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# Projective finitely supported $M$-sets

Khadijeh Keshvardoost and Mahdieh Haddadi

**Abstract.** The purpose of this paper is to provide simple characterizations of the projective objects in the category of finitely supported $M$-sets. To do so, first, we introduce the notion of zero-retraction monoid and then characterize projective finitely supported $M$-sets where $M$ contains a zero-retraction monoid.

## 1. Introduction

Take $\mathbb{D}$ to be a countable infinite set. A permutation over $\mathbb{D}$ is said to be finite if it changes only a finite number of elements of $\mathbb{D}$. Let $G = \mathrm{Perm}(\mathbb{D})$ be the group of finite permutations. A nominal set is a $G$-set such that for each element $x$ one can find a finite set of $\mathbb{D}$ supporting $x$.

The notion of nominal sets (finitely supported $G$-sets) was introduced by Fraenkel in 1922, and developed by Mostowski in the 1930s in order to prove the independence of the axiom of choice and other axioms in classical Zermelo-Fraenkel set theory. In computer science, nominal sets were used in order to properly model the syntax of formal systems involving variable-binding operations (cf. [5]). Nominal sets also have been used in game theory [1], Logic [10], topology [9] and in proof theory [13].

Pitts [12] generalized finite permutations to finite substitutions and introduced the monoid $Cb$. He has shown that this category is equivalent to a particular category of presheaves named cubical sets.

The question of projectivity, as the dual notion of injectivity, is one which arises in many areas of mathematics, and concerns the possibility of lifting a given morphism defined in to a structure through the epimorphisms.

A projective $M$-set, a set equipped with an action of a monoid (or a group) $M$, generalizes the concept of the free $M$-set (cf. [8]). In fact, a projective $M$-set is a retract of a free $M$-set. Indecomposable projective $M$-sets are cyclic (cf. Proposition 17.7.III, [8]). Also a characterization of a

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projective $M$-set in terms of indecomposable projective $M$-sets is given by Knauer (cf. Theorem 17.8.III, [8]).

Throughout the paper, $\mathbb{D}$ and $\mathrm{End}(\mathbb{D})$ are both fixed. The set $\mathbb{D}$ is an infinite countable set and $\mathrm{End}(\mathbb{D})$ is the monoid of all maps from $\mathbb{D}$ to itself with respect to composition. As we mentioned, a projective $M$-set is a retract of a free $M$-set. However in categories of finitely supported $M$-sets there exists no free finitely supported $M$-sets over sets (see Theorem 3.1). Here, we observe that although the category of nominal sets has no projective object (see Corollary 3.4), but projective finitely supported $M$-sets exist, in which $M$ is a submonoid of $\mathrm{End}(\mathbb{D})$. In [8], it is proved that every singleton $M$-set is projective if and only if $M$ contains some zero elements. This result fails in the category of finitely supported $Cb$-sets. The monoid $Cb$ has no zero element (see Lemma 2.15) while every singleton $Cb$-set (which is also a finitely supported $Cb$-set) in the category of finitely supported $Cb$-sets is projective, by Proposition 3.5.

These facts motivate us to study projective finitely supported $M$-sets where $M$ is a submonoid of $\mathrm{End}(\mathbb{D})$. We introduce the notion of zero-retraction monoids (Definition 3.6) and then we characterize projective finitely supported $M$-sets where $M$ is a zero-retraction monoid. In fact, we consider those monoids to behave almost like $Cb$. Finally, using the functor introduced in [6], we characterize projective finitely supported $N$-sets where $N$ contains a zero-retraction monoid $M$.

## 2. Preliminaries

In this section, the preliminary facts about (finitely supported) $M$-sets are given where $M \leqslant \mathrm{End}(\mathbb{D})$. For more information see [2, 3, 8, 12].

### 2.1 $M$-sets

An (left) *$M$-set* for a monoid $M$ with identity $id_{\mathbb{D}}$ is a set $X$ equipped with a map $M \times X \to X, (m, x) \rightsquigarrow mx$, called an *action* of $M$ on $X$, such that $id_{\mathbb{D}} x = x$ and $m(m'x) = (mm')x$, for all $x \in X$ and $m, m' \in M$.

The set $\mathbb{D}$ is an $M$-set with the action given by $md = m(d)$ for all $m \in M$ and $d \in \mathbb{D}$.

The set $\mathbb{D}^k = \{(d_1, \ldots, d_k) : d_1, \ldots, d_k \in \mathbb{D}\}$ is an $M$-set with the action $m(d_1, \ldots, d_k) = (md_1, \ldots, md_k)$.

An *equivariant map* from an $M$-set $X$ to an $M$-set $Y$ is a map $f : X \to Y$ with $f(mx) = mf(x)$, for all $x \in X, m \in M$.

An element $x$ of an $M$-set $X$ is called a *zero* (*fixed* or *equivariant*) element if $mx = x$, for all $m \in M$. We denote the set of all zero elements of an $M$-set $X$ by $\mathcal{Z}(X)$. The $M$-set $X$ all of whose elements are zero is called a *discrete $M$-set*.

A subset $Y$ of an $M$-set $X$ is an $M$-*subset* of $X$ if $my \in Y$, for all $m \in M$ and $y \in Y$. Given an $M$-set $X$, the set $\mathcal{Z}(X)$ is in fact an $M$-subset of $X$.

For the family $\{X_i\}_{i\in I}$ of $M$-sets, the cartesian product $\prod_{i\in I} X_i$ with the component wise action, $m(x_i)_{i\in I} = (mx_i)_{i\in I}$, is the product of $X_i$'s in the category of $M$-sets. The coproduct of the family $\{X_i\}_{i\in I}$ is their disjoint union $\coprod_{i\in I} X_i = \bigcup_{i\in I}(X_i \times \{i\})$ with the action of $M$ defined by $m(x, i) = (mx, i)$, for every $m \in M$ and $x \in X_i$.

An element $m \in M$ is called *idempotent* if $mm = m$.

An $M$-set $X$ is *decomposable* if there exist two $M$-subsets $Y, Z$ of $X$ with $X = Y \cup Z$ and $Y \cap Z = \emptyset$. In this case $X = Y \cup Z$ is called a *decomposition* of $X$. An $M$-set $X$ is called *indecomposable* if it has no decomposition.

Every $M$-set has a decomposition into indecomposable $M$-subsets (cf. Theorem 5.10.I, [8]).

A cyclic $M$-set $X$ is an $M$-set which is generated by only one element. That is $X = Mx$, for some $x \in X$.

## 2.2 Projective $M$-sets

The following facts about projective $M$-sets are needed in the sequel. For more details see [8].

An $M$-set $P$ is said to be *projective* if for each epimorphism (equivariant surjective map) $h : A \twoheadrightarrow B$ and each equivariant map $f : P \to B$, there exists an equivariant map $\varphi : P \to A$ with $h\varphi = f$.

Also, an $M$-subset $A$ of an $M$-set $B$ is called a *retract* of $B$ if there exists an equivariant map $f : B \to A$ with $fi = id_A$, in this case, $f$ is said to be a *retraction*.

**Remark 2.1.** (cf. Proposition 17.2.III, [8])

(1) A free $M$-set is projective.

(2) Any retract of a projective $M$-set is projective.

(3) Any monoid $M$ is a free $M$-set.

**Proposition 2.2.** (cf. [8]) *Let $X$ be an $M$-set. Then,*

(i) (Proposition 17.1.III) *$X$ is projective if and only if $X = \coprod_{i\in I} X_i$,* *where $X_i$'s are projective $M$-sets.*

(ii) (Theorem 17.8.III) $X$ *is projective if and only if* $X = \coprod_{i \in I} X_i$ *with* $X_i \cong Me$, *where* $Me$ *is a cyclic* $M$*-subset of* $M$, *and* $e \in M$ *is an idempotent element.*

(iii) (Proposition 17.4.III) $X$ *is projective if and only if it is a retract of a free* $M$*-set.*

### 2.3 Finitely supported $M$-sets

In this subsection, we give some needed facts about finitely supported $M$-sets. For more information see [2, 12].

**Definition 2.3.** (cf. [12]) Suppose $X$ is an $M$-set and $x \in X$.

(a) A subset $C \subseteq \mathbb{D}$ *supports* $x$ if, for every $m, m' \in M$,

$$(m(c) = m'(c), (\forall c \in C)) \Rightarrow mx = m'x.$$

If there is a finite (possibly empty) support $C$ then we say that $x$ is *finitely supported.*

(b) If every element of $X$ has a finite support, then $X$ is called a *finitely supported $M$-set.*

(c) A subset $C \subseteq \mathbb{D}$ *strongly supports* $x$ if, for every $m, m' \in M$,

$$(m(c) = m'(c), (\forall c \in C)) \Leftrightarrow mx = m'x.$$

We denote the category of all $M$-sets with equivariant maps between them by $M$-**Set**, and its full subcategory of all finitely supported $M$-sets by $(M\text{-}\mathbf{Set})_{\mathrm{fs}}$.

**Remark 2.4.** Suppose $C \subseteq \mathcal{Z}(\mathbb{D})$ is a finite subset. If $X$ is a finitely supported $M$-set and $x \in X$, then

(1) $B \subseteq \mathbb{D}$ supports $x$ if and only if $B - C$ supports $x$.

(2) $B \subseteq C$ supports $x$ if and only if $x$ is a zero element.

**Example 2.5.** (1) A discrete $M$-set is a finitely supported $M$-set, because the empty set is a finite support for each element.

(2) For each $M$-set $X$, the set

$$X_{\mathrm{fs}} = \{x \in X : x \ \text{ has a finite support in } X\},$$

is a finitely supported $M$-subset of $X$. Also, $\mathcal{Z}(X) = \mathcal{Z}(X_{\mathrm{fs}})$.

(3) The sets $\mathbb{D}$ and

$$\mathbb{D}^k = \{(d_1, \ldots, d_k) : d_1, \ldots, d_k \in \mathbb{D}\}$$

are finitely supported $M$-sets. In fact, $\{d\}$ is a finite support of $d$ and $\{d_1, \ldots, d_k\}$ is a finite support for $(d_1, \ldots, d_k)$.

The following example shows that there exists an $M$-set which is not a finitely supported $M$-set.

**Example 2.6.** (Exercise 2.4, [11]) For each natural number $k$, let $X_k = \mathbb{D}$. Take the element $(x_k)_{k\in\mathbb{N}}$ in $A = \prod_{k\in\mathbb{N}} X_k$ such that for every $d \in \mathbb{D}$ there exists $k \in \mathbb{N}$ with $d = x_k$. Then, this element has no finite support. So, $A_{\text{fs}} \neq A$.

**Remark 2.7.** (1) The product of a family of finitely supported $M$-sets $X_i$'s is $(\prod_{i\in I} X_i)_{\text{fs}}$.

(2) Coproducts in the category of finitely supported $M$-sets are constrcuted just as in the catgeory of $M$-sets. Hence, for a family of finitely supported $M$-sets $X_i$ indexed by a set $I$, disjoint union of $X_i$'s is the coproduct of them, and denoted by $\coprod_{i\in I} X_i$. For each element $t \in \coprod_{i\in I} X_i$, there exists $j \in I$ with $t \in X_j$. Hence, if $S$ is a finite support of $t$ in $X_j$, then $S$ is a finite support of $t$ in $\coprod_{i\in I} X_i$. For more details cf. Section 2.2, [12] and Chapter II, [8].

**Definition 2.8.** (cf. [2, 12]) Let $X$ be a finitely supported $M$-set and $x \in X$. Then,

(a) $x$ admits least support if the set $\bigcap\{C : C \text{ is a finite support of } x\}$ supports $x$. We denote the least support of $x \in X$ with $\operatorname{supp} x$.

(b) $X$ admits least support if every $x \in X$ has the least support.

**Remark 2.9.** (1) For the given $M$-set $X$ and $x \in X$, if $C$ (strongly) supports $x$, then $m(C)$ (strongly) supports $mx$.

(2) Suppose $f : X \to Y$ is an equivariant map and $x \in X$. If $C$ is a finite support of $x$, then $C$ is a finite support of $f(x)$.

**Definition 2.10.** (a) A permutation (bijection map) $\pi : \mathbb{D} \to \mathbb{D}$ is said to be *finite* if $\{d \in \mathbb{D} \mid \pi(d) \neq d\}$ is finite. The set $\text{Perm}(\mathbb{D})$ is the group of all finite permutations on $\mathbb{D}$.

(b) A finitely supported $\text{Perm}(\mathbb{D})$-set $X$ is called a *nominal set.*

**Example 2.11.** The set $\mathbb{D}^{(k)} = \{(d_1, \ldots, d_k) \in \mathbb{D}^k : (\forall i \neq j),\ d_i \neq d_j\}$ is a nominal set.

## 2.4 Finitely supported $Cb$-sets

We give some basic facts about the monoid $Cb$ and finitely supported $Cb$-sets. For more information one can see [4, 7, 12].

Also, we take $2 = \{0, 1\}$ with $0, 1 \notin \mathbb{D}$.

**Definition 2.12.** (a) A map $\sigma : \mathbb{D} \to \mathbb{D} \cup 2$ is called an *injective finite substitution* if $\{d \in \mathbb{D} \mid \sigma(d) \neq d\}$ is finite and

$$(\forall d, d' \in \mathbb{D}),\ \sigma(d) = \sigma(d') \notin 2 \Rightarrow d = d'.$$

(b) If $d \in \mathbb{D}$ and $b \in 2$, a basic substitution $(b/d)$ maps $d$ to $b$, and is the identity mapping on all the other elements of $\mathbb{D}$.

(c) If $d, d' \in \mathbb{D}$ then each transposition $(d\ d')$ is called a *transposition substitution.*

**Definition 2.13.** (a) The monoid $Cb$ is the monoid whose elements are injective finite substitutions with the monoid operation given by $\sigma \cdot \sigma' = \hat{\sigma}\sigma'$, where $\hat{\sigma} : \mathbb{D} \cup 2 \to \mathbb{D} \cup 2$ maps 0 to 0, 1 to 1, and on $\mathbb{D}$ is defined the same as $\sigma$. The identity element of $Cb$ is the inclusion $\iota : \mathbb{D} \hookrightarrow \mathbb{D} \cup 2$.

(b) The subsemigroup of $Cb$ generated by basic substitutions is denoted by $S$. Each element of $S$ is of the form $\delta = (b_1/d_1) \cdots (b_k/d_k) \in S$ for some $d_i \in \mathbb{D}$ and $b_i \in 2$, and we denote the set $\{d_1, \ldots, d_k\}$ by $\mathbb{D}_\delta$.

**Theorem 2.14.** (Theorem 2.4, [7]) *For the monoid $Cb$, we have*

$$Cb = \mathrm{Perm}(\mathbb{D})(S \cup \{\iota\}).$$

**Lemma 2.15.** *The monoid $Cb$ (as a $Cb$-set) has no zero element.*

*Proof.* On the contrary, assume that there exists a zero element $\sigma' \in Cb$. We must show that $\sigma\sigma' = \sigma'$, for all $\sigma \in Cb$. By Theorem 2.14, $\sigma' \in \mathrm{Perm}(\mathbb{D})$ or $\sigma' \in \mathrm{Perm}(\mathbb{D})S$. For the first case, let $\sigma = (0/d)$. Then,

$$d = \sigma'(\sigma'^{-1}d) = \sigma\sigma'(\sigma'^{-1}d) = \sigma d = (0/d)d = 0,$$

which is a contradiction. Now, suppose $\sigma' \in \mathrm{Perm}(\mathbb{D})S$. So, there exist $\pi' \in \mathrm{Perm}(\mathbb{D})$ and $\delta' \in S$ with $\sigma' = \pi'\delta'$. Let $\sigma = (0/d)\pi'^{-1}$ with $d \notin \mathbb{D}_{\delta'}$. Then,

$$\pi' d = \pi'\delta' d = \sigma' d = \sigma\sigma' d = (0/d)\pi'^{-1}\pi'\delta' d = (0/d)d = 0$$

which is impossible. $\square$

**Proposition 2.16.** (i) (Lemma 2.4, [12]) *Suppose $X$ is a $Cb$-set, $x \in X$ and $b \in 2$. Also, let $C$ be a finite subset of $\mathbb{D}$. Then, $C$ is a support of $x$ if and only if*

$$(\forall d \in \mathbb{D}) \quad d \notin C \Rightarrow (b/d)x = x.$$

(ii) *The set $\{d \in \mathbb{D} : (0/d)x \neq x\}$ is the least finite support of $x$.*

**Lemma 2.17.** (cf. Lemma 3.4, [4]) *Let $X$ be a $Cb$-set and $x \in X$. Then,*

(i) *$S_x = \{\delta \in S \mid \delta x \neq x\}$ and $S'_x = S - S_x$ are two subsemigroups of $S$.*

(ii) *If $x$ has the least finite support, then $\operatorname{supp} \delta x \subseteq (\operatorname{supp} x) \setminus \mathbb{D}_\delta$, for all $\delta \in S'_x$.*

(iii) *Let $\delta \in S$. Then, $\delta x \neq x$ if and only if $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$.*

**Remark 2.18.** Let $\delta, \delta' \in S$. Then, $\delta = \delta'$ if and only if $\mathbb{D}_\delta = \mathbb{D}_{\delta'}$ with $\delta(d) = \delta'(d)$, for all $d \in \mathbb{D}_\delta$.

**Lemma 2.19.** (Lemma 4.5, [4]) *Let $Cbx$ be a cyclic finitely supported $Cb$-set. Then, $Cbx = \mathrm{Perm}(\mathbb{D})S'_x x \cup \mathrm{Perm}(\mathbb{D})x$.*

## 3. Projective finitely supported $M$-sets

In this section, we give a characterization of projective finitely supported $M$-sets, where $M$ is a zero-retraction monoid. To do so, first we show that although free objects over sets do not exist in the categories of finitely supported $Cb$-sets and nominal sets, but this is not true about projectivity. In fact, we show that the singleton finitely supported $Cb$-sets are projective while no nominal sets are projective. This fact happens because of a property of finite substitutions in $Cb$. So then we generalize this property and introduce the notion of zero-retraction monoids. Then, using this notion we find the projective finitely supported $M$-sets when $M$ is a zero-retraction monoid or contains a zero-retraction monoid.

Let us begin this section with the following theorem which shows that, analogous to the categories nominal sets and $Cb$-sets, the forgetful functor

$$V : (M\text{-}\mathbf{Set})_{\text{fs}} \to \mathbf{Set}$$

has no left adjoint and so free finitely supported $M$-sets over sets do not exist.

**Theorem 3.1.** *The forgetful functor $V : (M\text{-}\mathbf{Set})_{\text{fs}} \to \mathbf{Set}$ has no left adjoint.*

*Proof.* Let $L$ be a left adjoint of $V$. Then, since right adjoints preserves limits, we get that $V$ preserves arbitrary products. Consider the finitely supported $M$-set $A_{\text{fs}}$ in Example 2.6. So, $V(A_{\text{fs}}) = A_{\text{fs}}$ is the product of the family of $X_k$'s. But, the product of $X_k$'s is $A$. □

**Corollary 3.2.** *No free finitely supported $M$-sets exists over sets.*

**Lemma 3.3.** *No indecomposable nominal set is projective.*

*Proof.* Let $X$ be indecomposable. Then $X = \mathrm{Perm}(\mathbb{D})x$, for some $x \in X$. Notice that, indecomposable nominal sets are cyclic. Take $k$ to be a natural number with $k > |\operatorname{supp} x|$, $h : \mathbb{D}^{(k)} \to \{\theta\}$ to be a surjective constant equivariant map, and $f : X \to \{\theta\}$ is an equivariant map. If $X$ is projective, then there exists an equivariant map $\varphi : X \to \mathbb{D}^{(k)}$ with $h\varphi = f$. Now, we have $\varphi(x) = (d_1, \ldots, d_k) \in \mathbb{D}^{(k)}$. Since $\varphi$ is equivariant, we get that $\operatorname{supp} \varphi(x) \subseteq \operatorname{supp} x$. Thus, $k = |\{d_1, \ldots, d_l\}| \leqslant |\operatorname{supp} x|$ which is a contradiction. □

**Corollary 3.4.** *No nominal set is projective.*

*Proof.* Follows from Proposition 2.2(i). □

**Proposition 3.5.** *The singleton finitely supported $Cb$-set $\{\theta\}$ is projective.*

*Proof.* Suppose $h : A \to B$ is a surjective equivariant map. Take $f : \{\theta\} \to B$ to be an equivariant map with $f(\theta) = \theta' \in \mathcal{Z}(B)$. Notice that, finitely supported $Cb$-sets have zero elements. We show that there exists an equivariant map $\varphi : \{\theta\} \to A$ with $h\varphi = f$. Since $h$ is surjective, there exists $a \in A$ with $h(a) = \theta'$. If $a \in \mathcal{Z}(A)$, then define $\varphi(\theta) = a$, and so, $h\varphi(\theta) = h(a) = \theta' = f(\theta)$. If $\operatorname{supp} a \neq \emptyset$, then take $\delta \in S$ with $\mathbb{D}_\delta = \operatorname{supp} x$. Now, by Lemma 2.17(ii), $\operatorname{supp} \delta x = \emptyset$, and so, taking $\varphi(\theta) = \delta a$, we get that $h\varphi(\theta) = h(\delta a) = \delta h(a) = \delta\theta' = \theta' = f(\theta)$. □

## 3.1 Retraction-monoid

**Definition 3.6.** (a) Let $A$ and $B$ be two finite subsets of $\mathbb{D}$ with $A \subseteq B$. Then, $A$ is called an *$M$-zero-retraction* of $B$ if $A \cup \mathcal{Z}(\mathbb{D})$ is a retraction of $B$; that is there exists $m \in M$ with $m(B) \subseteq A \cup \mathcal{Z}(\mathbb{D})$ and $m|_A = id|_A$.

(b) $A$ is an *absolutely $M$-zero-retraction* if $A$ is an $M$-zero-retraction of every $B$ that contains $A$; that is $A \subseteq B$.

The monoid $M$ is called *zero-retraction*, if every finite subset $A$ of $\mathbb{D}$ is an absolutely $M$-zero-retraction.

**Proposition 3.7.** *Let $M$ be a zero-retraction monoid. Then,*

(i) *$\mathcal{Z}(\mathbb{D})$ is non-empty.*

(ii) *There exists $C \subseteq \mathcal{Z}(\mathbb{D})$ such that $m|_C = id|_C$, for every $m \in M$. In other words, $M$ is a submonoid of $M_C$ where*

$$M_C = \{m \in \mathrm{End}(\mathbb{D}) : m|_C = id|_C\},$$

*for some $C \subseteq \mathcal{Z}(\mathbb{D})$.*

*Proof.* (i). Suppose $B \subseteq \mathbb{D}$ is a non-empty finite subset. Notice that, $m(B) \neq \emptyset$, for every $m \in M$. Since $M$ is zero-retraction and $\emptyset \subseteq B$, there exists $m_0 \in M$ with $m_0(B) \subseteq \mathcal{Z}(\mathbb{D})$. Now, since $m_0(B) \neq \emptyset$, we get $\mathcal{Z}(\mathbb{D}) \neq \emptyset$.

(ii). By (i), $C = m_0(B) \subseteq \mathcal{Z}(\mathbb{D})$. Let $m \in M$. Then, $m|_C = id|_C$ and so $m \in M_C$. □

**Remark 3.8.** (1) The nominal set $\mathbb{D}$ has no zero elements.

(2) The group $G = \mathrm{Perm}(\mathbb{D})$ is not a zero-retraction monoid. This is because, if $A \subsetneq B$ finite subsets of $\mathbb{D}$ and there exists $\pi \in G$ with $\pi(B) \subseteq A$, then $A = B$ which is a contradiction.

(3) The monoid $Cb$ is a zero-retraction monoid (cf. Lemma 4.1). Notice that, $\mathcal{Z}(\mathbb{D}) = 2$.

## 3.2 Finitely supported $\mathbb{D}^A$

The following example plays an important role in characterizing projective finitely supported $M$-sets.

For a finite subset $A$ of $\mathbb{D}$, the set $\mathbb{D}^A = \{m|_A : m \in M\}$ is an $M$-set with the action defined by $m' * m|_A := (m'm)|_A$, for all $m, m' \in M$, where $* : M \times \mathbb{D}^A \to \mathbb{D}^A$ (cf. Example 2.4, [2]).

**Lemma 3.9.** *Let $A$ be a finite subset of $\mathbb{D}$. Then, $\mathbb{D}^A$ is a cyclic finitely supported $M$-set.*

*Proof.* First we show that $m(A)$ is a finite support of $m|_A$. Indeed, if $m_1, m_2 \in M$ and $m_1(a) = m_2(a)$, for all $a \in m(A)$, then $m_1 m(d) = m_2 m(d)$, for all $d \in A$. Hence $(m_1 m)|_A = (m_2 m)|_A$, and so $m_1 * m|_A = m_2 * m|_A$.

Now we note that $\mathbb{D}^A = M id|_A$. That is $\mathbb{D}^A$ is cyclic and we are done. □

**Corollary 3.10.** *The map* $\varphi : M \to \mathbb{D}^A$ *defined by* $\varphi(m) = m|_A$ *is a surjective equivariant map.*

**Proposition 3.11.** *Given a finitely supported* $M$*-set* $Y$ *and a finite subset* $A \subseteq \mathbb{D}$*, there exists an equivariant map from* $\mathbb{D}^A$ *to* $Y$ *if and only if* $A$ *is a finite support of some* $y \in Y$*.*

*Proof.* Suppose $\varphi : \mathbb{D}^A \to Y$ is an equivariant map. Then, we consider $\varphi(id|_A) = y \in Y$. Since $\varphi$ is equivariant, we get that $A$ is a finite support of $y$, by Remark 2.9.

To prove the other part, it is sufficient to define $\varphi : \mathbb{D}^A \to Y$ by $\varphi(m|_A) = my$, where $A$ is a finite support of $y$. □

**Lemma 3.12.** *Let* $X = Mx$ *be a cyclic finitely supported* $M$*-set and* $A$ *be a finite support of* $x$*. Then,* $Mx$ *is isomorphic to* $\mathbb{D}^A$ *if and only if* $A$ *strongly supports* $x$*.*

*Proof.* First notice that if $A = \emptyset$, then $X \cong \mathbb{D}^{\emptyset}$. Let $x$ be non-zero. Then, $\varphi : \mathbb{D}^A \to Mx$ defined by $\varphi(m|_A) = mx$ is a surjective equivariant map using Proposition 3.11. Now, $\varphi$ is an injective map if and only if

$$(\forall m, m' \in M)\,(mx = m'x \Leftrightarrow m|_A = m'|_A)$$

if and only if $A$ strongly supports $x$. □

## 3.3 Projective finitely supported $M$-sets

In this subsection, we take $M$ to be a zero-retraction monoid and then characterize projective finitely supported $M$-sets.

**Proposition 3.13.** *If* $X$ *is a finitely supported* $M$*-set, then* $X$ *has some zero elements.*

*Proof.* First, notice that $M$ is a submonoid of $M_C$, for some finite subset $C$ of $\mathbb{D}$, by Proposition 3.7. Now suppose $x \in X$ and $B$ is a finite support of $x$. The set $B - C$ is a support of $x$, by Remark 2.4(2). If $B - C = \emptyset$, then $x$ is zero. If $\emptyset \subsetneq B - C$, then there exists $m_0 \in M$ with $m_0(B - C) \subseteq \mathcal{Z}(\mathbb{D})$. By Remark 2.9(1), $m_0(B - C)$ is a finite support of $m_0 x$. Thus, $m_0 x \in \mathcal{Z}(X)$, by Remark 2.4(2). □

The following lemma is the key to show that cyclic finitely supported $M$-sets $\mathbb{D}^A$ are projective (see Lemma 3.16).

**Lemma 3.14.** *Suppose $f : X \to Y$ is an equivariant map between finitely supported $M$-sets, and $A$ is a finite support of $f(x)$, for some $x \in X$. Then, there exists $m \in M$ with $f(mx) = f(x)$ such that $A$ is a finite support of $mx$.*

*Proof.* Let $A$ be a finite support of $y = f(x)$. Then, by Remark 2.4(2), $A_1 = A - \mathcal{Z}(\mathbb{D})$ supports $y$. If $A_1$ supports $x$, then taking $m = id$ we get the result. Otherwise, take $B_1 = B - \mathcal{Z}(\mathbb{D})$ to be a finite support of $x$. So, $A_1 \subseteq A_1 \cup B_1$ and since $M$ is a zero-retraction monoid, there exists $m \in M$ with $[m(A_1 \cup B_1)] \subseteq A_1 \cup \mathcal{Z}(\mathbb{D})$ and $m|_{A_1} = id|_{A_1}$. Since $m(A_1 \cup B_1)$ supports $mx$, we have $A_1 \cup \mathcal{Z}(\mathbb{D})$ supports $mx$ and so $A_1$ supports $mx$, by Remark 2.4(2). Also, $m|_{A_1} = id|_{A_1}$ implies that $f(mx) = mf(x) = f(x) = y$. $\square$

As a result of Lemma 3.14, we have the following corollary for finitely supported $M$-sets admit least supports.

**Corollary 3.15.** *Let $f : X \to Y$ be an equivariant map between finitely supported $M$-sets admit least supports. Then, for every $y \in f(X)$ there exist $x \in X$ and $m \in M$ with $f(mx) = y$ and $\operatorname{supp} y = \operatorname{supp} mx$.*

*Proof.* Let $y = f(x)$, for some $x \in X$. Since $f$ is equivariant, we have $\operatorname{supp} y \subseteq \operatorname{supp} x$. Since $M$ is zero-retraction, there exists $m_0 \in M$ with $m_0(\operatorname{supp} x) \subseteq \operatorname{supp} y \cup \mathcal{Z}(\mathbb{D})$ and $m_0|_{\operatorname{supp} y} = id|_{\operatorname{supp} y}$. Now, $f(m_0 x) = m_0 y = y$. Also, $\operatorname{supp} y \subseteq \operatorname{supp} m_0 x \subseteq [m_0(\operatorname{supp} x) - \mathcal{Z}(\mathbb{D})] \subseteq \operatorname{supp} y$ implies that $\operatorname{supp} m_0 x = \operatorname{supp} y$. $\square$

**Lemma 3.16.** *If $A$ is a finite subset of $\mathbb{D}$, then $\mathbb{D}^A$ is a projective finitely supported $M$-set.*

*Proof.* Let $f : X \to Y$ be a surjective equivariant map and $g : \mathbb{D}^A \to Y$ be an equivariant map. Then, we show that there exists an equivariant map $\varphi : \mathbb{D}^A \to X$ with $f\varphi = g$. To do so, applying Proposition 3.11, we find an element in $X$ such that $A$ supports it. We have $g(id|_A) \in Y$ and $f$ is surjective. So, there exists $x \in X$ with $f(x) = g(id|_A)$. Since $g$ is equivariant and $A = id(A)$ is a finite support of $id|_A$, we get that $A$ is a finite support of $g(id|_A)$. Hence, by Lemma 3.14, there exists $m_0 \in M$ with $f(m_0 x) = g(id|_A)$ and $A$ supports $m_0 x$. Therefore, $\varphi(m|_A) = mm_0 x$ is a required equivariant map by Proposition 3.11. Also,

$$f\varphi(m|_A) = f(mm_0 x) = mf(m_0 x) = mg(id|_A) = g(m|_A).$$

This completes the proof. $\square$

**Corollary 3.17.**

(i) *Every singleton finitely supported $M$-set is projective.*

(ii) *Every discrete finitely supported $M$-set is projective.*

*Proof.* (i). $\mathbb{D}^{\emptyset}$ is isomorphic to a singleton finitely supported $M$-set.

(ii). Follows from (i) and Proposition 2.2(i). □

**Theorem 3.18.** *Suppose $X$ is a finitely supported $M$-set. Then, there exists a surjective equivariant map from $P = \coprod_{x\in X} \mathbb{D}^{A_x}$ to $X$, where $A_x$ is a finite support of $x$.*

*Proof.* For each element $x \in X$, take $A_x$ to be a finite support of $x$. Then, by Proposition 3.11 there exists an equivariant map $\varphi_x : \mathbb{D}^{A_x} \to X$ with $\varphi_x(m|_{A_x}) = mx$. Now, the universal property of coproduct, ensures that there exists a unique equivariant map $\varphi : P \to X$ by $\varphi(a) = \varphi_x(a)$, for every $a \in \mathbb{D}^{A_x}$. Also, for each $x \in X$, there exists an element $id|_{A_x} \in \mathbb{D}^{A_x}$ with $\varphi(id|_{A_x}) = \varphi_x(id|_{A_x}) = x$ which means that $\varphi$ is surjective. □

**Lemma 3.19.** *Let $X$ be a finitely supported $M$-set. Then, there exists a projective finitely supported $M$-set $P$ such that $X$ is a surjective equivariant image of $P$.*

*Proof.* If $X = Z(X)$, then by Corollary 3.17(ii), $X$ is projective, and so, in this case $P = X$. If $X$ is non-discrete, then applying Lemma 3.16, Proposition 2.2(i), and Theorem 3.18 we get the result. □

**Lemma 3.20.** *Let $X$ be a finitely supported $M$-set. Then, $X$ is indecomposable and projective if and only if $X$ is cyclic and $X \cong \mathbb{D}^A$, for some finite subset $A \subseteq \mathbb{D}$.*

*Proof.* Necessity. First, notice that, applying Theorem 3.18 there exists a surjective equivariant map $\varphi : P \to X$, where $P = \coprod_{x\in X} \mathbb{D}^{A_x}$ and $\varphi|_{\mathbb{D}^{A_x}} = \varphi_x : \mathbb{D}^{A_x} \to X$. Now since $X$ be projective, there exists an equivariant map $\psi : X \to P$ such that $\varphi\psi = id_X$, where $id_X : X \to X$. Thus, $\varphi\psi(X) = X$. Since $X$ is indecomposable, we have $\psi(X) \subseteq \mathbb{D}^{A_x}$, for some $x \in X$. Now, $X = \varphi\psi(X) \subseteq \varphi(\mathbb{D}^{A_x}) \subseteq X$, and so, $M\varphi(id|_{A_x}) = \varphi(\mathbb{D}^{A_x}) = X$ which means that $X$ is cyclic. Notice that, $X = M\varphi(id|_{A_x}) = M\varphi_x(id|_{A_x}) = \mathbb{D}^{A_x}$. Also, since $\psi$ is an injective equivariant map, we get that

$$X \cong \psi(X) = \psi(\mathbb{D}^{A_x}) = M\psi\varphi(id|_{A_x}) = M\psi(id|_{A_x}) = \mathbb{D}^{A_x}.$$

Sufficiency. Follows from Lemma 3.9 and Lemma 3.16. □

**Theorem 3.21.** *Let $X$ be a finitely supported $M$-set. Then, $X$ is projective if and only if $X = \coprod_{i\in I} X_i$, where every $X_i$ is isomorphic to $\mathbb{D}^A$, for some finite subset $A$.*

*Proof.* Suppose $X$ is projective. Take $X = \coprod X_i$ to be a coproduct of indecomposable finitely supported $M$-sets $X_i$. Then, since each $X_i$ has a zero element, it is a retract of $X$. So, by Remark 2.1(2), we get that $X_i$'s are projective. Now, applying Theorem 3.20, every $X_i$ is isomorphic to $\mathbb{D}^A$, for some finite subset $A$.

To prove the other side, by Lemma 3.16, cyclic $\mathbb{D}^A$'s are projective, and so, every $X_i$ is projective. Now, applying Proposition 2.2(i), any coproduct of projective finitely supported $M$-sets is projective. □

**Corollary 3.22.** *Every projective finitely supported $M$-set is a surjective equivariant image of a free $M$-set.*

*Proof.* Let $X$ be a projective finitely supported $M$-set. Then, by Theorem 3.21, we get that $X \cong \coprod_{x\in X} \mathbb{D}^{A_x}$, where $A_x$ supports $x$. On the other hand, by Corollary 3.10, for every $x \in X$ there exists a surjective equivariant map $\varphi_x : M \to \mathbb{D}^{A_x}$. Now, the map $\varphi : \coprod_{x\in X} M \to \coprod_{x\in X} \mathbb{D}^{A_x}$ defined by $\varphi(m) = \varphi_x(m)$ for some $x \in X$ and $m \in M$ is a required surjective equivariant map. □

# 4. Projective finitely supported $Cb$-sets

The monoid $Cb$ is isomorphic to a submonoid of $\mathrm{End}(\mathbb{D})$. It is sufficient to take two fixed elements $a, b \in \mathbb{D}$ instead of $\mathsf{0}$ and $\mathsf{1}$ and replace $\mathbb{D}$ with $\mathbb{D} - \{a, b\}$. Then, by the same scheme of Pitts, one can get a submonoid of $\mathrm{End}(\mathbb{D})$ isomorphic to $Cb$. As an example of a zero-retraction monoid, we can mention the monoid $Cb$.

**Lemma 4.1.** *The monoid $Cb$ is a zero-retraction monoid.*

*Proof.* Suppose $A \subseteq B$ are two finite subsets of $\mathbb{D}$. If $A = B$ or $B = \emptyset$, then $\iota(B) - \mathsf{2} = \iota(B) = \iota(A) = A$.

If $A \subsetneq B$ and $A = \emptyset$, then take $\delta \in S$ with $\mathbb{D}_\delta = B$. So, $\delta(B) - \mathsf{2} = \emptyset = A$.

If $A \subsetneq B$ and $A \neq \emptyset$, then take $\delta \in S$ with $\mathbb{D}_\delta = B - A$. Hence $\delta(B) = A \cup \mathsf{2}$ and so $\delta(B) - \mathsf{2} = A$ and $\delta|_A = id|_A$. □

Lemma 3.15 holds in the category of finitely supported $Cb$-sets. In the following proposition we prove it more specifically.

**Proposition 4.2.** *Suppose $X$ and $Y$ are two finitely supported $Cb$-set. If $f : X \to Y$ is an equivariant map, then for every $y \in f(X)$ there exists $x \in X$ with* $\operatorname{supp} x = \operatorname{supp} y$.

*Proof.* Let $y \in f(X)$. Then, there exists $x \in X$ with $y = f(x)$. If $\operatorname{supp} x = \emptyset$, then $\operatorname{supp} y = \emptyset$. Suppose $x \in X$ with $\operatorname{supp} x \neq \emptyset$. If $\operatorname{supp} x = \operatorname{supp} y$, then we get the result. If $\operatorname{supp} x \subsetneq \operatorname{supp} y$, then take $\delta_0 \in S$ with $\mathbb{D}_{\delta_0} = (\operatorname{supp} y) - \operatorname{supp} x$. Thus $\delta_0 x = x$ and so $\operatorname{supp} x \subseteq \operatorname{supp} \delta_0 y$. On the other hand, $\operatorname{supp} y = (\operatorname{supp} y - \operatorname{supp} x) \cup \operatorname{supp} x = \mathbb{D}_{\delta_0} \cup \operatorname{supp} x$. So, $\operatorname{supp} \delta_0 y \subseteq \operatorname{supp} y - \mathbb{D}_{\delta_0} = \operatorname{supp} x$. □

## 4.1 Max-zero cyclic finitely supported $Cb$-sets

In this subsection, we construct a particular cyclic finitely supported $Cb$-set, and show that it is isomorphic to a finitely supported $Cb$-set $\mathbb{D}^A$, for some finite subset $A \subseteq \mathbb{D}$.

First, we give the following needed remark and lemma.

**Remark 4.3.** Let $Cbx$ be a non-singleton cyclic finitely supported $Cb$-set. Then,

(1) The $Cb$-subset $\mathcal{Z}(Cbx)$ of $Cbx$ is a subset of $S'_x x$. This is because, by Lemma 2.19, we have $\mathcal{Z}(Cbx) \cap \operatorname{Perm}(\mathbb{D})x = \emptyset$.

(2) If $\delta \in S'_x$ with $\mathbb{D}_\delta = \operatorname{supp} x$, then, by Lemma 2.17(ii), we get that $\operatorname{supp} \delta x \subseteq (\operatorname{supp} x) - \mathbb{D}_\delta = \emptyset$. So, $\delta x \in \mathcal{Z}(Cbx)$.

(3) If $\delta \in S'_x$, then there exists $\delta' \in S'_x$ such that $\mathbb{D}_{\delta'} \subseteq \operatorname{supp} x$ and $\delta x = \delta' x$. Furthermore, $\operatorname{supp} x \setminus \mathbb{D}_\delta = \operatorname{supp} x - \mathbb{D}_{\delta'}$. To show this, let $\delta \in S'_x$. Then, $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$. Suppose $\delta = \delta_1\delta_2$ with $\mathbb{D}_{\delta_2} = \mathbb{D}_\delta \cap \operatorname{supp} x \subseteq \operatorname{supp} x$ and $\mathbb{D}_{\delta_1} \cap \operatorname{supp} x = \emptyset$. Then, $\delta x = \delta_2 x$. Now, we show that $\operatorname{supp} x - \mathbb{D}_\delta = \operatorname{supp} x - \mathbb{D}_{\delta'}$. Notice that, since $\mathbb{D}_{\delta_1} \cap \operatorname{supp} x = \emptyset$, we get that $\operatorname{supp} x - \mathbb{D}_{\delta_1} = \operatorname{supp} x$. So,

$$\operatorname{supp} x - \mathbb{D}_\delta = \operatorname{supp} x - (\mathbb{D}_{\delta_1} \cup \mathbb{D}_{\delta_2}) = (\operatorname{supp} x - \mathbb{D}_{\delta_1}) - \mathbb{D}_{\delta_2} = \operatorname{supp} x - \mathbb{D}_{\delta_2}.$$

**Proposition 4.4.** *Suppose $X$ is a finitely supported $Cb$-set and $x \in X$. Let $\sigma, \sigma' \in Cb$ with $\sigma x = \sigma' x$. Then,*

(i) $\sigma, \sigma' \in \operatorname{Perm}(\mathbb{D}) \cup \operatorname{Perm}(\mathbb{D})S_x$ *or* $\sigma, \sigma' \in \operatorname{Perm}(\mathbb{D})S'_x$.

(ii) *there exists* $\pi \in \operatorname{Perm}(\mathbb{D})$ *with* $\pi x = x$ *or there exist* $\delta, \delta' \in S'_x$ *and* $\pi \in \operatorname{Perm}(\mathbb{D})$ *with* $\pi\delta x = \delta' x$.

*Proof.* (i). Since $S_x \cap S'_x = \emptyset$, it is sufficient to prove if $\sigma \in \mathrm{Perm}(\mathbb{D})S'_x$, then $\sigma' \in \mathrm{Perm}(\mathbb{D})S'_x$ and vice versa. Let $\sigma x = \sigma' x$ with $\sigma \in \mathrm{Perm}(\mathbb{D})S'_x$. Then, $\sigma = \pi\delta$, and so, by Lemma 2.17, we get that

$$|\mathrm{supp}\,\sigma' x| = |\mathrm{supp}\,\sigma x| = |\mathrm{supp}\,\pi\delta x| = |\mathrm{supp}\,\delta x| < |\mathrm{supp}\,x|.$$

Now, if $\sigma' \in \mathrm{Perm}(\mathbb{D}) \cup \mathrm{Perm}(\mathbb{D})S_x$, then $|\mathrm{supp}\,\sigma' x| = |\mathrm{supp}\,x|$, which is impossible.

(ii). By (i), we get $\sigma, \sigma' \in \mathrm{Perm}(\mathbb{D}) \cup \mathrm{Perm}(\mathbb{D})S_x$ or $\sigma, \sigma' \in \mathrm{Perm}(\mathbb{D})S'_x$. If $\sigma, \sigma' \in \mathrm{Perm}(\mathbb{D})S'_x$, then $\sigma = \pi_1\delta$ and $\sigma' = \pi_2\delta'$, and so, we get that $\pi_1\delta x = \pi_2\delta' x$. In this case, taking $\pi_2^{-1}\pi_1 = \pi$, we have $\pi\delta x = \delta' x$.

Notice that, if $\sigma \in \mathrm{Perm}(\mathbb{D})S_x$, then $\sigma x = \pi\delta x = \pi x$. So, if both $\sigma, \sigma' \in \mathrm{Perm}(\mathbb{D}) \cup \mathrm{Perm}(\mathbb{D})S_x$, then $\sigma x = \pi x$, and $\sigma' x = \pi' x$ which means that $\pi x = \pi' x$. Thus, $\pi_1 x = x$ where $\pi_1 = \pi'^{-1}\pi$. □

**Remark 4.5.** (1) For given $c_1, \ldots, c_k \in \{0, 1\}$, the decimal number is denoted by $(c_k c_{k-1} \cdots c_1)_2$ and computed as $c_k \times 2^{k-1} + c_{k-1} \times 2^{k-2} + \cdots + c_2 \times 2^1 + c_1 \times 2^0$.

(2) If $(c_k c_{k-1} \cdots c_1)_2 = (c'_k c'_{k-1} \cdots c'_1)_2$, then $c_i = c'_i$, for all $i = 1, \ldots, k$.

**Note**. The substitutions $\mathsf{0} \neq \mathsf{1}$ are just symbols, and they do not belong to $\mathbb{D}$. In the following lemma, if $b_i$ is the substitution $\mathsf{1}$, then take $c_i$ to be the natural number 1, and if $b_i = \mathsf{0}$, then take $c_i$ to be the zero number 0, for all $i = 1, \ldots, k$.

**Lemma 4.6.** *Suppose $X$ is a finitely supported $Cb$-set with $x \in X$. Let* $\mathrm{supp}\,x = \{d_1, \ldots, d_k\}$ *and $b_1, \ldots, b_k \in \mathsf{2}$. Take $A = \{0, 1, 2, 3, \ldots, 2^{k-1}, x\}$ to be a set with $2^{k+1}$ elements. Define map $g_x : Cb \to A$ by*

$$g_x(\sigma) = \begin{cases} (c_k c_{k-1} \cdots c_1)_2, & \text{if} \quad \sigma \in Cb(b_1/d_1)\cdots(b_k/d_k) \\ \\ x, & \text{if} \qquad\qquad\qquad \text{otherwise}. \end{cases}$$

*Then,* $\mathrm{supp}\,g_x = \mathrm{supp}\,x$.

*Proof.* First applying Proposition 2.16, we show that $(\mathsf{0}/d)\, g_x = g_x$, for all $d \notin \mathrm{supp}\,x$. In fact, we show that $((\mathsf{0}/d)g_x)(\sigma) = g_x(\sigma)$, for all $\sigma \in Cb$. Suppose $\sigma \in Cb$.

If $\sigma \notin Cb(b_1/d_1)\cdots(b_k/d_k)$, then $\sigma(\mathsf{0}/d) \notin Cb(b_1/d_1)\cdots(b_k/d_k)$, because otherwise if $\sigma(\mathsf{0}/d) = \sigma'(b_1/d_1)\cdots(b_k/d_k)$, then

$$\hat{\sigma}(d_i) = (\hat{\sigma}(\mathsf{0}/d))(d_i) = \hat{\sigma'}(b_1/d_1)\cdots(b_k/d_k)(d_i) \in \mathsf{2},$$

for all $d_i \in \{d_1, \ldots, d_k\}$. This implies that $\sigma \in Cb(b_1/d_1)\cdots(b_k/d_k)$, which is impossible. Thus, in this case, for all $\sigma \in Cb$, we get that

$$((0/d)g_x)(\sigma) = g_x(\sigma(0/d)) = x = g_x(\sigma).$$

Now, let $\sigma \in Cb(b_1/d_1)\cdots(b_k/d_k)$. Then, for some $\sigma_1 \in Cb$, we have $\sigma = \sigma_1(b_1/d_1)\cdots(b_k/d_k)$. Hence,

$$\begin{aligned}\sigma(0/d) &= \sigma_1(b_1/d_1)\cdots(b_k/d_k)(0/d)\\ &= \sigma_1(0/d)(b_1/d_1)\cdots(b_k/d_k) \in Cb(b_1/d_1)\cdots(b_k/d_k),\end{aligned}$$

and so, $g_x(\sigma(0/d)) = g_x(\sigma)$.

Therefore, $g_x \in (A^{Cb})_{\mathrm{fs}}$, and so $\operatorname{supp} g_x \subseteq \operatorname{supp} x$.

Now, we show that $\operatorname{supp} x \subseteq \operatorname{supp} g_x$. To prove this, by Proposition 2.16, first, we prove that $(0/d)\, g_x \neq g_x$, for all $d \in \operatorname{supp} x$.

Let $\alpha = (0/d_1)\cdots(0/d_{i-1})(1/d_i)(0/d_{i+1})\cdots(0/d_k)$, and $d = d_i$. Then,

$$\begin{aligned}((0/d_i)g_x)(\alpha) &= g_x(\alpha(0/d_i))\\ &= g_x((0/d_1)\cdots(0/d_{i-1})(0/d_i)(0/d_{i+1})\cdots(0/d_k))\\ &= (0\cdots0)_2 = 0\\ &\neq 2^{i-1} = (0\cdots1\cdots0)_2\\ &= g_x(\alpha).\end{aligned}$$

Thus, $(0/d_i)g_x \neq g_x$. $\square$

**Remark 4.7.** The element $g_x$ in Lemma 4.6 belongs to $(A^{Cb})_{\mathrm{fs}}$. Thus, $Cbg_x$ is a $Cb$-subset of $(A^{Cb})_{\mathrm{fs}}$.

In the following proposition, we give all needed information about $Cbg_x$.

**Proposition 4.8.** *Consider $Cbg_x$ constructed in Lemma 4.6. The following statements hold:*

(i) *Suppose $\delta, \delta' \in S'_{g_x}$ with $\mathbb{D}_\delta = \mathbb{D}_{\delta'} = \operatorname{supp} g_x$. Then $\delta(d) = \delta'(d)$, if $\delta g_x = \delta' g_x$, for all $d \in \operatorname{supp} g_x$.*

(ii) *For all $\delta \in S'_{g_x}$, we have $\operatorname{supp} \delta g_x = (\operatorname{supp} g_x) - \mathbb{D}_\delta$.*

(iii) *For all $\delta \in S'_{g_x}$ with $\mathbb{D}_\delta \subsetneq \operatorname{supp} g_x$, we have $\delta g_x \notin Z(Cbg_x)$.*

(iv) *$\mathcal{Z}(Cbg_x) = \{\delta g_x : \mathbb{D}_\delta = \operatorname{supp} g_x\}$.*

(v) *$Cbg_x$ has exactly $2^k$ zero elements.*

*Proof.* (i). On the contrary, suppose there exists some $d \in \operatorname{supp} g_x$ with $\delta(d) = 0 \neq 1 = \delta'(d)$. We show that $\delta g_x \neq \delta' g_x$. Let $\delta = (b_1/d_1)\cdots(0/d_k)$ and $\delta' = (b_1'/d_1)\cdots(b_{k-1}'/d_{k-1})(1/d_k)$. Then,

$$\delta g_x(\iota) = g_x(\iota\delta) = (0c_{k-1}\cdots c_0)_2 \neq (1c_{k-1}'\cdots c_0')_2 = g_x(\iota\delta') = \delta' g_x(\iota).$$

(ii). Let $\delta \in S_{g_x}'$. Then, by Remark 4.3(3), there exists $\delta_1 \in S_{g_x}'$ with $\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} g_x$ and $\delta g_x = \delta_1 g_x$. Also, $\operatorname{supp} g_x \setminus \mathbb{D}_\delta = \operatorname{supp} g_x - \mathbb{D}_{\delta_1}$. If $\mathbb{D}_{\delta_1} = \operatorname{supp} g_x$, then by Lemma 2.17(ii), $\operatorname{supp} \delta_1 g_x \subseteq (\operatorname{supp} g_x) - \mathbb{D}_{\delta_1} = \emptyset$. So, in this case, $\operatorname{supp} \delta g_x = \operatorname{supp} \delta_1 g_x = (\operatorname{supp} g_x) - \mathbb{D}_{\delta_1} = \operatorname{supp} g_x - \mathbb{D}_\delta$. Let $\mathbb{D}_{\delta_1} \subsetneq \operatorname{supp} g_x$. In this case, we also show that $\operatorname{supp} \delta g_x = (\operatorname{supp} g_x) - \mathbb{D}_\delta$. To prove this, it is sufficient to show that $\operatorname{supp} \delta_1 g_x = (\operatorname{supp} g_x) - \mathbb{D}_{\delta_1}$. On the contrary, suppose the equality does not hold. Take $\delta_1', \delta_2' \in S$ with $\mathbb{D}_{\delta_1'} = \mathbb{D}_{\delta_2'} = [(\operatorname{supp} g_x) - \mathbb{D}_{\delta_1})] - \operatorname{supp} \delta_1 g_x$, and $\delta_1'(d) \neq \delta_2'(d)$, for some $d \in \mathbb{D}_{\delta_1'}$. Then, $\delta_1'\delta_1 g_x = \delta_1 g_x = \delta_2'\delta_1 g_x$. We have the following cases;

Case (1): Suppose $\operatorname{supp} \delta_1 g_x \neq \emptyset$. Let $\delta_0 \in S$ with $\mathbb{D}_{\delta_0} = \operatorname{supp} \delta_1 g_x$. Then, $\mathbb{D}_{\delta_0\delta_1'\delta_1} = \operatorname{supp} g_x = \mathbb{D}_{\delta_0\delta_2'\delta_1}$. Now, since there exists some $d \in \mathbb{D}_{\delta_1'}$ with $\delta_1'(d) \neq \delta_2'(d)$, we get that $\delta_0\delta_1'\delta_1(d) \neq \delta_0\delta_2'\delta_1(d)$. So applying (i), we get that $\delta_0\delta_1'\delta_1 g_x \neq \delta_0\delta_2'\delta_1 g_x$ which is a contradiction. This is because the equality of $\delta_1'\delta_1 g_x = \delta_1 g_x = \delta_2'\delta_1 g_x$ implies that $\delta_0\delta_1'\delta_1 g_x = \delta_0\delta_1 g_x = \delta_0\delta_2'\delta_1 g_x$.

Case (2): Let $\operatorname{supp} \delta_1 g_x = \emptyset$. Then, $\mathbb{D}_{\delta_1'\delta_1} = \operatorname{supp} g_x = \mathbb{D}_{\delta_2'\delta_1}$. Now, since there exists some $d \in \mathbb{D}_{\delta_1'}$ with $\delta_1'(d) \neq \delta_2'(d)$, we get that $\delta_1'\delta_1(d) \neq \delta_2'\delta_1(d)$. So, applying (i), we get that $\delta_1'\delta_1 g_x \neq \delta_2'\delta_1 g_x$ which is a contradiction. This is because $\delta_1'\delta_1 g_x = \delta_1 g_x = \delta_2'\delta_1 g_x$.

(iii). Since $\mathbb{D}_\delta \subsetneq \operatorname{supp} g_x$, we get that $\operatorname{supp} g_x - \mathbb{D}_\delta \neq \emptyset$. Now, applying (ii), we have $\operatorname{supp} \delta g_x \neq \emptyset$, and so, $\delta g_x \notin \mathcal{Z}(Cbg_x)$.

(iv). Let $\mathcal{Z} = \{\delta g_x : \mathbb{D}_\delta = \operatorname{supp} g_x\}$. Then, we show that $\mathcal{Z} = \mathcal{Z}(Cbg_x)$. Let $a \in \mathcal{Z}$. Then, $a = \delta g_x$ with $\mathbb{D}_\delta = \operatorname{supp} g_x$. Thus, by Lemma 2.17(ii), $\operatorname{supp} \delta g_x \subseteq (\operatorname{supp} g_x) - \mathbb{D}_\delta = \emptyset$, and so, $\delta g_x \in \mathcal{Z}(Cbg_x)$. Now, let $a \in \mathcal{Z}(Cbg_x)$. Then, $a = \sigma g_x$ for some $\sigma \in Cb$. By Remark 4.3(1), $\sigma \in S_{g_x}'$. Thus, $a = \delta g_x$ where $\delta \in S_{g_x}'$. First, we show that $\mathbb{D}_\delta \subseteq \operatorname{supp} g_x$. Notice that, since $\delta \in S_{g_x}'$, applying Remark 4.3(3), there exists $\delta_1 \in S_{g_x}'$ with $\mathbb{D}_{\delta_1} \subseteq \operatorname{supp} g_x$ and $\delta g_x = \delta_1 g_x$. Also, $\operatorname{supp} g_x - \mathbb{D}_\delta = \operatorname{supp} g_x - \mathbb{D}_{\delta_1}$. If $\mathbb{D}_{\delta_1} \subsetneq \operatorname{supp} g_x$, then by part (iii), $\delta_1 g_x \notin \mathcal{Z}(Cbg_x)$. So, $\delta g_x \notin \mathcal{Z}(Cbg_x)$. Thus, $\mathbb{D}_{\delta_1} = \operatorname{supp} g_x$, and so, $\operatorname{supp} g_x - \mathbb{D}_\delta = \operatorname{supp} g_x - \mathbb{D}_{\delta_1} = \emptyset$. Therefore,

$\mathbb{D}_\delta \subseteq \operatorname{supp} g_x$. Now, if $\mathbb{D}_\delta \subsetneq \operatorname{supp} g_x$, then using part (iii) we get that $\delta g_x \notin \mathcal{Z}(Cbg_x)$ which is impossible. So, $\mathbb{D}_\delta = \operatorname{supp} g_x$.

(v). This follows by (iv). $\square$

**Lemma 4.9.** *Let $\pi\delta_1 g_x = \delta_2 g_x$ where $\mathbb{D}_{\delta_1}, \mathbb{D}_{\delta_2} \subseteq \operatorname{supp} g_x$ and $\pi \in \mathrm{Perm}(\mathbb{D})$. Then,*

(i) $|\mathbb{D}_{\delta_1}| = |\mathbb{D}_{\delta_2}|$.

(ii) $\mathbb{D}_{\delta_1} = \mathbb{D}_{\delta_2}$.

(iii) $\delta_1 = \delta_2$.

*Proof.* (i). Notice that, by Proposition 4.8(ii), $\operatorname{supp} \delta_1 g_x = (\operatorname{supp} g_x) - \mathbb{D}_{\delta_1}$, and $\operatorname{supp} \delta_2 g_x = (\operatorname{supp} g_x) - \mathbb{D}_{\delta_2}$. Now, since $|\operatorname{supp} \delta_1 g_x| = |\operatorname{supp} \delta_2 g_x|$, we get that $|\mathbb{D}_{\delta_1}| = |\mathbb{D}_{\delta_2}|$.

(ii). On the contrary, suppose $\mathbb{D}_{\delta_1} \neq \mathbb{D}_{\delta_2}$. So, there exists some $d \in \mathbb{D}_{\delta_1} - \mathbb{D}_{\delta_2}$ or $d \in \mathbb{D}_{\delta_2} - \mathbb{D}_{\delta_1}$. Assuming $d \in \mathbb{D}_{\delta_2} - \mathbb{D}_{\delta_1}$, we prove the result. The other case is proved similarly. Notice that, $d \in \mathbb{D}_{\delta_2}$ implies that $\delta_2(d) \in \mathsf{2}$, say, $\delta_2(d) = \mathsf{0}$. Take $\delta \in S$ with $\mathbb{D}_\delta = \operatorname{supp} \delta_1 g_x$ and $\delta(d) = \mathsf{1}$. Now, $\pi\delta\delta_1 g_x = \delta''\pi\delta_1 g_x = \delta''\delta_2 g_x$. Applying Proposition 4.8(ii), since $\operatorname{supp} \delta\delta_1 g_x = (\operatorname{supp} \delta_1 g_x) - \mathbb{D}_\delta = \emptyset$, we get that $\delta\delta_1 g_x \in \mathcal{Z}(Cbg_x)$. Thus, by Proposition 4.8(i), we have $\delta\delta_1(d) = \delta''\delta_2(d)$ for all $d \in \operatorname{supp} g_x$ which is a contradiction. This is because $\delta''\delta_2(d) = \mathsf{0}$ while $\delta\delta_1(d) = \delta(d) = \mathsf{1}$.

(iii). By part (ii), we have $\mathbb{D}_{\delta_1} = \mathbb{D}_{\delta_2}$. Now, we show that $\delta_1(d) = \delta_2(d)$, for all $d \in \mathbb{D}_{\delta_1}$. Similar to the proof of (ii), take $\delta \in S$ with $\mathbb{D}_\delta = \operatorname{supp} \delta_1 g_x$. Then, we get that $\pi\delta\delta_1 g_x = \delta''\delta_2 g_x$, and so, $\delta\delta_1(d) = \delta''\delta_2(d)$ for all $d \in \operatorname{supp} g_x$. Let $d \in \mathbb{D}_{\delta_1}$. Then, $\delta_1(d) \in \mathsf{2}$, say $\delta_1(d) = \mathsf{0}$. So, $\delta''\delta_2(d) = \delta\delta_1(d) = \mathsf{0}$. Now, since $\mathbb{D}_{\delta_1} = \mathbb{D}_{\delta_2}$, we get that $d \in \mathbb{D}_{\delta_2}$, and so, $\delta_2(d) = \mathsf{0}$. Thus, $\delta_1 = \delta_2$. $\square$

The following lemma shows that $g_x$ is a strongly finitely supported element of $Cbg_x$.

**Lemma 4.10.** *Let $\sigma g_x = \sigma' g_x$ where $\sigma, \sigma' \in Cb$. Then, $\sigma|_{\operatorname{supp} g_x} = \sigma'|_{\operatorname{supp} g_x}$.*

*Proof.* Let $\sigma g_x = \sigma' g_x$. Then, by Proposition 4.4, we have the following cases:

Case (1): Suppose $\pi g_x = g_x$. In this case, for all $d \in \operatorname{supp} g_x$, we show that $\pi(d) = d$. We have $\pi(\operatorname{supp} g_x) = \operatorname{supp} \pi g_x = \operatorname{supp} g_x$. Take $d \in \operatorname{supp} g_x$. Since $\pi g_x = g_x$, we get that $\pi(0/d) g_x = (0/\pi d) g_x$. Now, by Lemma 4.9, $\pi d = d$.

Case (2): If $\pi\delta g_x = \delta' g_x$, then applying Lemma 4.9, we have $\delta = \delta'$. So, $\pi\delta g_x = \delta g_x$. Notice that, by Proposition 4.8(ii), $\operatorname{supp}\delta g_x = (\operatorname{supp} g_x) - \mathbb{D}_\delta$. If $\operatorname{supp}\delta g_x = \emptyset$, then $\operatorname{supp} g_x = \mathbb{D}_\delta$. So, in this case, it s clear that $\pi\delta(d) = \delta(d)$ for all $d \in \operatorname{supp} g_x$. Suppose $\operatorname{supp}\delta g_x \neq \emptyset$. First, we show $\pi|_{\operatorname{supp}\delta g_x} = \iota|_{\operatorname{supp}\delta g_x}$. Let $d \in \operatorname{supp}\delta g_x$ Then, $\pi d \in \operatorname{supp}\delta g_x$, and so, $\pi(0/d)\delta g_x = (0/\pi d)\delta g_x$. Applying Lemma 4.9, $\pi d = d$. Therefore, for all $d \in \operatorname{supp}\delta g_x$, we have $\pi d = d$. Now, we prove the result. Take $d \in \operatorname{supp} g_x$. If $d \in \mathbb{D}_\delta$, then the result holds. If $d \notin \mathbb{D}_\delta$, then $\sigma d = \pi\delta d = \pi d$ and $\delta' d = \delta d = d$. On the other hand, since $d \in (\operatorname{supp} g_x - \mathbb{D}_\delta) = \operatorname{supp}\delta g_x$, we get that $\pi d = d$. $\square$

**Corollary 4.11.** *Max-zero cyclic finitely supported $Cb$-sets are projective.*

*Proof.* If $X = Cbx$ is a max-zero cyclic finitely supported $Cb$-sets, then $X$ is isomorphic to $\mathbb{D}^{\operatorname{supp} x}$, by Lemma 3.12 and Lemma 4.10. So, applying Lemma 3.16 we get that $X$ is projective. $\square$

## 5. Conclusions

In this section, we assume that $M$ and $N$ are two submonoids of $\operatorname{End}(\mathbb{D})$ such that $M$ is a submonoid of $N$.

**Note**. An $N$-equivariant ($M$-equivariant) map is an equivariant map between finitely supported $N$-sets ($M$-sets). (cf. Example 2.4, [2])

In [6], we proved that free finitely supported $N$-sets exist over finitely supported $M$-sets (Theorem 5.1). By Theorem 5.1, we show that the functor $F$ preserves projective objects and then we characterize projective finitely supported $N$-sets in which $N$ contains a zero-retraction submonoid $M$.

**Theorem 5.1.** (cf. [6]) *The forgetful functor $U : (N\text{-}\mathbf{Set})_{\text{fs}} \to (M\text{-}\mathbf{Set})_{\text{fs}}$ has a left adjoint $F : (M\text{-}\mathbf{Set})_{\text{fs}} \to (N\text{-}\mathbf{Set})_{\text{fs}}$.*

**Remark 5.2.** (cf. [6]) (1) Since $M \leqslant N$, every finitely supported $N$-set can be considered as a finitely supported $M$-set.

(2) The set $N \times X$ together with the action $(n,(n',x)) \mapsto (nn',x)$ is an $N$-set, for each finitely supported $M$-set $X$.

(3) $F(X) = (N \times X)/\sim$ is a finitely supported $N$-set where $X$ is a finitely supported $M$-set and the relation $\sim$ over $N \times X$ is the smallest equivariant equivalence relation generated by $R$ defined as follows:

$$(n,x)R(n',x') \Leftrightarrow \exists m \in M;\ mx = x' \text{ and } n'm|_S = n|_S,$$

where $S$ is a finite support of $x$.

The equivalence class of $(n, x)$ denoted by $[n, x]$.

(4) If $X$ is a finitely supported $M$-set, then $\eta_X : X \longrightarrow F(X)$ defined by $\eta_X(x) = [id, x]$ is an equivariant map.

(5) Using (1), for every finitely supported $N$-set, there exists a surjective equivariant map $\varphi : F(U(X)) = F(X) \longrightarrow X$ defined by $\varphi([n, x]) = nx$.

Now, by Theorem 5.1, we can characterize projective finitely supported $N$-sets.

**Proposition 5.3.** *If $X$ is a projective finitely supported $M$-set, then $F(X)$ is a projective finitely supported $N$-set.*

*Proof.* Let $g : Y \longrightarrow Z$ be a surjective $N$-equivariant map and $f : F(X) \longrightarrow Z$ be an $N$-equivariant map. Then, $f\eta_X : X \longrightarrow Z$ is an $M$-equivariant map, since $\eta_X : X \longrightarrow F(X)$ is an $M$-equivariant map. Now, since $X$ is projective, there exists an $M$-equivariant map $h : X \longrightarrow Y$ with $gh = f\eta_X$. On the other hand, since $F(X)$ is free over $X$, there exists an $N$-equiavriant map $\bar{f} : F(X) \longrightarrow Y$ with $\bar{f}\eta_X = h$. Now, we show that $g\bar{f} = f$. We have

$$g\bar{f}[n, x] = ng\bar{f}[id, x] = ng\bar{f}\eta_X(x) =$$
$$ngh(x) = nf\eta_X(x) = nf([id, x]) = f([n, x]). \qquad \square$$

**Corollary 5.4.** *If $M$ is a zero-retraction submonoid of $N$ and $A \subseteq \mathbb{D}$ is a finite subset, then $F(\mathbb{D}^A)$ is a projective finitely supported $N$-set.*

*Proof.* Follows from Lemma 3.16 and Proposition 5.3. $\square$

**Lemma 5.5.** *Let $\mathbb{D}^A$ be a finitely supported $N$-set where $A$ is a finite subset of $\mathbb{D}$. Then, $\mathbb{D}^A$ is a retract of $F(\mathbb{D}^A)$.*

*Proof.* First, notice that, by Remark 5.2(2), there exists a surjective $N$-equivariant $\varphi : F(\mathbb{D}^A) \longrightarrow \mathbb{D}^A$ defined by $\varphi[n, id|_A] = nid|_A = n|_A$. Suppose $n \in N$. Define $h(n|_A) = [n, id|_A]$. We show that $h$ is an $N$-equivariant and commutes the following diagram; that is, $\varphi h = id$.

$$\begin{tikzcd}
& \mathbb{D}^A \arrow[dl, dotted, "h"'] \arrow[d, "id"] \\
F(\mathbb{D}^A) \arrow[r, two heads, "\varphi"] & \mathbb{D}^A
\end{tikzcd}$$

To do so, let $n, n' \in N$ with $n|_A = n'|_A$. Then, since $A$ is a finite support of $id|_A$, by Remark 5.2(2), we get that $(n, id|_A)R(n', id|_A)$. Now, since $R \subseteq \sim$, we get that $[n, id|_A] = [n', id|_A]$. If $n_1 \in N$, then

$$n_1 h(n|_A) = n_1[n, id|_A] = [n_1 n, id|_A] = h(n_1 n|_A).$$

Also, $\varphi h(n|_A) = \varphi([n, id|_A] = n|_A = id(n|_A)$. □

**Corollary 5.6.** *For every finite subset $A \subseteq \mathbb{D}$, finitely supported $N$-set $\mathbb{D}^A$ is projective.*

*Proof.* Follows from Proposition 5.3 and Lemma 5.5. □

**Theorem 5.7.** *Let $X$ be a finitely supported $N$-set. Then,*

(i) *$X$ is indecomposable and projective if and only if it is cyclic and isomorphic to $\mathbb{D}^A$ for some finite subset $A \subseteq \mathbb{D}$.*

(ii) *$X$ is projective if and only if $X = \coprod_{i \in I} X_i$, where every $X_i$ is iso–morphic to $\mathbb{D}^A$ for some finite $A \subseteq \mathbb{D}$.*

*Proof.* (i). Follows from Lemma 3.20 and Corollary 5.6.
(ii). Follows from (i), Proposition 2.2 and Corollary 5.6. □

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Kh. Keshvardoost
Faculty of Mathematics, Statistics and Computer Sciences, Velayat University, Iranshahr, Sistan and Balouchestan, Iran

E-mail: khadijeh.keshvardoost@gmail.com, kh.keshvardoost@velayat.ac.ir

M. Haddadi
Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, Semnan, Iran

E-mail: m.haddadi@semnan.ac.ir, haddadi_1360@yahoo.com