S is a quasi-primry ideal of S. Therefore if I is a 2-prime ideal of S, then $\sqrt{I} = P$ is a prime ideal of S.*

Remark 2.3. For a 2-prime ideal I of a semiring S, we refer to the prime ideal $P = \sqrt{I}$ as the associated prime ideal of I and I is referred to as a *P-2-prime ideal of S.*

The following examples show that converses of above lemmas are not true.

Example 2.4. Consider the ideal $I = \{m \in \mathbb{N} \cup \{0\} : m \geqslant 3\}$ in the semiring $S = \{\mathbb{N} \cup \{0\}, +, \cdot\}$. Clearly, I is 2-prime but not a prime ideal of S, since $2.2 \in I$ but $2 \notin I$.

Example 2.5. Consider the ideal $I = (\{X_n^n\}_{n=1}^{\infty})$ in the semiring $S = \mathbb{Z}_2[\{X_i\}_{i=1}^{\infty}]$. Clearly I is quasiprimary ideal of S, since $\sqrt{I}$ is a prime ideal of S. But I is not a 2-prime ideal of S, as $X_6^2 \cdot X_6^4 = X_6^6 \in I$ and neither $(X_6^2)^2 \notin I$ nor $(X_6^4)^2 \notin I$.

If S is a semiring with unity, then every maximal ideal of S is prime ([1], Theorem 11) and hence 2-prime. If S is a semiring without unity then maximal ideal of S need not be prime for example see ([1], Example 12) but there is a relation between maximal and 2-prime ideal of S, as follows

Theorem 2.6. *Let S be semiring without unity and assume maximal ideal exists. Then every maximal ideal of S is a 2-prime ideal of S.*

Proof. Let $xy \in M$ with $x^2 \notin M$ for some $x, y \in S$, where M is a maximal ideal of S. If $y^2 \notin M$, then clearly $x, y \in S - M$. Hence $M + (x) = M + (y) = S$. Since $x \in S$, $x^2 = (p + s_1x + n_1x)(q + s_2y + n_2y)$ for some $p, q \in M$, $s_1, s_2 \in S$ and $n_1, n_2 \in \mathbb{Z}$, implies $x^2 \in M$, a contradiction. Consequently, $y^2 \in M$. Hence M is a 2-prime ideal of S. □

Proposition 2.7. *Let I be an ideal of a semiring S.*

(1) *If I is a 2-prime ideal of S, then there is exactly one prime ideal of S that is minimal over I.*

(2) *If I is a prime ideal of S, then I^2 is a 2-prime ideal of S.*

(3) *An ideal I of S is prime if and only if it is both 2-prime and semi-prime.*

(4) *If I is a 2-prime ideal of S and $J_1, J_2, \ldots, J_n$ are ideals of S such that $\bigcap J_i \subseteq \sqrt{I}$, then $J_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, \ldots, n\}$.*
In particular, if $\bigcap J_i = \sqrt{I}$, then $J_i = \sqrt{I}$ for some $i \in \{1, 2, \ldots, n\}$.

(5) *If I is a P-2-prime ideal of S, then $(I : a^2)$ is a 2-prime ideal of S, for all $a \in S$ such that $a^2 \notin I$.*
In particular $(I : a^2)$ is a P-2-prime ideal of S for all $a \in S - \sqrt{I}$.

(6) *If I is a 2-prime ideal of S and $(I : a) = (I : a^2)$ for all $a \in S - I$, then $(I : a)$ is a 2-prime ideal of S.*

(7) *I is a proper ideal of S and A be a multiplicatively closed subset of S, then the following statements hold.*
   (*i*) *If I is a 2-prime ideal of S such that $I \cap A = \phi$, then $A^{-1}I$ is a 2-prime ideal of $A^{-1}S$.*
   (*ii*) *If $A^{-1}I$ is a 2-prime ideal of $A^{-1}S$ with $Zd_S(I) \cap S = \phi$, then I is a 2-prime ideal of S.*

(8) *If I is a P-primary ideal for some prime ideal P of S such that $P^2 \subseteq I$. Then I is a 2-prime ideal of S.*

Proof. (1). If possible, let J_1 and J_2 be two distinct prime ideal that are minimal over I. Hence there exists $j_1 \in J_1 - J_2$ and $j_2 \in J_2 - J_1$. By Theorem 1.1 there is $a_1 \notin J_1$ and $a_2 \notin J_2$ such that $a_1 j_1^n \in I$ and $a_2 j_2^m \in I$ for some integer $m, n \geqslant 1$. Since $j_1, j_2 \notin I \subseteq J_1 \cap J_2$ and I is 2-prime , hence $a_1^2 \in I \subseteq J_1 \cap J_2$ and $a_2^2 \in I \subseteq J_1 \cap J_2$. Therefore $a_1^2 \in J_1$. Since J_1 is prime so $a_1 \in J_1$, a contradiction. Similarly if $a_2^2 \in J_2$ then $a_2 \in J_2$, a contradiction. Hence there is exactly one prime ideal minimal over I.

(2). Since $I^2 \subseteq I$ for any ideal I of S, it is clear.

(3). If an ideal I is prime, then clearly it is 2-prime and semiprime.

Conversely, let $ab \in I$ for some a, $b \in S$. Since I is 2-prime we have $a^2 \in I$ or $b^2 \in I$, which implies $a \in I$ or $b \in I$, since I is semprime also. Consequently I is a prime ideal of S.

(4). Let $J_i \nsubseteq \sqrt{I}$ for all $i \in \{1, 2, \ldots, n\}$. Then there exists $a_i \in J_i$ but $a_i \notin \sqrt{I}$ for all $i \in \{1, 2, \ldots, n\}$. Let $x = a_1 a_2 \cdots a_n$. Then $x \in \bigcap J_i$ but $x \notin \sqrt{I}$, since $\sqrt{I}$ is a prime ideal of S, a contradiction . Hence $J_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, \ldots, n\}$.
Again if, $\bigcap J_i = \sqrt{I}$, then $\sqrt{I} \subseteq J_i$ for all $i \in \{1, 2, \ldots, n\}$. Hence $J_i = \sqrt{I}$ for some $i \in \{1, 2, \ldots, n\}$.

(5). Let $xy \in (I : a^2)$ with $x^2 \notin (I : a^2)$ for x, $y \in S$. Then $xya^2 = (xa)(ya) \in I$. Hence $(ya)^2 = y^2a^2 \in I$, since I is a 2-prime ideal of S and $x^2a^2 \notin I$. Consequently $(I : a^2)$ is a 2-prime ideal of S.
Again let $a \in S - P$ and $x \in (I : a^2)$. Then $a^2x \in I \subseteq P$. Hence $x^2 \in I$, since $a \notin P$ and I is a 2-prime ideal of S. Thus $I \subseteq (I : a^2) \subseteq P$, which implies $P = \sqrt{I} \subseteq \sqrt{(I : a^2)} \subseteq \sqrt{P} = P$. Consequently $(I : a^2)$ is a P-2-prime ideal of S.

(6). Clearly follows from (5).

(7). (i) Let $(a/s)(b/t) \in A^{-1}I$ for some a, $b \in S$ and s, $t \in A$. Then there exists $u \in A$ such that $abu \in I$. Then $a^2 \in I$ or $b^2u^2 \in I$, since I is a 2-prime ideal of S. If $a^2 \in I$, then $(a/s)^2 = (ua^2/us^2) \in A^{-1}I$ and if $b^2u^2 \in I$ then $(b/s)^2 = (b^2u^2/s^2u^2) \in A^{-1}I$. Therefore $A^{-1}I$ is a 2-prime ideal of $A^{-1}S$.

(ii) Let $xy \in I$ for some x, $y \in S$. Then $\frac{x}{1}\frac{y}{1} \in A^{-1}I$ implies $\frac{x^2}{1} \in A^{-1}I$ or $\frac{y^2}{1} \in A^{-1}I$. Hence $ax^2 \in I$ or $by^2 \in I$ for some a, $b \in S$. Since $A \cap Zd_S(I) = \phi$, we have either $x^2 \in I$ or $y^2 \in I$, as desired.

(8). Let $ab \in I$ for some a, $b \in S$, where I is a P-primary ideal of S such that $P^2 \subseteq I$. Then either $a \in I$ or $b \in \sqrt{I} = P$. If $a \in I$ then $a^2 \in I^2$ and if $b \in P$ then $b^2 \in P^2 \subseteq I$. Consequently I is a 2-prime ideal of S. $\square$

Theorem 2.8. *Let P be a proper ideal of a semiring S. Then the following statements are equivalent:*

(1) *P is a 2-prime ideal of S.*

(2) *for any ideals J, K of S with $JK \subseteq P$ implies either $J_2 \subseteq P$ or $K_2 \subseteq P$, where $J_2 = (\{x^2 : x \in J\})$ and $K_2 = (\{k^2 : k \in K\})$.*

(3) *For every $s \in S$, either $(s) \subseteq (P : s)$ or $(P : s) \subseteq \sqrt[2]{P}$.*

(4) *For any ideals A and B of S with $AB \subseteq P$ implies either $A_2 \subseteq P$ or $B \subseteq \sqrt[2]{P}$.*

(5) *For every $s \in S$, either $s^2 \in P$ or $(P : s)_2 \subseteq P$.*

Proof. (1) $\Rightarrow$ (2). Let P be a 2-prime ideal of a semiring S and $JK \subseteq P$ for some ideal J, K of S with $J_2 \nsubseteq P$. Then there exists an element $p \in J$ such that $p^2 \notin P$. Since $pK \subseteq P$ and $p^2 \notin P$, we conclude $K_2 \subseteq P$ (Proposition 2.7 ????).

(2) $\Rightarrow$ (1). Let $ab \in P$ for some $a,b \in S$ and $a^2 \notin P$. Let $J = (a)$ and $K = (b)$. Then $JK \subseteq P$ and $J_2 \nsubseteq P$, otherwise $a^2 \in P$. Hence $K_2 \subseteq P$ implies $b^2 \in P$. Consequently, P is a 2-prime ideal of S.

(1) $\Rightarrow$ (3) Let $s \in S$. If $s^2 \in P$, then $s \in (P : s)$ implies $(s) \subseteq (P : s)$. Let $s^2 \notin P$ and $r \in (P : s)$ for some $r \in S$. Hence $rs \in P$ implies $r^2 \in P$, since P is 2-prime and $s^2 \notin P$. Consequently, $(P : s) \subseteq \sqrt[2]{P}$.

(3) $\Rightarrow$ (4). Let $AB \subseteq P$ for some ideals A, B of S. Let $B \nsubseteq \sqrt[2]{P}$. Then there exists $b \in B - \sqrt[2]{P}$ and $ab \in P$ for all $a \in A$. Since $b \in (P : a) - \sqrt[2]{P}$, we have $(P : a) \nsubseteq \sqrt[2]{P}$. Hence by hypothesis, $(a) \subseteq (P : a)$ implies $a^2 \in P$. Consequently $A_2 \subseteq P$.

(4) $\Rightarrow$ (5). Let $s \in S$. If $s^2 \in P$, there is nothing to prove. So let $s^2 \notin P$ and $A = (P : s)$, $B = (s)$. Then $AB = (P : s)(s) \subseteq P$. Since $B \nsubseteq \sqrt[2]{P}$, we have $A_2 = (P : s)_2 \subseteq P$.

(5) $\Rightarrow$ (1). Let $xy \in P$ with $x^2 \notin P$ for some $x, y \in S$. Then $y \in (P : x)$. Hence by hypothesis, $y^2 \in (P : s)_2 \subseteq P$, as desired. □

The concept of valuation semiring has been defined by P. Nasehpour in [10], here we define valuation ideal of a semiring, as follows

Definition 2.9. Let S be a semidomain and K be its semifield of fractions. Then an ideal I in S is a *valuation ideal* if I is the intersection of S with an ideal of a valuation semiring S_v containing S. Moreover if v is the corresponding M-valuation we say I is a *valuation ideal associated with the M-valuation v* or I is a *v-ideal.*

Lemma 2.10. *Let v be an M-valuation on K and I an ideal of a semidomain S. Then the followings are equivalent*

(1) *I is a valuation ideal.*

(2) *For each $x \in S$, $y \in I$, the inequality $v(x) \geqslant v(y)$ implies $x \in I$.*

(3) *I is of the form $I = S_v I \cap S$.*

Proof. The proof is similar to ([15], page 340). □

Theorem 2.11. *Let S be a semidomain. Then the following are equivalent*

(1) *Every ideal of S is 2-prime.*

(2) *Every principal ideal of S is* 2*-prime.*

(3) *S is a valuation semiring.*

Proof. (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (3). Let $x \in K - \{0\}$, where K is the semifield of fractions of S. Then $x = \frac{a}{b}$ for some a, $b \in S - \{0\}$. Let $I = (ab)$ be a principal ideal of S so 2-prime and since $ab \in (ab) = I$, we have $a^2 \in I$ or $b^2 \in I$. If $a^2 \in I$, then there exists an element $c \in S$ such that $a^2 = cab$, hence $x = \frac{a}{b} = c \in S$. Similarly, if $b^2 \in I$, we have $x^{-1} \in S$. Consequently, S is a valuation semiring ([10], Theorem 2.4).

(3) $\Rightarrow$ (1). Let I be a v-ideal on S where v is a valuation on S. Let $xy \in I$ for some x, $y \in S$. If $v(x) \geqslant v(y)$, we get $v(x^2) \geqslant v(xy)$ and as I is a v-ideal we have $x^2 \in I$. Similarly, $v(y) \geqslant v(x)$ implies $y^2 \in I$. Consequently I is a 2-prime ideal of S. $\square$

The following lemmas are obvious, hence we omit the proof

Lemma 2.12. *Let S be a semidomain and a, $b \in S - \{0\}$. Then a and b are associates if and only if $(a) = (b)$.*

Lemma 2.13. *Let S be a semidomain and $p \in S - \{0\}$. Then p is an irreducible element of S if and only if (p) is a maximal ideal of S.*

Lemma 2.14. *Let I be a P-primary ideal of a semiring S. Then P is the unique minimal prime ideal of I in S.*

Proof. Let Q be another minimal prime of I in S. Then $I \subseteq Q$ implies $P = \sqrt{I} \subseteq \sqrt{Q} = Q$. Hence P is the unique minimal prime ideal of I in S. $\square$

Theorem 2.15. *Let I be a proper ideal of a principal ideal semidomain S. Then the followings are equivalent*

(1) *I is a quasi-primary ideal of S.*

(2) *I is a primary ideal of S.*

(3) *I is of the form (p^n), where n is a postitive integer and $p = 0$ or an irreduicible element of S.*

(4) *I is a* 2*-prime ideal of S.*

Proof. (1) $\Rightarrow$ (2). Since every nonzero prime ideal of a principal ideal semidomain S is a maximal ideal ([11], Proposition 2.1), it follows claerly from ([1], Theorem 40).

(2) $\Rightarrow$ (1). It is obvious.

(2) $\Rightarrow$ (3). Let I be a nonzero primary ideal of S. Then $I = (a)$ for some nonzero nonunit element $a \in S$. Since every principal ideal semidomain is a unique factorization semidomain ([11], Theorem 3.2), a can written as a product of irreduicible elements of S. If a were divisible by two irreduicible elements x and y of S, which are not associates, then by Lemma 2.12 and 2.13 (x) and (y) would be distinct maximal ideal of S, they would both minimal prime ideal of (a), which contradicts Lemma 2.14. Hence $I = \{(p^n) : p = 0$ or p is an irreduicible elements of S and $n \in \mathbb{N}\}$.

(3) $\Rightarrow$ (2). Since S is a semidomain, $\{0\}$ is prime and hence primary. Let p be an irreduicible element of S and $n \in \mathbb{N}$, then by Lemma 2.13 (p^n) is a power of a maximal ideal so is a primary ideal of S ([1], Theorem 40).

(3) $\Leftrightarrow$ (4) The proof is similar as that of ([13], Theorem 2.3). □

Example 2.16. Let I be an ideal of a von neuman regular semiring S. Then $I = I^2 = \sqrt{I}$ ([14], Proposition 1). Hence the concepts of prime, primary, 2-prime and quasiprimary ideal coincide in a regular semiring S.

If R and S are semirings then a function $f : R \longrightarrow S$ is said to be a morphism of semirings ([4], p. 105) if (i)) $f(0_R) = 0_S$, (ii) $f(1_R) = 1_S$ and (iii) $f(r_1 + r_2) = f(r_1) + f(r_2)$ and $f(r_1r_2) = f(r_1)f(r_2)$ for all r_1, $r_2 \in R$.

Theorem 2.17. *Let $f : S_1 \to S_2$ be a morphism of semirings. Then the following statements holds:*

(1) *If J is a 2-prime ideal of S_2, then $f^{-1}(J)$ is a 2-prime ideal of S_1.*

(2) *If f is onto steady morphism such that $kerf \subseteq I$ and I is a 2-prime k-ideal of S_1, then $f(I)$ is a 2-prime k-ideal of S_2.*

Proof. (1). Let $ab \in f^{-1}(J)$ for some a, $b \in S_1$. Then $f(ab) \in J$, hence $f(a^2) \in J$ or $f(b^2) \in J$, since f is a morphism and J is a 2-prime of S_2. Therefore $a^2 \in f^{-1}(J)$ or $b^2 \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a 2-prime ideal of S_1.

(2). Let $xy \in f(I)$ for some x, $y \in S_2$. Then there exists a, $b \in S_1$ such that $f(a) = x$ and $f(b) = y$. Then $xy = f(a)f(b) = f(ab) \in f(I)$. Hence $f(ab) = f(r)$ for some $r \in I$. So we have $ab + s = r + t$ for some s, $t \in I$, since f is steady. Hence $ab \in I$, since $\ker f \subseteq I$ and I is a k-ideal of S_1. Hence either $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S_1. Thus either $f(a^2) \in f(I)$ or $f(b^2) \in f(I)$. Consequently, $f(I)$ is a 2-prime k-ideal of S_2. □

Corollary 2.18. *If $S \subseteq R$ is an extension of semiring and I is a 2-prime ideal of R, then $I \cap S$ is a 2-prime ideal of S.*

Theorem 2.19. *Let $S = S_1 \times S_2$ and $I = I_1 \times I_2$, where I_i are ideals of S_i for $i = 1, 2$. Then the following are equivalent*

(1) *I is a 2-prime ideal of S.*

(2) *$I_1 = S_1$ and I_2 is a 2-prime ideal of S_2 or $I_2 = S_2$ and I_1 is a 2-prime ideal of S_1.*

Proof. (1) $\Rightarrow$ (2). Let I be a 2-prime ideal of S. Then $\sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}$, is a prime ideal of S. Hence either $I_1 = S_1$ or $I_2 = S_2$. Let $I_2 = S_2$ and $ab \in I_1$ for some a, $b \in S_1$. Then $(a, 1)(b, 1) \in I$. Hence $(a, 1)^2 \in I$ or $(b, 1)^2 \in I$, since I is a 2-prime ideal of S. This implies $a^2 \in I_1$ or $b^2 \in I_1$. Consequently, I_1 is a 2-prime of S_1. Similarly, if $I_1 = S_1$, we can show that I_2 is a 2-prime ideal of S_2.

(2) $\Rightarrow$ (1). Assume $I_1 = S_1$ and I_2 is a 2-prime ideal of S_2. Let $(a, x)(b, y) \in I$ for some a, $b \in S_1$ and x, $y \in S_2$. Then $xy \in I_2$ and this implies $x^2 \in I_2$ or $y^2 \in I_2$. Hence $(a, x)^2 \in I$ or $(b, y)^2 \in I$, as desired. In a similar way, one can prove the other case. $\square$

Corollary 2.20. *Let $S = S_1 \times S_2 \times \ldots \times S_n$ and $I = I_1 \times I_2 \times \ldots \times I_n$, where I_i are ideals of S_i and $n \in \mathbb{N}$. Then the following are equivalent*

(1) *I is a 2-prime ideals of S.*

(2) *I_i is a 2-prime ideal of S_i for some $i \in \{1, 2, \ldots, n\}$ and $I_j = S_j$ for all $j \neq i$.*

Proof. By using Theorem 2.19 and induction on n, the proof is straightforward. $\square$

Let S be a semiring and M an S-semimodule. Then $S \times M$ equipped with the following two operations $(s_1, m_1) + (s_2, m_2) = (s_1 + s_2, m_1 + m_2)$ and $(s_1, m_1)(s_2, m_2) = (s_1 s_2, s_1 m_2 + s_2 m_1)$, forms a semiring, denoted by $S \tilde{\bigoplus} M$, is called the expectation semiring of the S-semimodule M ([12], Proposition 1.1).

If I is an ideal of S and N is an S-subsemimodule of M, then $I \tilde{\bigoplus} N$ is an ideal of $S \tilde{\bigoplus} M$ if and only if $IM \subseteq N$ ([12], Theorem 1.6(2)).

Theorem 2.21. *Let M be a S-semimodule, I a proper ideal of S and $N \neq M$ an S-subsemimodule of M. Then*

(1) *If $I \tilde{\bigoplus} N$ is a 2-prime ideal of $S \tilde{\bigoplus} M$, then I is a 2-prime ideal of S.*

(2) *If the ideal I of S is 2-prime and $\sqrt[2]{I}M \subseteq N$, then $I\tilde{\bigoplus}N$ is a 2-prime ideal of $S\tilde{\bigoplus}M$.*

Proof. (1). Let $ab \in I$ with $a^2 \notin I$ for some a, $b \in S$. Then $(a,0)(b,0) \in I\tilde{\bigoplus}N$ while $(a,0)^2 \notin I\tilde{\bigoplus}N$. Hence $(b,0)^2 \in I\tilde{\bigoplus}N$, since $I\tilde{\bigoplus}N$ is a 2-prime ideal of $S\tilde{\bigoplus}M$. Consequently, $b^2 \in I$, as desired.

(2). Let $(a,m)(b,n) \in I\tilde{\bigoplus}N$ for some a, $b \in S$, m, $n \in M$. This implies $ab \in I$ implies $a^2 \in I$ or $b^2 \in I$. If $a^2 \in I$, then $am \in \sqrt[2]{I}M \subseteq N$ and this yields $(a,m)^2 = (a^2, 2am) \in I\tilde{\bigoplus}N$. Again if $b^2 \in I$ we have $(b,m)^2 \in I\tilde{\bigoplus}N$. Consequently, $I\tilde{\bigoplus}N$ is a 2-prime ideal of $S\tilde{\bigoplus}N$. □

3. 2-P-semiring

Definition 3.1. A semiring S is said to be a 2-P-*semiring* if 2-prime ideals of S are prime.

Example 3.2. Clearly every idempotent semiring is a 2-P-semiring.

Theorem 3.3. *A semiring S is 2-P-semiring if and only if one of the following conditions holds:*

(1) *2-prime ideals are semiprime.*

(2) *Prime ideals are idempotent and every 2-prime ideal is of the form A^2, where A is a prime ideal of S.*

Proof. (1). If S is a 2-P-semiring, clearly 2-prime ideals are semiprime. Converse follows easily from Proposition 2.7(3).

(2). Let P be a prime ideal of a 2-P-semiring S. Then P^2 is a prime ideal of S (Proposition 2.7(2)) and hence $P \subseteq P^2$. Clearly $P^2 \subseteq P$. Therefore prime ideals of S are idempotent. Again, let I be a 2-prime ideal of S. Then I is prime and hence $I = I^2$.

Conversely, let I be a 2-prime ideal of S. Then $I = P^2 = P$ for some prime ideal P of S. Consequently, S is a 2-P semiring. □

Lemma 3.4. *Let (S, M) be a local semiring. Then for every prime ideal I of S, IM is a 2-prime ideal of S. Furthermore, IM is prime if and only if $IM = I$*

Proof. Let $xy \in IM \subseteq I$. Then either $x \in I$ or $y \in I$, since I is a prime ideal of S. Let $x \in I$ implies $x^2 \in IM$, since $I \subseteq M$. Hence IM is a 2-prime ideal of S. □

Definition 3.5. Let I be an ideal of a semiring S. We define a 2-prime ideal P to be a *minimal* 2*-prime ideal over* I if there is not a 2-prime ideal K of S such that $I \subseteq K \subset P$. We denote the set of minimal 2-prime ideals over I by 2-Min$_S(I)$.

Theorem 3.6. *Let S be a subtructive semiring with unique maximal ideal M such that $(\sqrt{I})^2 \subseteq I$ for every 2-prime ideal I of S. Then the following statements are equivalent.*

(1) *S is a 2-P-semiring.*

(2) *If P is the minimal prime ideal over a 2-prime ideal I, then $IM = P$.*

(3) *For every prime ideal P of S, 2-Min$_S(P^2) = \{P\}$.*

Proof. (1) $\Rightarrow$ (2). Let P be the minimal prime ideal over a 2-prime ideal I of a 2-P-semiring S. Then clearly $IM = P$ (Lemma 3.4).

(2) $\Rightarrow$ (1). Let I be a 2-prime ideal of a subtructive semiring S with unique maximal ideal M and P is the minimal prime ideal over I such that $IM = P$. Then $I \subseteq P = IM \subseteq I \cap M = I$ implies $I = P$. Hence S is a 2-P-semiring.

(2) $\Rightarrow$ (3) Let P be a prime ideal of S and I be a 2-prime ideal of S such that $I \in$ 2-Min$_S(P^2)$. Let J be a prime ideal of S such that $I \subseteq J \subseteq P$. Clearly, $P^2 \subseteq I \subseteq J \subseteq P$. Let $a \in P$ then $a^2 \in P^2$. Therefore $a^2 \in J$ implies $a \in J$, since J is prime. Hence $J = P$. Now by hypothesis, $IM = P$ implies $P = IM \subseteq I \subseteq P$. Consequently, 2-Min$_S(P^2) = \{P\}$.

(3) $\Rightarrow$ (2). Let P is the minimal prime ideal over a 2-prime ideal I of S. Then $\sqrt{I} = P$. Hence by hypothesis $P^2 \subseteq I \subseteq P$. Therefore 2-Min$_S(P^2) = \{P\}$. Clearly $I = P$ implies IM is 2-prime (Lemma 3.4). Now $P^2 \subseteq PM \subseteq P$ so $IM = PM = P$. $\square$

Theorem 3.7. *Let $S \subseteq R$ be an extension of semiring and spec(S)=spec(R), where spec(S) and spec(R) denotes set of all prime ideals of S and R respectively. If S is a 2-P-semiring, then R is 2-P-semiring.*

Proof. Let I be a 2-prime ideal of R. Then $\sqrt{I} = P \in spec(R) = spec(S)$. Clearly $I \subseteq P$. Also $I \cap S$ is a 2-prime ideal of S (Corollary 2.18), hence prime, since S is 2-P-semiring. Therefore $I \cap S = \sqrt{I \cap S} = P$ and $P^2 \subseteq I \cap S$. Let $x \in P$. Then $x^2 \in P^2 \subseteq I \cap S \in spec(A)$. Hence $x \in I \cap S \subseteq I$. Consequently, $I = P$, as desired. $\square$

4. n-weakly 2-prime ideal

Definition 4.1. A proper ideal I of a semiring S is said to be *n-weakly 2-prime* if for a, $b \in S$, $ab \in I - I^n$ implies that $a^2 \in I$ or $b^2 \in I$.

Definition 4.2. A proper ideal I of a semiring S is said to be a *weakly 2-prime ideal* of S if $0 \neq xy \in I$ for some x, $y \in S$ implies $x^2 \in I$ or $y^2 \in I$.

The following lemmas are obvious, hence we omit the proof.

Lemma 4.3.

(1) *Every 2-prime ideal of S is a weakly 2-prime ideal of S.*

(2) *Every weakly prime ideal of S is a weakly 2-prime ideal of S.*

(3) *Every weakly 2-prime ideal of S is a n-weakly 2-prime ideal of S.*

(4) *An n-weakly 2-prime is a $(n-1)$-weakly 2-prime ideal, for all $n \geqslant 3$.*

Proposition 4.4. *Let I be a subtructive ideal of a semiring S. Then*

(1) *If I is weakly 2-prime but not a 2-prime ideal of S, then*
  (i) $I^2 = 0$.
  (ii) $\sqrt{I} = \sqrt{0}$.

(2) *Let (S, M) be a local semiring with $M^2 = 0$. Then every proper subtructive ideal of S is a weakly prime and hence weakly 2-prime ideal of S.*

(3) *Let P be a weakly prime ideal of S and Q be an ideal of S containg P, then PQ is a weakly 2-prime ideal of S. In particular, for every weakly prime ideal P of S, P^2 is a weakly 2-prime ideal of S.*

(4) *$\sqrt{I}$ is a prime (resp. weakly prime) ideal of S if and only if $\sqrt{I}$ is a 2-prime (resp. weakly 2-prime) ideal of S.*

(5) *Let I be a n-weakly 2-prime ideal of S and A be a multiplicatively closed subset of S with $A \cap I = \phi$ and $A^{-1}I^n \subseteq (A^{-1}I)^n$. Then $A^{-1}I$ is a n-weakly 2-prime ideal of $A^{-1}S$.*

Proof. (1)(i). We first show that if $ab = 0$ for some a, $b \in S - I$, then we have $aI = bI = 0$. Let $ai \neq 0$ for some $i \in I$. Then $0 \neq a(b + i) \in I$. Since I is a subtructive weakly 2-prime ideal of S, either $a^2 \in I$ or $b^2 \in I$, a contradiction. Therefore $aI = 0$. Similarly we can show $Ib = 0$. Now let $xy \neq 0$ for some x, $y \in I$ and $ab = 0$ for some a, $b \notin I$. Then we have

$(a+x)(b+y) = xy \neq 0$. Since I is subtructive weakly 2-prime ideal of S, either $a^2 \in I$ or $b^2 \in I$, a contradiction. Hence $I^2 = 0$.

(ii). By (i), $I^2 = \{0\}$. So we have $I \subseteq \sqrt{0}$ implies $\sqrt{I} \subseteq \sqrt{0}$. Also we have $\sqrt{0} \subseteq \sqrt{I}$. Therefore $\sqrt{I} = \sqrt{0}$.

(2). Let I be a proper ideal of a local semiring (S, M) such that $M^2 = 0$ and $0 \neq ab \in I$ for some a, $b \in S$. Then either $a \in M$ or $b \in M$ but both a, b does not belongs to M, otherwise $ab \in M^2 = 0$, a contradiction. Hence a or b must be semi-unit, let a be a semi-unit of S. Then there exists p, $q \in S$ such that $1 + pa = qa$ implies $b + pab = qab \in I$. Also $pab \in I$ implies $b \in I$, since I is a subtructive ideal of S. Similarly if b is a semi-unit then $a \in I$. Consequently I is a weakly 2-prime ideal of S, as desired.

(3). Let $0 \neq ab \in PQ$ for some a, $b \in I$. Since $PQ \subseteq P$ and P is weakly prime ideal of S, we have either $a \in P \subseteq Q$ or $b \in P \subseteq Q$. Hence either $a^2 \in PQ$ or $b^2 \in PQ$. Consequently, PQ is a weakly 2-prime ideal of S, in particular, P^2 is a weakly 2-prime ideal of S.

(4). Since $\sqrt{\sqrt{I}} = I$ for any ideal I of S, it is clear.

(5). Let a, $b \in S$ and x, $y \in A$ such that $\frac{a}{x}\frac{b}{y} \in A^{-1}I - (A^{-1}I)^n$. Then there exists $u \in A$ such that $uab \in I$. Again $vab \notin I^n$ for any $v \in A$ because if $vab \in I^n$ then $\frac{a}{x}\frac{b}{y} \in A^{-1}I \subseteq (A^{-1}I)^n$, a contradiction. So $abu \in I - I^n$, implies $a^2 \in I$ or $b^2u^2 \in I$, since I is a n-weakly 2-prime ideal of S. Hence $(\frac{a}{x})^2 \in A^{-1}I$ or $(\frac{b}{y})^2 \in A^{-1}I$. Thus $A^{-1}I$ is a n-weakly 2-prime ideal of $A^{-1}S$. □

The following is a characterization of a semiring in which every proper ideal is weakly 2-prime.

Theorem 4.5. *Let S be a semiring. Then every proper ideal of S is weakly 2-prime if and only if $(a^2) \subseteq (ab)$ or $(b^2) \subseteq (ab)$ or $ab = 0$, for any a, $b \in S$ such that $(ab) \neq S$.*

Proof. Let every proper ideal of a semiring S is weakly 2-prime and a, $b \in S$ such that $(ab) \neq S$. If $ab \neq 0$, then $0 \neq ab \in (ab)$ and (ab) is weakly 2-prime, hence $a^2 \in (ab)$ or $b^2 \in (ab)$. Consequently, $(a^2) \subseteq (ab)$ or $b^2 \subseteq (ab)$.

Conversely, let I be a proper ideal of a semiring S and $0 \neq ab \in I$ for some a, $b \in S$. Then $0 \neq ab \in (ab) \subseteq I$ implies $a^2 \in (a^2) \subseteq (ab) \subseteq I$ or $b^2 \in (b^2) \subseteq (ab) \subseteq I$. Hence, I is weakly 2-prime ideal of S, as desired. □

Theorem 4.6. *Let I be a subtructive ideal of a semiring S with $I^2 \nsubseteq I^n$. Then I is a 2-prime ideal of S if and only if I is a n-weakly 2-prime ideal.*

Proof. Let I be a subtructive n-weakly 2-prime ideal of S such that $I^2 \subseteq I^n$ and $ab \in I$ for some a, $b \in S$. If $ab \notin I^n$, then $a^2 \in I$ or $b^2 \in I$, since I is n-weakly 2-prime. So we assume $ab \in I^n$. First we suppose $aI \nsubseteq I^n$. Then for some $i \in I$, $ai \notin I^n$ implies $a(b+i) \notin I^n$, since I is subtructive and $ab \in I^n$. Hence $a(b+i) \in I - I^n$ implies $a^2 \in I$ or $b^2 \in I$. So we can assume $aI \subseteq I^n$. Similarly we can assume $Ib \subseteq I^n$. Now since $I^2 \nsubseteq I^n$, there exists a_1, $b_1 \in I$ such that $a_1 b_1 \notin I^n$. Hence $(a+a_1)(b+b_1) \in I - I^n$ because if $(a+a_1)(b+b_1) \in I^n$ then $a_1b_1 = (a+a_1)(b+b_1) = (ab+aa_1+bb_1+a_1b_1) \in I^n$, which contradicts that $a_1b_1 \notin I^n$. Hence $(a+a_1)^2 \in I$ or $(b+b_1)^2 \in I$, since I is n-weakly 2-prime ideal of S. Therefore $a^2 \in I$ or $b^2 \in I$, since I is subtructive ideal of S, as desired. The other part is obvious. □

Proposition 4.7. *Let S be a semiring and $x \in S$. Then the following statements holds.*

(1) *If Sx is a subtructive ideal of S and $Ann(x) \subseteq Sx$. Then Sx is a 2-prime ideal of S if and only if Sx is a n-weakly 2-prime ideal.*

(2) *If Sx is a subtructive ideal of S and $Ann(x) \subseteq xI$ for some subtrative ideal I of S. Then xI is a 2-prime ideal of S if and only if xI is a n-weakly 2-prime ideal of S.*

Proof. (1). Let Sx be a subtructive n-weakly 2-prime ideal of S and $ab \in Sx$ for some a, $b \in S$. If $ab \notin (Sx)^n$, then I is 2-prime ideal, since Sx is n-weakly 2-prime ideal of S. So we assume $ab \in (Sx)^n$. Clearly $a(b+x) \in Sx$. If $a(b+x) \notin (Sx)^n$, then $a^2 \in Sx$ or $b^2 \in Sx$, since Sx is n-weakly 2-prime ideal of S. So we assume $a(b+x) \in (Sx)^n$. Since $ab \in (Sx)^n$ and $(Sx)^n$ is subtructive, we have $ax \in (Sx)^n$ implies $ax = tx$ for some $t \in (Sx)^n \subseteq Sx$. Hence $a - t \subseteq Ann(x) \subseteq Sx$ implies $a^2 \in Sx$. Consequently, Sx is a 2-prime ideal of S. The converse part is obvious.

(2). Let xI be a subtructive n-weakly 2-prime ideal of S and $ab \in xI$ for some a, $b \in S$. If $ab \in (xI)^n$, then xI is a 2-prime ideal of S. Hence we assume $ab \in (xI)^n$. Clearly, $a(b+x) \in xI$. If $a(b+x) \notin (xI)^n$, then $a^2 \in xI$ or $b^2 \in xI$, since xI is subtructive n-weakly 2-prime ideal of S. Hence xI is n-weakly 2-prime ideal of S. Now suppose $a(b+x) \in (xI)^n$. Since $ab \in (xI)^n$, we have $ax = yx$ for some $y \in (aI)^n \subseteq aI$. This implies $(a-y)x = 0$. Hence $a-y \in Ann(x) \subseteq xI$. Therefore $a^2 \in xI$. Consequently, xI is a 2-prime ideal of S. □

Definition 4.8. A proper ideal I of a semiring S is said to be a *strong ideal*, if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$.

Proposition 4.9. *Let $f : S \to S_1$ be an epimorphism of semirings such that $f(0) = 0$ and I be a subtructive strong ideal of S. Then*

(1) *If I is a weakly 2-prime ideal of S such that $kerf \subseteq I$, then $f(I)$ is a weakly 2-prime ideal of S_1.*

(2) *If I is a 2-prime ideal of S such that $kerf \subseteq I$, then $f(I)$ is a 2-prime ideal of S_1.*

Proof. (1). Let $a_1, b_1 \in S_1$ be such that $0 \neq a_1b_1 \in f(I)$. So there exists an element $p \in I$ such that $0 \neq a_1b_1 = f(p)$. Also there exist $a, b \in S$ such that $f(a) = a_1$, $f(b) = b_1$, since f is an epimorphism. Since I is a strong ideal of S and $p \in I$, there exists $q \in I$ such that $p + q = 0$. This implies $f(p + q) = 0$, that is, $f(ab + q) = 0$, implies $ab + q \in kerf \subseteq I$, Hence $0 \neq ab \in I$, as I is a subtructive ideal of S and if $ab = 0$, then $f(p) = 0$, a contradiction. Thus $a^2 \in I$ or $b^2 \in I$, since I is a weakly 2-prime ideal of S. Thus $a_1^2 \in f(I)$ or $b_1^2 \in f(I)$. Hence, $f(I)$ is a weakly 2-prime ideal of S.

(2). It is clear from (1). □

Proposition 4.10. *Let S_1 and S_2 be two semirings and I be a proper ideal of S_1. Then the followings are equivalent:*

(1) *I is a 2-prime ideal of S_1.*

(2) *$I \times S_2$ is a 2-prime ideals of $S_1 \times S_2$.*

(3) *$I \times S_2$ is a weakly 2-prime ideals of $S_1 \times S_2$.*

Proof. (1) $\Rightarrow$ (2). Let $(a_1, b_1)(c_1, d_1) \in I \times S_2$ for some $(a_1, b_1) \in S_1 \times S_2$ and $(c_1, d_1) \in S_1 \times S_2$. Then $(a_1c_1, b_1d_1) \in I \times S_2$ implies $a_1^2 \in I$ or $c_1^2 \in I$, since I is a 2-prime ideal of S_1. Now if $a_1^2 \in I$, then $(a_1, b_1)^2 = (a_1^2, b_1^2) \in I \times S_2$. Similarly if $c_1^2 \in I$, then $(c_1, d_1)^2 = (c_1^2, d_1^2) \in I \times S_2$. Consequently, $I \times S_2$ is a 2-prime ideal of $S_1 \times S_2$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1). Let $ab \in I$ for some $a, b \in S$. Then $(0, 0) \neq (a, 1)(b, 1) \in I \times S_2$. This implies $(a^2, 1) \in I \times S_2$ or $(b^2, 1) \in I \times S_2$, since $I \times S_2$ is a 2-prime ideal of $S_1 \times S_2$. Hence, $a^2 \in I$ or $b^2 \in I$, as desired. □

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