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On right bases of partially ordered ternary semigroups

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Abstract. We investigate the results of a partially ordered ternary semigroup containing right bases and characterize when a non-empty subset of a partially ordered ternary semigroup is a right base. Moreover, we give a characterization of a right base of a partially ordered ternary semigroup to be a ternary subsemigroup and we show that the right bases of a partially ordered ternary semigroup have same cardinality. Finally, we show that the complement of the union of all right bases of a partially ordered ternary semigroup is a maximal proper left ideal.

1. Introduction

A ternary semigroup is a particular case of n-ary semigroup introduced by Kasner [5], i.e. it is a non-empty set T with an operation $T \times T \times T \to T$, written as $(a, b, c) \to [abc]$, such that $[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in T$. Any ternary semigroup can be embedded into some binary semigroup (called a covering semigroup) in this way that $[abc] = abc$ for $a, b, c \in T$ [1]. Based on the notion of one-sided ideals of a semigroup generated by a non-empty set, the notion of one-sided bases of a semigroup was first introduced by Tamura [6]. Later, this concept was studied by Fabrici [2]. Moreover, the concept of one-sided bases were introduced and discussed in ternary semigroups by Changphas and Kummon [7]. In this paper, we introduce the concept of right bases of a partially ordered ternary semigroup. We study the structure of a partially ordered ternary semigroup containing right bases and extend the conclusions obtained by Thongkam and Changphas [7] to the results in partially ordered ternary semigroups, where by a *partially ordered ternary semigroup* (shortly: *ternary po-semigroup*) is mean

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a ternary semigroup with a partial order such that

$$a \leqslant b \Rightarrow [axy] \leqslant [bxy], \quad [xay] \leqslant [xby] \text{ and } [xya] \leqslant [xyb]$$

for all $a, b, x, y \in T$. In the last years ternary semigroups (also partially ordered) were studied by many authors (see for example [3, 4]).

We shall assume throughout this paper that T stands for a ternary po-semigroup. For non-empty subsets A, B and C of a ternary po-semigroup T, we denote

$$[ABC] := \{[abc] \mid a \in A, b \in B, c \in C\} \text{ and}$$

$$(A] := \{t \in T \mid t \leqslant a \text{ for some } a \in A\}.$$

If $A = \{a\}$, we write $[\{a\}BC]$ as $[aBC]$ and $(\{a\}]$ as $(a]$. For any other cases can be defined analogously. For the sake of simplicity, we write $[ABC]$ as ABC and $[abc]$ as abc.

A non-empty subset A of a ternary po-semigroup T is called a *left* (resp. *right*) *ideal* if (1) $TTA \subseteq A$ (resp. $ATT \subseteq A$) (2) if $x \in A$ and $y \in T$ such that $y \leqslant x$, then $y \in A$. A left ideal A of T is said to be *proper* if $A \subset T$. A proper left ideal A of T is said to be *maximal* if there is no a proper left ideal B of A such that $A \subset B$. Note that the union of left ideals of T is a left ideal of T, and the intersection of left ideals of T is a left ideal of T, if it is non-empty. By $L(A)$ we denote the smallest left ideal of T containing A, that is $L(A) = (A \cup TTA]$. In particular case, for $a \in T$, we write $L(a)$ instead of $L(\{a\})$, called the *principal left ideal* of T generated by a, and it is the from $L(a) = (a \cup TTa]$.

As in [7], we define the *quasi-ordering* on a partially ordered ternary semigroup T by for any $a, b \in T$,

$$a \leqslant_L b \text{ if and only if } L(a) \subseteq L(b).$$

The symbol $a <_L b$ stands for $a \leqslant_L b$ and $a \neq b$ i.e., $L(a) \subset L(b)$.

Let A, B, C be non-empty subsets of T. Then

(1) $A \subseteq (A]$ and $((A]] = (A]$.
(2) If $A \subseteq B$, then $(A] \subseteq (B]$.
(3) $(A](B](C] \subseteq (ABC]$.
(4) $(A] \cup (B] = (A \cup B]$.
(5) $(TTA]$ is a left ideal of T.
(6) For any $a \in T$, $(TTa]$ is a left ideal of T.

2. Main results

In this section we characterize right bases of a ternary po-semigroup and extend the results from [7].

Definition 2.1. A non-empty subset A of a ternary po-semigroup T is called a *right base* of T if:

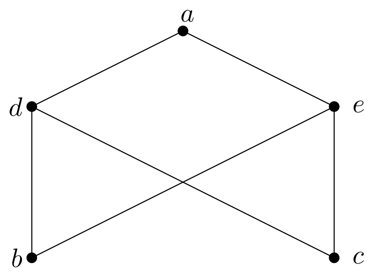
(1) $T = (A \cup TTA]$, i.e., $T = L(A)$;

(2) if B is a subset of A such that $T = L(B)$, then $B = A$.

Example 2.2. Let $T = \{a, b, c, d, e\}$ be a ternary po-semigroup with the operation $xyz = z$ and the partial order $a \leqslant c \leqslant e \leqslant b$ where d is a separate element. Then $\{b, d\}$ is a right base of T, but $\{b\}$ and $\{d\}$ are not right bases of T.

Example 2.3. Let $T = \{a, b, c, d, e\}$ be a ternary po-semigroup with the operation $abc = a * (b * c)$, where $(T, *, \leqslant)$ is a po-semigroup defined by the following table and graph:

$*$	a	b	c	d	e
a	a	e	e	a	e
b	d	b	b	d	b
c	d	b	b	d	b
d	d	b	b	d	b
e	a	e	e	a	e



The right bases of T are $\{a\}$ and $\{d\}$. But $\{a, d\}$ is not a right base.

Lemma 2.4. Let T be a ternary po-semigroup. For any $a, b \in T$, if $a \leqslant b$, then $a \leqslant_L b$.

Lemma 2.5. Let A be a right base of a ternary po-semigroup T, and $a, b \in A$. If $a \in (TTb]$, then $a = b$.

Proof. Let $a, b \in A$ be such that $a \in (TTb]$. Suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Then $B \subset A$. We claim that $L(B) = T$. Clearly, $L(B) \subseteq T$. Next, let $x \in T$. Then, by $L(A) = T$, we have $x \in (A \cup TTA]$. Thus, $x \leqslant y$ for some $y \in A \cup TTA$. There are two cases to consider:

Case 1: $y \in A$. We have $y \neq a$ or $y = a$. If $y \neq a$, then $y \in B \subseteq L(B)$. If $y = a$, then

$$y = a \in (TTb] \subseteq (TTB] \subseteq L(B).$$

Case 2: $y \in TTA$. We set $y = t_1t_2a_1$ for some $t_1, t_2 \in T$ and $a_1 \in A$. If $a_1 \neq a$, then

$$y = t_1t_2a_1 \in TTB \subseteq L(B).$$

If $a_1 = a$, then

$$y = t_1t_2a_1 \in TT(TTb] \subseteq (T](T](TTB] \subseteq ([TTT]TB] \subseteq (TTB] \subseteq L(B).$$

From both cases, we obtain that $y \in L(B)$. Since $x \leqslant y$ and $y \in L(B)$, then $x \in (L(B)] = L(B)$. Thus, $T \subseteq L(B)$. Hence $L(B) = T$. This is a contradiction. Therefore, $a = b$. □

Theorem 2.6. *A non-empty subset A of a ternary po-semigroup T is a right base of T if and only if*

(1) *for any $x \in T$ there exists $a \in A$ such that $x \leqslant_L a$;*

(2) *for any $a, b \in A$, if $a \neq b$, then neither $a \leqslant_L b$ nor $b \leqslant_L a$.*

Proof. Assume that A is a right base of T. We have $L(A) = T$. First, to show that (1) holds. Let $x \in T$. Then $x \in (A \cup TTA]$. Thus, $x \leqslant y$ for some $y \in A \cup TTA$. If $y \in A$ and $x \leqslant y$, by Lemma 2.4, we obtain $x \leqslant_L y$. If $y \in TTA$, then $y = t_1t_2a$ for some $t_1, t_2 \in T$ and $a \in A$. Since $x \leqslant y$ and $y = t_1t_2a \in TTa \subseteq L(a)$, we obtain $x \in (L(a)] = L(a)$. It follows that $L(x) \subseteq L(a)$. Thus, $x \leqslant_L a$ where $a \in A$. This shows that (1) holds.

To prove (2) let $a, b \in A$ be such that $a \neq b$. Suppose that $a \leqslant_L b$. We set $B = A \setminus \{a\}$. Then $b \in B$ and $B \subset A$. Let $x \in T$, by (1), there exists $c \in A$ such that $x \leqslant_L c$ i.e., $L(x) \subseteq L(c)$. Since $c \in A$, we have $c \neq a$ or $c = a$. If $c \neq a$, then $c \in B$. We obtain $x \in L(x) \subseteq L(c) \subseteq L(B)$. If $c = a$, then $x \leqslant_L c = a \leqslant_L b$ and so $x \leqslant_L b$. We obtain $x \in L(x) \subseteq L(b) \subseteq L(B)$. Hence, $T \subseteq L(B)$ and so $T = L(B)$. This is a contradiction. The case $b \leqslant_L a$ can be proved similarly. Thus, $a \leqslant_L b$ and $b \leqslant_L a$ are false.

Conversely, assume that the conditions (1) and (2) hold. We will show that A is a right base of T. Clearly, $L(A) \subseteq T$. By (1), we obtain $T \subseteq L(A)$. Thus, $T = L(A)$. Next, suppose that $T = L(B)$ for some $B \subset A$. Let $a \in A \setminus B$. We have $a \in A \subseteq T = L(B) = (B] \cup (TTB]$. If $a \in (B]$, then $a \leqslant b$ for some $b \in B$. By Lemma 2.4, we obtain $a \leqslant_L b$ where $a, b \in A$. This contradicts to (2). If $a \in (TTB]$, then $a \leqslant t_1t_2b_1$ for some $t_1, t_2 \in T$ and $b_1 \in B$. Since $a \leqslant t_1t_2b_1$ and $t_1t_2b_1 \in TTb_1$, we have $a \in (TTb_1] \subseteq L(b_1)$. It follows that $L(a) \subseteq L(b_1)$. Thus, $a \leqslant_L b$ where $a, b_1 \in A$. This contradicts to (2). Hence, A is a right base of T. □

Definition 2.7. A ternary po-semigroup T is said to be *right singular* if $xyz = z$ for all $x, y, z \in T$.

In general, a right base of a ternary po-semigroup need not be a ternary subsemigroup. Thus, the following theorem is a requirement for a right base to be a ternary subsemigroup.

Theorem 2.8. *Let A be a right base of a ternary po-semigroup T. Then A is a ternary subsemigroup of T if and only if A is right singular.*

Proof. Assume that A is a ternary subsemigroup of T. Let $a, b, c \in A$. By assumption, we have $abc \in A$. Since $abc \in A$, then there exists $x \in A$ such that $x = abc$. Then $x = abc \in TTc \subseteq (TTc]$. By Lemma 2.5, $x = c$. Thus, $abc = c$. Therefore, A is right singular. The converse statement is obvious. □

In Example 2.2 and Example 2.3, it is observed that the cardinality of right bases are the same. However, it turns out that this is true in general, and we will prove in the following theorem.

Theorem 2.9. *Let A and B be any right bases of a ternary po-semigroup T. Then A and B have the same cardinality.*

Proof. Let $a \in A$. Since B is a right base of T, by Theorem 2.6(1), we have $a \leqslant_L b$ for some $b \in B$. Similarly, since A is a right base of T, we have $b \leqslant_L a^*$ for some $a^* \in A$. Thus, $a \leqslant_L b \leqslant_L a^*$ and so $a \leqslant_L a^*$. By Theorem 2.6(2), we obtain $a = a^*$. Hence, $a \leqslant_L b \leqslant_L a$ and so $L(a) = L(b)$. Define a mapping

$$f : A \to B \quad \text{by} \quad f(a) = b \quad \text{for all } a \in A.$$

If $a_1, a_2 \in A$ be such that $a_1 = a_2$, $f(a_1) = b_1$ and $f(a)_2 = b_2$ for some $b_1, b_2 \in B$, we have $L(a_1) = L(a_2)$, $L(a_1) = L(b_1)$ and $L(a_2) = L(b_2)$. Thus, $L(a_1) = L(a_2) = L(b_1) = L(b_2)$ i.e., $b_1 \leqslant_L b_2$ and $b_2 \leqslant_L b_1$. By Theorem 2.6(2), we obtain $b_1 = b_2$. Hence, f is well-defined. Next, to show that f is one-to-one. Let $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2) = b$ for some $b \in B$. Then $a_1 \leqslant_L b$ and $a_2 \leqslant_L b$. Since A is a right base of T, we have $b \leqslant_L a$ for some $a \in A$. Thus, $a_1 \leqslant_L b \leqslant_L a$, $a_2 \leqslant_L b \leqslant_L a$ and so $a_1 \leqslant_L a$, $a_2 \leqslant_L a$. By Theorem 2.6(2), we obtain $a_1 = a = a_2$. Hence, f is one-to-one. Finally, we will show that f is onto. Let $b \in B$. To show that $f(a) = b$ for all $a \in A$, it suffices to show $L(a) = L(b)$ for all $a \in A$. Since A is a right base of T, by Theorem 2.6(1), we have $b \leqslant_L a$ for some $a \in A$. Similarly, since B is a right base of T, we have $a \leqslant_L b^*$ for some $b^* \in B$. Thus, $b \leqslant_L a \leqslant_L b^*$ and so $b \leqslant_L b^*$. By Theorem 2.6(2), $b = b^*$. This implies that $L(a) = L(b)$. Therefore, f is onto. □

Theorem 2.10. *Let A be a right base of a ternary po-semigroup T, and $a \in A$. If $L(a) = L(b)$ for some $b \in T$ and $a \neq b$, then b is an element of a right base of T which is distinct from A.*

Proof. Assume that $L(a) = L(b)$ for some $b \in T$ and $a \neq b$. Let $B = (A \setminus \{a\}) \cup \{b\}$. Then $B \neq A$. We will show that B is a right base of T, it suffices to show that B satisfies the conditions (1) and (2) of Theorem 2.6. First, let $x \in T$. Since A is a right base of T, we have $x \leqslant_L c$ for some $c \in A$. If $c \neq a$, then $c \in B$. If $c = a$, then $L(c) = L(a) = L(b)$. Thus, $L(x) \subseteq L(c) = L(b)$. Hence, $x \leqslant_L b$ where $b \in B$. This means that satisfies the condition (1) of Theorem 2.6. Next, let $b_1, b_2 \in B$ be such that $b_1 \neq b_2$. We consider four cases:

Case 1: $b_1 \neq b$ and $b_2 \neq b$. Then $b_1, b_2 \in A$. This implies neither $b_1 \leqslant_L b_2$ nor $b_2 \leqslant_L b_1$.

Case 2: $b_1 \neq b$ and $b_2 = b$. Then $L(b_2) = L(b)$. If $b_1 \leqslant_L b_2$, we have

$$L(b_1) \subseteq L(b_2) = L(b) = L(a).$$

Thus, $b_1 \leqslant_L a$ where $b_1, a \in A$. This is a contradiction. If $b_2 \leqslant_L b_1$, we have

$$L(a) = L(b) = L(b_2) \subseteq L(b_1).$$

Thus, $a \leqslant_L b_1$ where $b_1, a \in A$. This is a contradiction.

Case 3: $b_1 = b$ and $b_2 \neq b$. Then $L(b_1) = L(b)$. If $b_1 \leqslant_L b_2$, we have

$$L(a) = L(b) = L(b_1) \subseteq L(b_2).$$

Thus, $a \leqslant_L b_2$ where $b_2, a \in A$. This is a contradiction. If $b_2 \leqslant_L b_1$, we have

$$L(b_2) \subseteq L(b_1) = L(b) = L(a).$$

Thus, $b_2 \leqslant_L a$ where $b_2, a \in A$. This is a contradiction.

Case 4: $b_1 = b$ and $b_2 = b$. Then $b_1 = b_2$. This contradicts to $b_1 \neq b_2$. This means that B satisfies the condition (2) of Theorem 2.6. Therefore, B is a right base of T. □

The following corollary follows directly from Theorem 2.10.

Corollary 2.11. *Let A be a right base of a ternary po-semigroup T, and $a \in A$. If $L(a) = L(b)$ for some $b \in T$ and $a \neq b$, then T contains at least two right bases.*

Theorem 2.12. *Let R be the union of all right bases of a ternary po-semigroup T. If $T \setminus R$ is non-empty, then it is a left ideal of T.*

Proof. Assume that $T \setminus R$ is non-empty. Let $x, y \in T$ and $a \in T \setminus R$. Suppose that $xya \notin T \setminus R$. Then $xya \in R$. Thus, $xya \in A$ for some a right base A of T. We set $xya = b$ for some $b \in A$. Then $b = xya \in TTa \subseteq L(a)$. This implies that $L(b) \subseteq L(a)$. Thus, $b \leqslant_L a$. If $L(b) = L(a)$, by Theorem 2.10, $a \in R$. This contradicts to $a \in T \setminus R$. Hence, $L(b) \neq L(a)$. Since A is a right base of T, we have $a \leqslant_L c$ for some $c \in A$. If $c = b$, then $L(a) \subseteq L(c) = L(b) \subseteq L(a)$. Thus, $L(a) = L(b)$. This is a contradiction. Hence, $c \neq b$. Since $b \leqslant_L a$ and $a \leqslant_L c$, we have $b \leqslant_L c$ where $b \neq c$ and $b, c \in A$. This contradicts to the condition (2) of Theorem 2.6. Thus, $xya \in T \setminus R$. Next, let $x \in T \setminus R$ and $y \in T$ such that $y \leqslant x$. By Lemma 2.4, $y \leqslant_L x$. To show that $y \in T \setminus R$, suppose that $y \notin T \setminus R$. Then $y \in R$ and so $y \in B$ for some a right base B of T. Since B is a right base of T, we have $x \leqslant_L z$ for some $z \in B$. Since $y \leqslant_L x$ and $x \leqslant_L z$, then $y \leqslant_L z$ where $y, z \in B$. If $y = z$, we have $x \leqslant_L z = y \leqslant_L x$. By Theorem 2.6(2), $x = y$. This is a contradiction. Thus, $y \neq z$ and $y \leqslant_L z$. This contradicts to the condition (2) of Theorem 2.6. Hence, $y \in T \setminus R$. Therefore, $T \setminus R$ is a left ideal of T. □

Theorem 2.13. *Let R be the union of all right bases of a ternary po-semigroup T such that $R \neq \varnothing$. Then $T \setminus R$ is a maximal proper left ideal of T if and only if $R \neq T$ and $R \subseteq L(a)$ for all $a \in R$.*

Proof. Assume that $T \setminus R$ is a maximal proper left ideal of T. We have $T \setminus R \subset T$ and so $R \neq T$. Let $a \in R$. Suppose that $R \nsubseteq L(a)$. Then there exists $x \in R$ such that $x \notin L(a)$. Since $x \notin T \setminus R$ and $x \notin L(a)$, then $(T \setminus R) \cup L(a) \subset T$. So, we have $(T \setminus R) \cup L(a)$ is a proper left ideal of T, and $(T \setminus R) \subset (T \setminus R) \cup L(a)$. This contradicts to the maximality of $T \setminus R$. Thus, $R \subseteq L(a)$.

Conversely, assume that $R \neq T$ and $R \subseteq L(a)$ for all $a \in R$. We will show that $T \setminus R$ is a maximal proper left ideal of T. Since $\varnothing \neq R \subset T$, then $\varnothing \neq T \setminus R \subset T$. By Theorem 2.12, $T \setminus R$ is a proper left ideal of T. Next, let L is a proper left ideal of T such that $T \setminus R \subset L \subset T$. Then there exists $x \in L$ such that $x \notin T \setminus R$ i.e., $x \in R$. Thus, $R \cap L \neq \varnothing$. Let $a \in R \cap L$.

Then $a \in R$ and $a \in L$. So, we have $R \subseteq L(a)$ and $L(a) \subseteq L$. Hence, $R \subseteq L$ and so

$$T = (T \setminus R) \cup R \subseteq L \subset T.$$

Thus, $T = L$. This is a contradiction. Therefore, $T \setminus R$ is a maximal proper left ideal of T. $\square$

Theorem 2.14. *Let R be the union of all right bases of a ternary po-semigroup T such that $\varnothing \neq R \subset T$, and let L^* be a proper left ideal of T containing every proper left ideal of T. Then the following statements are equivalent:*

(1) *$T \setminus R$ is a maximal proper left ideal of T;*

(2) *$R \subseteq L(a)$ for all $a \in R$;*

(3) *$T \setminus R = L^*$;*

(4) *every right base of T is singleton set.*

Proof. (1) $\Leftrightarrow$ (2). This follows from Theorem 2.13.

(3) $\Leftrightarrow$ (4). Assume that $T \setminus R = L^*$. Then $T \setminus R$ is a maximal proper left ideal of T. Let $a \in R$. By Theorem 2.13, we have $R \subseteq L(a)$. If $T \setminus R \nsubseteq L(a)$ for some $a \in R$, we have $L(a) \neq T$ and so $L(a)$ is a proper left ideal of T. Thus, $a \in L(a) \subseteq L^* = T \setminus R$ and so $a \in T \setminus R$. This contradicts to $a \in R$. Hence, $T \setminus R \subseteq L(a)$. Since $R \subseteq L(a)$ and $T \setminus R \subseteq L(a)$ for all $a \in R$, it follows that

$$T = (T \setminus R) \cup R \subseteq L(a) \subseteq T.$$

Thus, $T = L(a)$ for all $a \in R$. Hence, $\{a\}$ is a right base of T. Next, let A be a right base of T. To show that $a = b$ for all $a, b \in A$ suppose that there exists $a, b \in A$ such that $a \neq b$. Then $a, b \in A \subseteq R$. So, we obtain $T = L(a)$. Since $b \in T = L(a) = (a \cup TTa]$ we have $b \leqslant a$ or $b \in (TTa]$. If $b \leqslant a$, by Lemma 2.4, $b \leqslant_L a$. This contradicts to the condition (2) of Theorem 2.6. Thus, $b \in (TTa]$. By Lemma 2.5, $b = a$. This is a contradiction. Hence, $a = b$ for all $a, b \in A$. Therefore, every right base of T is singleton set. Conversely, assume that every right base of T is singleton set. To show that $T \setminus R = L^*$, it suffices to show $A \subseteq T \setminus R$ for all a proper left ideal A of T. Suppose that A is a proper left ideal of T such that $A \nsubseteq T \setminus R$. Then there exists $x \in A$ such that $x \notin T \setminus R$ i.e., $x \in R$. Since $x \in A$, it follows that $L(x) \in A$. Since $x \in R$, by assumption, $T = L(x)$ and so $T = L(x) \subseteq A \subset T$. Thus, $T = A$. This is a contradiction. Hence, $A \subseteq T \setminus R$. Therefore, $T \setminus R = L^*$.

(1) $\Leftrightarrow$ (3). Assume that $T \setminus R$ is a maximal proper left ideal of T. To show that $T \setminus R = L^*$. Let A be a left ideal of T such that $A \nsubseteq T \setminus R$. Then there exists $x \in A \cap R$. By Theorem 2.13, we have $R \subseteq L(x) \subseteq A$. Thus, $A = R \cup B$ for some $B \subseteq T \setminus R$. For any $a \in T$, there exists $b \in R$ such that $a \leqslant_L b$. Since $b \in R$, then $L(b) \in R$. Thus, $a \in L(a) \subseteq L(b) \subseteq R \subseteq A$. Hence, $T = A$. Therefore, $T \setminus R = L^*$. The converse statement is obvious. □

Theorem 2.15. *Let R be the union of all right bases of a ternary po-semigroup T such that $\varnothing \neq R \subset T$. If $T \setminus R$ is a maximal proper left ideal of T, then one of the following conditions holds:*

(1) $(TTA] = T$ *(i.e., $L(A) = (TTA]$) for every right base A of T;*

(2) *there is unique a right base A of T such that $A \subseteq T \setminus (TTA]$.*

Proof. Assume that $T \setminus R$ is a maximal proper left ideal of T and suppose that the condition (1) is false. By Theorem 2.14, we have a right base $A = \{a\}$ of T and $(TTA] \neq T$. If $a \in (TTa]$, then $(a] \subseteq ((TTa]] = (TTa]$. So, we have $(TTa] = (a] \cup (TTa] = (a \cup TTa] = T$. This is a contradiction. Thus, $a \notin (TTa]$. Hence, $A \subseteq T \setminus (TTA]$. Next, suppose that T contains at least two right bases, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$ such that $a_1 \notin (TTa_1]$,$a_2 \notin (TTa_2]$ and $a_1, a_2 \in R$. We claim that $\{a_1\} = (a_1]$. Suppose that $b \in T \setminus A_1$ such that $b \in (a_1]$. Then $b \leqslant a_1$, by Lemma 2.4, we have $b \leqslant_L a_1$. Thus, $b \in L(b) \subseteq L(a_1) \subseteq L(A_1)$. Clearly, if $x \in T \setminus A_1$ such that $x \notin (a_1]$, then $x \in L(A_1)$. So, we obtain $T \setminus A_1 \subseteq L(A_1)$. Since $A_1 \subseteq L(A_1)$ and $T \setminus A_1 \subseteq L(A_1)$, we have $T \setminus L(A_1) \subseteq T \setminus A_1 \subseteq L(A_1)$. This is a contradiction. Thus, $\{a_1\} = (a_1]$. Since $A_1 \subseteq R$, we have

$$T \setminus R \subseteq T \setminus A_1 = (a_1 \cup TTa_1] \setminus \{a_1\} = ((a_1] \cup (TTa_1]) \setminus (a_1] = (TTa_1].$$

Since $a_2 \in T = (a_1 \cup TTa_1]$, then $a_2 \in (TTa_1]$. Thus, $T \setminus R \subset (TTa_1]$. This contradicts to the maximality of $T \setminus R$. Hence, there is unique a right base A of T such that $A \subseteq T \setminus (TTA]$. □

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