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Translatable isotopes of translatable quasigroups

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Abstract. We determine the structure of translatable isotopes of translatable quasigroups. Necessary and sufficient conditions are found for a bijection between two such isotopes to be an isomorphism. It is also proved that in a left cancellative, k-translatable magma, the value of k is unique.

1. Introduction

This paper is motivated by the following question: *What is the structure of translatable isotopes of a left cancellative translatable magma?* In Theorem 3.1 below we start with a quasigroup that is k-translatable with respect to the natural order. The elements of a quasigroup $(Q,\cdot)$ that is translatable with respect to a particular ordering of Q can be re-labelled so that $(Q,\cdot)$ is translatable with respect to the natural ordering, so that starting with the natural ordering is no limitation. We then determine all bijections α and β on Q such that $(Q,*)$, defined by $l' * m' = \alpha l' \cdot \beta m'$, is h-translatable with respect to the ordering $1', 2', 3', \ldots, n'$ of Q. This ordering is arbitrary, except for the fact that $1' = 1$, the first element of the natural ordering. Using perhaps repeated applications of Lemma 2.6 below, such an ordering is always possible.

That is, we have determined the form of all h-translatable isotopes of any k-translatable quasigroup. As a Corollary, it follows that a k-translatable quasigroup of order n has h-translatable isotopes of every value relatively prime to n. In addition, such translatable isotopes exist for every possible ordering of Q.

We also give a correct proof of the fact that a left cancellative k-translatable magma is translatable for a unique value of k and explain why the proof of this given in [4], Theorem 3.3, is not valid.

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2. Preliminary results, definitions and notation

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All magmas (groupoids) considered here are of finite order n. That is, $Q = \{1, 2, \ldots, n\}$ and $1, 2, 3, \ldots, n$ is the natural ordering. We denote $\{1, 2, \ldots, n\}$ by $\overline{\{1, n\}}$. The value of t modulo n is denoted by $[t]_n$. If $i \equiv j(\mathrm{mod}\, n)$ we write $[i]_n = [j]_n$. Recall that $t \in \overline{\{1, n\}}$ is relatively prime to n if and only if there exists $\tilde{t} \in \overline{\{1, n\}}$ such that $[\tilde{t}t]_n = 1$, if and only if $[tx]_n = [ty]_n$ implies $x = y$ for all $x, y \in \overline{\{1, n\}}$. We denote this by $(t, n) = 1$.

Definition 2.1. (cf. [2]) A finite magma is called *k-translatable* (for fixed k, $1 \leqslant k < n$) if its Cayley table is obtained by the following rule: If the first row of the table is $a_1, a_2, \ldots, a_n$ then the q^{th} row is obtained from the $(q-1)^{th}$ row by taking the last k entries in the $(q-1)^{th}$ row and inserting them as the first k entries of the q^{th} row and by taking the first $(n-k)$ entries from the $(q-1)^{th}$ row and inserting them as the last $(n-k)$ entries of the q^{th} row, where $q \in \{2, 3, \ldots, n\}$. Then, the (ordered) sequence $a_1, a_2, \ldots, a_n$ is called a *k-translatable sequence of Q* with respect to the natural ordering $1, 2, 3, \ldots, n$. A magma is called *translatable* if it has a k-translatable sequence for some $k \in \{1, 2, \ldots, n-1\}$.

Example 2.2. Consider the following magma represented by three Cayley tables with different orderings.

·	1	2	3	4	5
1	2	5	3	4	1
2	3	4	1	2	5
3	1	2	5	3	4
4	5	3	4	1	2
5	4	1	2	5	3

·	4	2	3	1	5
4	1	3	4	5	2
2	2	4	1	3	5
3	3	2	5	1	4
1	4	5	3	2	1
5	5	1	2	4	3

·	4	2	5	3	1
4	1	3	2	4	5
2	2	4	5	1	3
5	5	1	3	2	4
3	3	2	4	5	1
1	4	5	1	3	2

Notice that $(Q, \cdot)$ is 3-translatable with respect to the natural ordering and with respect to the ordering $4, 2, 5, 3, 1$. But it is not translatable with respect to the ordering $4, 2, 3, 1, 5$.

Example 2.3. *The magma $(Q, \cdot)$, where $Q = \{1, 2, 3, 4\}$ and $x \cdot y = 1$, is k-translatable for every $k \in \{1, 2, 3, 4\}$. Its k-translatable sequence has the form* $1, 1, 1, 1$.

The following lemmas are stated without proof, as the proofs are elsewhere, as referenced.

Lemma 2.4. (cf. [2, Lemma 2.5]) *Let $a_1, a_2, \ldots, a_n$ be the first row of the Cayley table of the magma $(Q, \cdot)$ of order n. Then $(Q, \cdot)$ is k-translatable with respect to the natural ordering if and only if for all $i, j \in \overline{\{1, n\}}$ one of the following (equivalent) conditions is satisfied:*

(*i*) $i \cdot j = a_{[k-ki+j]_n}$,

(*ii*) $i \cdot j = [i+1]_n \cdot [j+k]_n$,

(*iii*) $i \cdot [j-k]_n = [i+1]_n \cdot j$.

Lemma 2.5. *Suppose that $Q = \{1', 2', \ldots, n'\}$ is a set of order n. In the magma $(Q, \cdot)$, let $a_i = 1' \cdot i'$ for all $i \in \overline{\{1, n\}}$. Then $(Q, \cdot)$ is k-translatable with respect to the ordering $1', 2', \ldots, n'$ if and only if for all $i, j \in \overline{\{1, n\}}$ one of the following (equivalent) conditions is satisfied:*

(*i*) $i' \cdot j' = a_{[k-ki+j]_n}$,

(*ii*) $i' \cdot j' = [i+1]'_n \cdot [j+k]'_n$,

(*iii*) $i' \cdot [j-k]'_n = [i+1]'_n \cdot j'$.

Note that in Lemma 2.5, $i' \cdot j' \neq a_{[k-ki'+j']_n}$. This is because $[k-ki+j]_n$ marks the position of the entry $a_{[k-ki+j]_n}$. For example, in the third Cayley table in Example 2.2, $(Q, \cdot)$ is 3-translatable with respect to the ordering $1', 2', 3', 4', 5'$ where $1' = 4$, $2' = 2$, $3' = 5$, $4' = 3$ and $5' = 1$. Then, $1' \cdot 2' = 4 \cdot 2 = 3 = 4' \neq 2 = a_3 = a_{[-2]_5} = a_{[3-(3 \cdot 4)+2]_5} = a_{[3-3(1')+2']_5}$ while $1' \cdot 2' = 3 = a_2 = a_{[3-3(1)+2]_5}$.

Lemmas 2.4 and 2.5 above will be applied throughout the rest of the paper, at times without reference. Lemma 2.5 accounts, in part, for the error in the proof of the fact that the value of the translatability of a left cancellative translatable magma is unique, in [4], Theorem 3.3. An error in the proof there is that $a_{j''} = 1 \cdot j'' = 1'' \cdot j'' = c_j \neq c_{j''}$, because as we have just seen, j'' is not necessarily equal to j for all $j \in \overline{\{1, n\}}$, except in the natural ordering.

We now list some previously proved results that, along with the proof of the converse of Lemma 2.7 from [2] will be used as lemmas to give a valid proof that the value of translatability of left cancellative magmas is unique.

Lemma 2.6. (cf. [2, Lemma 2.7]) *Let $(Q, \cdot)$ be a k-translatable magma with respect to the natural ordering $1, 2, \ldots, n$, with k-translatable sequence*

$a_1, a_2, \ldots, a_n$. *Then* $(Q, \cdot)$ *is* k*-translatable with respect to the ordering* $n, 1, 2, \ldots, n-2, n-1$, *with* k*-translatable sequence*

$$a_k, a_{k+1}, \ldots, a_{n-1}, a_n, a_1, a_2, \ldots, a_{k-1}.a_{k-1}.$$

Lemma 2.7. (cf. [4, Lemma 2.6]) *Let* $(Q, \cdot)$ *be a naturally ordered* k*-translatable magma of order* n *with* k*-translatable sequence* $a_1, a_2, \ldots, a_n$ *and suppose that* $(t, n) = 1$. *Then* $(Q, \cdot)$ *is* k*-translatable with respect to the ordering*

$$1, [1+t]_n, [1+2t]_n, [1+3t]_n, \ldots, [1-2t]_n, [1-t]_n$$

with k*-translatable sequence*

$$a_1, a_{[1+t]_n}, a_{[1+2t]_n}, \ldots, a_{[1-2t]_n}, a_{[1-t]_n}.$$

The following result, the converse of Lemma 2.7, is new.

Lemma 2.8. *A magma* $(Q, \cdot)$ *of order* n *is a* k*-translatable with respect to the natural ordering if and only if it is* k*-translatable with respect to the ordering* $1, [1+t]_n, [1+2t]_n, \ldots, [1-2t]_n, [1-t]_n$ *for any* t *relatively prime to* n.

Proof. $(\Rightarrow)$. This is Lemma 2.7.

$(\Leftarrow)$. Let the magma $(Q, \cdot)$ be k-translatable with respect to the ordering $1', 2', 3', \ldots, n'$, where $i' = [1 + (i-1)t]_n$ for all $i \in \overline{\{1, n\}}$ and where $(t, n) = 1$.

Define $t_q = [(q-1)\tilde{t}\,]_n$ for any $q \in \overline{\{1, n\}}$. Note that $[1 + t_q t\,]_n = q = [t_q + 1]'_n$. Then, for all $i, j \in \overline{\{1, n\}}$, by Lemma 2.5(*iii*)

$$\begin{aligned}
[i+1]_n \cdot [j+k]_n &= [1 + (t_i + \tilde{t}\,)t]_n \cdot [1 + (t_j + \tilde{t} + t_k)t]_n \\
&= [t_i + \tilde{t} + 1]'_n \cdot [t_j + \tilde{t} + t_k + 1]'_n \\
&= [t_i + 1]'_n \cdot [t_j + \tilde{t} + t_k + 1 - \tilde{t}k]'_n \\
&= i \cdot [1 + (t_j + \tilde{t} + t_k - \tilde{t}k)t]_n = i \cdot j.
\end{aligned}$$

So, by Lemma 2.4(*ii*), $(Q, \cdot)$ is k-translatable with respect to the natural ordering. □

From Lemma 2.5 (*i*) it follows that if a k-translatable magma $(Q, \cdot)$ of order n is left cancellative, then all elements of its k-translatable sequence (consequently, elements in each row of its Cayley table) are different because $\varphi_i(x) = i \cdot x$ is a bijection. But, in general, such a magma is not a quasigroup. It is a quasigroup if and only if $(k, n) = 1$. A quasigroup of order n can be k-translatable only for $(k, n) = 1$ [2, Lemma 2.15].

Corollary 2.9. *If* $(Q,\cdot)$ *is a* k*-translatable quasigroup, then* $(k,n)=1$.

Theorem 2.10. *A left cancellative magma* $(Q,\cdot)$ *can be* k*-translatable only for one value of* k.

Proof. Suppose that $(Q,\cdot)$ is k-translatable of order n. Then, by a renumeration of the elements of Q, we can consider $(Q,\cdot)$ as k-translatable with respect to the natural ordering. Suppose that $(Q,\cdot)$ is h-translatable with respect to some ordering. Then using Lemma 2.6, perhaps repeatedly, $(Q,\cdot)$ is h-translatable with respect to some ordering $1',2',\dots,n'$, where $1'=1$. Suppose that $(Q,\cdot)$ has the k-translatable sequence $a_1,a_2,\dots,a_n$ with respect to the natural ordering and that $(Q,\cdot)$ has the h-translatable sequence $c_1,c_2,\dots,c_n$ with respect to the ordering $1,2',3',\dots,n'$. Then, for any $i\in Q$, by Lemma 2.4 (i) and Lemma 2.5 (i),

$$a_{i'}=1\cdot i'=1'\cdot i'=c_i. \tag{1}$$

For $i\in Q$, define $s_i=i'$. Then, $s_1=1'=1$ and for any $i,j\in Q$, by the definition of h-translatability $i'\cdot j'=s_i\cdot s_j=a'_{[k-ks_i+s_j]_n}=c_{[h-hi+j]_n}\overset{(1)}{=}a'_{[h-hi+j]_n}$, and, since $(Q,\cdot)$ is left cancellative, we have

$$[k-ks_i+s_j]_n=[h-hi+j]'_n. \tag{2}$$

Then, for any $i\in Q$, $[h+i]'_n=[h-h\tilde{h}+i+1]'_n\overset{(2)}{=}[k-ks_{\tilde{h}}+s_{i+1}]_n=[h-hn+i]'_n\overset{(2)}{=}[k-ks_n+s_i]_n$. So, for all $i\in\overline{\{1,n\}}$,

$$[s_{[i+1]_n}-s_i]_n=[k(s_{\tilde{h}}-s_n)]_n. \tag{3}$$

It is then straightforward to prove by induction on i that

$$s_i=[1+(i-1)k(s_{\tilde{h}}-s_n)]_n. \tag{4}$$

Since $\{s_1,s_2,\dots,s_n\}=\{1',2',\dots,n'\}=\{1,2,\dots,n\}$, $(k(s_{\tilde{h}}-s_n),n)=1$ and the ordering $1',2',\dots,n'$ is of the form in our Lemma 2.8, with $t=[k(s_{\tilde{h}}-s_n)]_n$. Therefore, by Lemma 2.8, $(Q,\cdot)$ is both h and k-translatable with respect to the natural ordering. Applying Lemma 2.3 from [2] we obtain $h=k$. □

3. Translatable isotopes of a translatable magma

A magma $(Q,*)$ is an *isotope* of the magma $(Q,\cdot)$ if there are bijections α,β and γ of Q such that $\gamma(i*j)=\alpha i\cdot\beta j$. If γ is the identity map, then $(Q,*)$ is called a *principal isotope* of $(Q,\cdot)$. One can prove (see for example [1]) that every isotope of the magma $(Q,\cdot)$ is isomorphic to a principal isotope of this magma. Therefore, although our results on isotopes of translatable magma are actually results on the principal isotopes of a translatable magma, up to isomorphism they are results on all isotopes.

Theorem 3.1. *Suppose that a quasigroup $(Q,\cdot)$ of order n is k-translatable with respect to the natural ordering of Q. Then $(Q,*)$ is an isotope of $(Q,\cdot)$ and is h-translatable with respect to the ordering $1,2',3',\dots,n'$ if and only if there exist bijections α and β of Q and $c,d\in\overline{\{1,n\}}$ such that $\alpha c'=n=\beta d'$ and*

(*i*) $l'*m'=\alpha l'\cdot\beta m'$ *for all* $l,m\in\overline{\{1,n\}}$,

(*ii*) $\alpha([c+i]'_n)=i\alpha([c+1]'_n)$ *for all* $i\in\overline{\{1,n\}}$,

(*iii*) $\beta([d+ih]'_n)=ki\alpha([c+1]'_n)$ *for all* $i\in\overline{\{1,n\}}$, *and*

(*iv*) $(\alpha([c+1]'_n),n)=1$.

Proof. $(\Rightarrow)$. Let $a_1,a_2,\dots,a_n$ be the k-translatable sequence of a quasigroup $(Q,\cdot)$. Then since $(Q,\cdot)$ is left cancellative,

$$a_l=a_m \text{ if and only if } l=m. \tag{5}$$

Also, by Lemma 2.4 (i), $l\cdot m=a_{[k-kl+m]_n}$ for all $l,m\in\overline{\{1,n\}}$. Since $(Q,*)$ is an isotope of $(Q,\cdot)$, by definition there exist bijections α and β of Q such that (i) is valid. Hence, $l'*m'=\alpha l'\cdot\beta m'=a_{[k-k(\alpha l')+\beta m']_n}$ for all $l,m\in\overline{\{1,n\}}$.

Since α and β are bijections of Q there exist $c,d\in\overline{\{1,n\}}$ such that $\alpha c'=n=\beta d'$. Thus, for all $i\in\overline{\{1,n\}}$, using h-translatability of $(Q,*)$, by Lemma 2.5(ii) we obtain

$$\begin{aligned}a_k=n\cdot n=\alpha c'\cdot\beta d'=c'*d'=[c+i]'_n*[d+ih]'_n=\alpha([c+i]'_n)\cdot\beta([d+ih]'_n)\\=a_{[k-k\alpha([c+i]'_n)+\beta([d+ih]'_n)]_n}\end{aligned}$$

and so by (5),

$$k\alpha([c+i]'_n)=\beta([d+ih]'_n) \tag{6}$$

for all $i \in \overline{\{1, n\}}$.

Also,

$$\begin{aligned}[c+1]'_n * d' &= \alpha([c+1]'_n) \cdot \beta d' = a_{[k-k\alpha([c+1]'_n)]_n} = [c+1+i]'_n * [d+ih]'_n \\ &= a_{[k-k\alpha([c+1+i]'_n)+\beta([d+ih]'_n)}\end{aligned}$$

and so by (5),

$$k\alpha([c+1+i]'_n) - k\alpha([c+1]'_n) = \beta([d+ih]'_n) \overset{(6)}{=} k\alpha([c+i]'_n). \tag{7}$$

By induction on i we now prove (ii), that $\alpha([c+i]'_n) = i\alpha([c+1]'_n)$ for all $i \in \overline{\{1, n\}}$. Clearly, the statement is true for $i = 1$. If the statement is true for all $t \leqslant i-1$ then $\alpha([c+i-1]'_n) = (i-1)\alpha([c+1]'_n)$. Then by (7) for $i-1$ we have that

$$k\alpha([c+i]'_n) - k\alpha([c+1]'_n) = k\alpha([c+i-1]'_n)$$

and so $k\alpha([c+i]'_n) = ki\alpha([c+1]'_n)$. Since $(k, n) = 1$,

$$\alpha([c+i]'_n) = i\alpha([c+1]'_n) \tag{8}$$

for all $i \in \overline{\{1, n\}}$.

Now (iii) follows from (6) and (8), and (iv) follows from (8), the fact that α is a bijection of Q and the fact that $Q = \{1, 2', 3', \ldots, n'\}$.

($\Leftarrow$). Clearly, $(Q, *)$ is an isotope of $(Q, \cdot)$. We need only prove therefore that $(Q, *)$ is h-translatable with respect to the ordering $1, 2', 3', \ldots, n'$.

For any $l, m \in \overline{\{1, n\}}$, $l = [c+i_l]_n$ and $m = [d+i_m h]_n$ for some $i_l, i_m \in \overline{\{1, n\}}$. Then, $[l+1]'_n * [m+h]'_n = \alpha([c+i_l+1]'_n) \cdot \beta([d+(i_m+1)h]'_n) = a_w$, where,

$$\begin{aligned} w &= [k - k\alpha([c+i_l+1]'_n) + \beta([d+(i_m+1)h]'_n]_n \\ &= [k - k(i_l+1)\alpha([c+1]'_n) + k(i_m+1)\alpha([c+1]'_n)]_n && \text{by } (ii),\ (iii) \\ &= [k - ki_l\alpha([c+1]'_n) + ki_m\alpha([c+1]'_n)]_n \\ &= [k - ki_l\alpha([c+1]'_n) + \beta([d+i_m h]'_n)]_n) && \text{by } (iii) \\ &= [k - k\alpha l' + \beta m']_n. && \text{by } (ii)\end{aligned}$$

So,

$$[l+1]'_n * [m+h]'_n = a_w = a_{[k-k\alpha l'+\beta m']_n} = \alpha l' \cdot \beta m' = l' * m'$$

and, by Lemma 2.5 (ii), $(Q, *)$ is h-translatable with respect to the ordering $1, 2', 3', \ldots, n'$. $\square$

Corollary 3.2. *A k-translatable quasigroup of order n has h-translatable isotopes for all values of h relatively prime to n. It has such isotopes for every ordering.*

Corollary 3.3. *$(\mathbb{Z}_n,*)$ is an h-translatable isotope of $(\mathbb{Z}_n,+)$ with respect to the ordering $1,2',3',\ldots,n'$ if and only if there exist bijections α and β of $\mathbb{Z}_n$ and $c,d\in\mathbb{Z}_n$ such that $\alpha c'=0=\beta d'$, $\alpha([c+i]'_n)=i\alpha([c+1]'_n)=-\beta([d+ih]'_n)$ for all $i\in\overline{\{1,n\}}$, $(\alpha([c+1]'_n),n)=1=(h,n)$ and $l'*m'=\alpha l'\cdot\beta m'$ for all $l,m\in\mathbb{Z}_n$.*

Corollary 3.4. *Suppose that $(Q,\cdot)$ is a k-translatable quasigroup with respect to the natural ordering, with k-translatable sequence $a_1,a_2,\ldots,a_n$. Suppose also that $(Q,*)$ is an h-translatable quasigroup with respect to the ordering $1,2',3',\ldots,n'$. Then $(Q,*)$ is an h-translatable isotope of $(Q,\cdot)$ if and only if there exist $c,d,t\in\overline{\{1,n\}}$ with $(t,n)=1$ and for all $q\in\overline{\{1,n\}}$, $1*q'=a_{x_q}$, where $x_q=[r+(q-1)\tilde{h}kt]_n$ and $r=[k+kct-kt+(1-d)\tilde{h}kt]_n$. Also, $(Q,*)$ is an h-translatable idempotent isotope of $(Q,\cdot)$ if and only if there exist $c,d,t\in\overline{\{1,n\}}$ with $(t,n)=1$ such that $q=a_{[k-kct+(q-d)\tilde{h}kt-ktq]_n}$ for all $q\in\overline{\{1,n\}}$.*

Proof. $(\Rightarrow)$. By Theorem 3.1, there exist bijections α and β of Q and $c,d\in\overline{\{1,n\}}$ such that $\alpha c'=n=\beta d'$ and (i), (ii), (iii) and (iv) of Theorem 3.1 are valid. Let $t=\alpha([c+1]'_n)$.

For any $m\in\overline{\{1,n\}}$ there exists $i_m\in\overline{\{1,n\}}$ such that $m=[d+i_mh]_n$ and $[m+1]_n=[d+(i_m+\tilde{h})h]_n$. Therefore,

$$\beta([m+1]'_n)\overset{(iii)}{=}[k(i_m+\tilde{h})t]_n=[ki_mt]_n+[k\tilde{h}t]_n=\beta m'+[k\tilde{h}t]_n.$$

By Theorem 3.1 again, for all $l,m\in\overline{\{1,n\}}$, we have $l'*m'=\alpha l'\cdot\beta m'=a_{[k-k(\alpha l')+\beta m']_n}$. Therefore, for any $q\in\overline{\{1,n\}}$, $1*q'=1'*q'=\alpha1\cdot\beta q'=a_{x_q}$, where

$$x_q=[k-k\alpha1+\beta q']_n=[k-k\alpha1+\beta1+(q-1)\tilde{h}kt]_n=[r+(q-1)\tilde{h}kt]_n$$

for $r=[k-k\alpha1+\beta1]_n$, whence, applying (ii) and (iii), we obtain

$$r=[k-k(1-c)t+(1-d)\tilde{h}kt]_n=[k+kct-kt+(1-d)\tilde{h}kt]_n.$$

$(\Leftarrow)$. For all $i\in\overline{\{1,n\}}$ we define $\alpha([c+i]'_n)=[it]_n$ and $\beta([d+i]'_n)=[i\tilde{h}kt]_n$. Then α and β are bijections of Q. For any $q\in\overline{\{1,n\}}$, define $b_q=a_w$, where $w=[r+(q-1)\tilde{h}kt]_n$. That is, $b_1,b_2,\ldots,b_n$ is the h-translatable sequence of $(Q,*)$ with respect to the ordering $1,2',3',\ldots,n'$.

By Lemma 2.5 (i), for all $l, m \in \overline{\{1, n\}}$, $l' * m' = b_{[h-hl+m]_n} = a_x$, where $x = [r + (h - hl + m - 1)\tilde{h}kt]_n$. Since $r = [k + kct - kt + (1 - d)\tilde{h}kt]_n$, $x = [k + kct - ktl - d\tilde{h}kt + m\tilde{h}kt]_n$ and $l' * m' = b_{[h-hi+m]_n} = a_x$. Then, since for any $m \in \overline{\{1, n\}}$, $\beta m' = \beta([d + (m - d)]'_n) = [(m - d)\tilde{h}kt]_n$ we have $\alpha l' \cdot \beta m' = a_y$, where $y = [k - k\alpha l' + \beta m']_n = [k - k(l - c)t + (m - d)\tilde{h}kt]_n = x$. Hence, $l' * m' = a_x = a_y = \alpha l' \cdot \beta m'$ and $(Q, *)$ is an h-translatable isotope of $(Q, \cdot)$.

The last sentence in the statement of Corollary 3.4 follows from the fact that $(Q, *)$ is idempotent if and only if $q = q * q = a_{x_{[h-hq+q]_n}}$. □

Example 3.5. The set $Q_8 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with the operation defined by table

$\cdot$	4	8	1	3	2	5	7	6
4	6	5	1	8	2	4	3	7
8	8	2	4	3	7	6	5	1
1	3	7	6	5	1	8	2	4
3	5	1	8	2	4	3	7	6
2	2	4	3	7	6	5	1	8
5	7	6	5	1	8	2	4	3
7	1	8	2	4	3	7	6	5
6	4	3	7	6	5	1	8	2

is a 5-translatable quasigroup. We can re-label the elements of Q_8 as follows: 4 becomes 1, 8 becomes 2, 1 becomes 3, 3 becomes 4, 2 becomes 5, 5 becomes 6, 7 stays as 7 and 6 becomes 8. Then, with this new labelling, $(Q_8, \cdot)$ is 5-translatable with respect to the natural ordering, as follows.

$\cdot$	1	2	3	4	5	6	7	8
1	8	6	3	2	5	1	4	7
2	2	5	1	4	7	8	6	3
3	4	7	8	6	3	2	5	1
4	6	3	2	5	1	4	7	8
5	5	1	4	7	8	6	3	2
6	7	8	6	3	2	5	1	4
7	3	2	5	1	4	7	8	6
8	1	4	7	8	6	3	2	5

Its translatable sequence has the form $8, 6, 3, 2, 5, 1, 4, 7$.

Using Corollary 3.4, we now construct an isotope $(Q_8, *)$ of $(Q_8, \cdot)$ that is 3-translatable with respect to the ordering $1, 2', 3', \ldots, n'$.

We have $k = 5$ and $h = 3 = \tilde{h}$. We choose $c = 4$ and $d = 7 = t = \alpha([4+1]'_8)$. Then we calculate that $r = [k + kct - kt + (1-d)\tilde{h}kt]_8 = 8$. Then, as in the proof of Corollary 3.4 ($\Leftarrow$), for all $l, m \in \overline{\{1, 8\}}$ is $l' * m' = b_{[3-3l+m]_8} = a_{[8+(3-3l+m-1)(105)]_8} = a_{[3-3l+m-1]_8}$. This gives the following 3-translatable sequence for $(Q_8, *)$, with respect to the ordering $1, 2', 3', \ldots, 8'$: $a_8, a_7, a_6, a_5, a_4, a_3,\ a_2, a_1$ or $7, 8, 6, 3, 2, 5, 1, 4$. This gives the following Cayley table for the 3-translatable isotope $(Q_8, *)$ of $(Q_8, \cdot)$.

$*$	1	$2'$	$3'$	$4'$	$5'$	$6'$	$7'$	$8'$
1	7	8	6	3	2	5	1	4
$2'$	5	1	4	7	8	6	3	2
$3'$	6	3	2	5	1	4	7	8
$4'$	4	7	8	6	3	2	5	1
$5'$	2	5	1	4	7	8	6	3
$6'$	8	6	3	2	5	1	4	7
$7'$	1	4	7	8	6	3	2	5
$8'$	3	2	5	1	4	7	8	6

According to Theorem 3.1, the mappings α and β that satisfy $l' * m' = \alpha l' \cdot \beta m'$, for all $l, m \in \overline{\{1, 8\}}$, are: $\alpha 1' = \alpha 1 = 3$, $\alpha 2' = 2$, $\alpha 3' = 1$, $\alpha 4' = 8$, $\alpha 5' = 7$, $\alpha 6' = 6$, $\alpha 7' = 5$, $\alpha 8' = 4$, and $\beta 1' = 2$, $\beta 2' = 3$, $\beta 3' = 4$, $\beta 4' = 5$, $\beta 5' = 6$, $\beta 6' = 7$, $\beta 7' = n$, $\beta 8' = 1$.

Using Lemma 2.6 we can see that 3-translatable isotopes of $(Q_8, \cdot)$ exist for every ordering of Q_8.

Example 3.6. Consider again the 5-translatable quasigroup of Example 3.5, with the following Cayley table:

$\cdot$	1	2	3	4	5	6	7	8
1	8	6	3	2	5	1	4	7
2	2	5	1	4	7	8	6	3
3	4	7	8	6	3	2	5	1
4	6	3	2	5	1	4	7	8
5	5	1	4	7	8	6	3	2
6	7	8	6	3	2	5	1	4
7	3	2	5	1	4	7	8	6
8	1	4	7	8	6	3	2	5

Consider also a quasigroup $(Q_8, \star)$ with ordering $5, 3, 8, 1, 7, 2, 6, 4$ and Cayley table as follows:

$\star$	5	3	8	1	7	2	6	4
5	8	3	6	5	4	1	7	2
3	3	7	4	6	2	8	5	1
8	6	4	8	1	3	5	2	7
1	5	6	1	2	8	7	4	3
7	4	2	3	8	7	6	1	5
2	1	8	5	7	6	2	3	4
6	7	5	2	4	1	3	6	8
4	2	1	7	3	5	4	8	6

Is the quasigroup $(Q_8, \star)$ a translatable isotope of $(Q_8, \cdot)$? If it is translatable then it must be 7-translatable, as all other possible values of translatability, $1, 3$ and 5, do not yield commutative quasigroups.

By (perhaps repeated) application of Lemma 2.6, if $(Q_8, \star)$ is 7-translatable then it is 7-translatable with respect to an ordering $1, 2', 3', 4', 5', 6', 7', 8'$, with 7-translatable sequence $2, b_2, b_3, b_4, b_5, b_6, b_7, b_8$ with $\{b_3, b_5, b_7\} = \{6, 7, 8\}$.

Assuming that $2' = 6$ and using Lemma 2.4 (ii) and (iii), we can calculate that $3' = 3$, $4' = 5$, $5' = 2$, $6' = 4$, $7' = 7$ and $8' = 8$. Using this ordering, $1, 6, 3, 5, 2, 4, 7, 8$ and the Cayley table given above for $(Q_8, \star)$, we can calculate that, in fact $(Q_8, \star)$, is 7-translatable, with 7-translatable sequence $2, 4, 6, 5, 7, 3, 8, 1$ or $a_4, a_7, a_2, a_5, a_8, a_3, a_1, a_6$. Then, by Corollary 3.4, we see that $(Q_8, \star)$ is not a 7-translatable isotope of $(Q_8, \cdot)$. That is because the subscripts of the a's in its translatable sequence must increase successively by the same value of $[\tilde{7} \cdot 5t]_8 = [35t]_8 = [3t]_8$. Although the subscripts start increasing by a value of 3, this does not continue when moving from b_6 to b_7.

4. Translatability and isomorphism

It is known that isomorphism preserves k-translatability; that is, if $(Q, \cdot)$ is k-translatable and isomorphic to $(Q, *)$ then $(Q, *)$ is k-translatable (cf. [3, Theorem 8.14]). However, k-translatable quasigroups of the same order are not necessarily isomorphic. An example of such quasigroups are 3-translatable quasigroups defined by the following tables. The first is without idempotents, in the second - all elements are idempotent. So, they cannot be isomorphic.

$\cdot$	1	2	3	4	5
1	2	4	3	5	1
2	3	5	1	2	4
3	1	2	4	3	5
4	4	3	5	1	2
5	5	1	2	4	3

$\cdot$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	1	5	3	5

So, when are two translatable quasigroups of the same order isomorphic? As already mentioned, we know by [3, Theorem 8.14] that if they are isomorphic then they must have equal value of translatability. Two idempotent, k-translatable quasigroups of the same order are isomorphic [2, Theorem 2.12]. The general problem remains. If $(Q,\cdot)$ and $(S,*)$ are both k-translatable quasigroups of the same order then when are they isomorphic?

Theorem 4.1. *Suppose that $(Q,\cdot)$ and $(S,*)$ are k-translatable quasigroups of the same order n. If $(Q,\cdot)$ is k-translatable with respect to the natural ordering, with k-traslatable sequence $a_1,a_2,a_3,\ldots,a_n$ and $(S,*)$ is k-translatable with respect to the ordering $1',2',3',\ldots,n'$, with k-translatable sequence $b'_1,b'_2,b'_3\ldots,b'_n$. Then $\Psi:Q\to S$ defined by $\Psi i=(s_i)'$, $i\in\overline{\{1,n\}}$, is an isomorphism if and only if*

(i) $s_i=[s_n+it]_n$ for all $i\in\overline{\{1,n\}}$, where $t=[k(s_1-s_{[1-\tilde{k}]_n})]_n$, $(t,n)=1$ and

(ii) $s'_{a_j}=b'_{[r+jt]_n}$ for all $j\in\overline{\{1,n\}}$, where $r=[k(1-s_n-t)+s_n]_n$.

Proof. $(\Rightarrow)$. By Lemma 2.4 (i) and Lemma 2.5 (i), $i\cdot j=a_{[k-ki+j]_n}$ and $i'*j'=b'_{[k-ki+j]_n}$ for all $i,j\in\overline{\{1,n\}}$. Since Ψ is an isomorphism, for all $i,j\in\overline{\{1,n\}}$, $\Psi a_{[k-ki+j]_n}=\Psi(i\cdot j)=\Psi i*\Psi j=s'_i*s'_j=b_{[k-ks_i+s_j]_n}$. But $1\cdot[k-ki+j]_n=a_{[k-ki+j]_n}$, $\Psi a_{[k-ki+j]_n}=b'_{[[k-ks_i+s_j]_n}=\Psi(1\cdot[k-ki+j]_n)=\Psi 1*\Psi[k-ki+j]_n=s'_1*s'_{[k-ki+j]_n}=b'_{[k-ks_1+s_{[k-ki+j]_n}]_n}$ and so

$$[k(s_1-s_i)]_n=[s_{[k-ki+j]_n}-s_j]_n,$$

which for $i=[1-\tilde{k}]_n$ gives

$$[k(s_1-s_{[1-\tilde{k}]_n})]_n=[s_{[j+1]_n}-s_j]_n,$$

for all $j\in\overline{\{1,n\}}$.

The last equation for $t=[k(s_1-s_{[1-\tilde{k}]_n})]_n$ implies $t=[s_1-s_n]_n=[s_2-s_1]_n=\ldots=[s_n-s_{n-1}]_n$. Hence

$$s_i=[s_n+it]_n \tag{9}$$

and $S = \{s_1, s_2, \ldots, s_n\}$ for $(t, n) = 1$.

Also, $\Psi(i \cdot j) = \Psi a_{[k-ki+j]_n} = s'_{a_{[k-ki+j]_n}}$ and $\Psi i * \Psi j = s'_i * s'_j = b'_{[k-ks_i+s_j]_n} \overset{(9)}{=} b'_{[k-ks_n-kit+s_n+jt]_n}$. Therefore, $s'_{a_{[k-ki+j]_n}} = b'_{[k-ks_n-kit+s_n+jt]_n}$. Hence, when $i = 1$, we have

$$s'_{a_j} = b'_{[k-ks_n-kt+s_n+jt]_n} = b'_{[k(1-s_n-t)+s_n+jt]_n} = b'_{[r+jt]_n}$$

for all $j \in \overline{\{1, n\}}$, where $r = [k((1 - s_n - t) + s_n]_n$. So, we have proved (i) and (ii), thereby proving necessity.

$(\Leftarrow)$. Assume that (i) and (ii) are valid. Let $j = [k - kl + m]_n$ for any $l, m \in \overline{\{1, n\}}$. By (ii), $s'_{a_{[k-ki+m]_n}} = b'_{[r+(k-kl+m)t]_n}$. Since $r = [k(1 - s_n - t) + s_n]_n$, $[r + (k - kl + m)t]_n = [k - ks_n + s_n - klt + mt]_n = [k - k(s_n + lt) + (s_n + mt)]_n \overset{(i)}{=} [k - ks_l + s_m]_n$ and so $s'_{a_{[k-kl+m]_n}} = b'_{[k-ks_l+s_m]_n}$ and $\Psi(l \cdot m) = \Psi a_{[k-kl+m]_n} = s'_{a_{[k-kl+m]_n}} = b'_{[k-ks_l-s_m]_n} = s'_l * s'_m = \Psi l * \Psi m$ for any $l, m \in \overline{\{1, n\}}$. Hence $(Q, \cdot)$ and $(S, *)$ are isomorphic. $\square$

Notice that, given the k-translatable quasigroups $(Q, \cdot)$ and $(Q, *)$, given the k-translatable sequence $a_1, a_2, \ldots, a_n$ of Q and a given t relatively prime to n, by (i) and (ii) every $s_n \in Q$ determines a k-translatable sequence $b'_1, b'_2, \ldots, b'_n$ for which $(Q.\cdot)$ and $(Q, *)$ are isomorphic.

Example 4.2. Let $(Q, \cdot)$, where $Q = \{1, 2, 3, 4, 5\}$, be a 2-translatable quasigroup with respect to the natural ordering, with 2-translatable sequence $3, 1, 5, 2, 4$. Let a quasigroup $(S, *)$ be 2-translatable with respect to the ordering $b'_1, b'_2, b'_3, b'_4, b'_5$. Let $s_5 = 5$ and $t = 3$, with $\Psi i = (s_i)'$, $i \in \overline{\{1, 5\}}$ and $s_1 = 3$, $s_2 = 1$, $s_3 = 4$, $s_4 = 2$, $s_5 = 5$. Suppose that $b_1 = 3$, $b_2 = 1$,$b_3 = 2$, $b_4 = 4$, $b_5 = 5$ Then, by Theorem 4.1, Ψ is not an isomorphism because although (i) is satisfied, $s_{a_5} = s_4 = 2 \neq b_x$, where $x = [k(1 - s_5 - t) + s_6 + 5t]_5 = 1$ and so, $2 \neq b_1 = 3$. Thus, (ii) is not satisfied and Ψ is not an isomorphism. However, if we consider the mapping when $s_5 = 2$ and $t = 3$, then $s_1 = 5$, $s_2 = 3$, $s_3 = 1$, $s_4 = 4$ and this satisfies (i) and (ii). So, $(Q, \cdot)$ and $(S, *)$ are isomorphic, with that mapping as the isomorphism; namely, $1 \mapsto 5'$, $2 \mapsto 3'$, $3 \mapsto 1'$, $4 \mapsto 4'$, $5 \mapsto 2'$.

Example 4.3. Here are the Cayley tables of the 2-translatable quasigroups of Example 4.2.

$\cdot$	1	2	3	4	5
1	3	1	5	2	4
2	2	4	3	1	5
3	1	5	2	4	3
4	4	3	1	5	2
5	5	2	4	3	1

$*$	$1'$	$2'$	$3'$	$4'$	$5'$
$1'$	b'_1	b'_2	b'_3	b'_4	b'_5
$2'$	b'_4	b'_5	b'_1	b'_2	b'_3
$3'$	b'_2	b'_3	b'_4	b'_5	b'_1
$4'$	b'_5	b'_1	b'_2	b'_3	b'_4
$5'$	b'_3	b'_4	b'_5	b'_1	b'_2

Using Theorem 4.1, we can determine all 2-translatable sequences $b'_1, b'_2, b'_3, b'_4, b'_5$ of $(S,*)$ such that $(Q,\cdot)$ and $(Q,*)$ are isomorphic.

Since $s_5 \in \overline{\{1,5\}}$ and $t \in \overline{\{1,4\}}$ there are 20 such 2-translatable sequences. Below we present these sequences for $t=2$.

s_5	t	r	s_1	s_2	s_3	s_4	b_1	b_2	b_3	b_4	b_5	$\Psi 1$	$\Psi 2$	$\Psi 3$	$\Psi 4$	$\Psi 5$
1	2	2	3	5	2	4	3	4	1	2	5	$3'$	$5'$	$2'$	$4'$	$1'$
2	2	1	4	1	3	5	5	2	3	1	4	$4'$	$1'$	$3'$	$5'$	$2'$
3	2	5	5	2	4	1	3	4	2	5	1	$5'$	$2'$	$4'$	$1'$	$3'$
4	2	4	1	3	5	2	5	3	1	2	4	$1'$	$3'$	$5'$	$2'$	$4'$
5	2	3	2	4	1	3	4	2	3	5	1	$2'$	$4'$	$1'$	$3'$	$5'$

We can check that the quasigroups in the table above are actually isomorphic to $(Q,\cdot)$ by using the mapping Ψ and re-ordering $(S,*)$ accordingly. In this sense, Ψ can be considered to be a mapping that re-orders S, giving it a 2-translatable Cayley table that is more clearly isomorphic to $(Q,\cdot)$. For example, for $s_5=3$, $t=2$ and $r=5$ we have the following:

$\cdot$	1	2	3	4	5
1	3	1	5	2	4
2	2	4	3	1	5
3	1	5	2	4	3
4	4	3	1	5	2
5	5	2	4	3	1

$*$	$1'$	$2'$	$3'$	$4'$	$5'$
$1'$	$3'$	$4'$	$2'$	$5'$	$1'$
$2'$	$5'$	$1'$	$3'$	$4'$	$2'$
$3'$	$4'$	$2'$	$5'$	$1'$	$3'$
$4'$	$1'$	$3'$	$4'$	$2'$	$5'$
$5'$	$2'$	$5'$	$1'$	$3'$	$1'$

$*$	$5'$	$2'$	$4'$	$1'$	$3'$
$5'$	$4'$	$5'$	$3'$	$2'$	$1'$
$2'$	$2'$	$1'$	$4'$	$5'$	$3'$
$4'$	$5'$	$3'$	$2'$	$1'$	$4'$
$1'$	$1'$	$4'$	$5'$	$3'$	$2'$
$3'$	$3'$	$2'$	$1'$	$4'$	$5'$

It is not at all obvious that the first and second quasigroups above are isomorphic. Whereas, using the mapping ψ that takes $1 \mapsto 5'$, $2 \mapsto 2'$, $3 \mapsto 4'$, $4 \mapsto 1'$ and $5 \mapsto 3'$, it is clear that the first and third Cayley tables are exactly the same, except for this re-labelling.

Definition 4.4. Suppose that $(Q,\cdot)$ is a k-translatable quasigroup with respect to the natural ordering, with k-translatable sequence $a_1, a_2, \ldots, a_n$. If $(Q,*)$ is an h-translatable quasigroup with respect to the ordering $1, 2', \ldots, n'$ and is also an isotope of $(Q,\cdot)$, then we write

$$(Q,*) = (Q,*,h,i',\cdot,k,c,d,t,a_1,a_2,\ldots,a_n),$$

where c, d and t are as in Corollary 3.4. If $(Q,*)$ is h-translatable with respect to the natural ordering then we write

$$(Q,*) = (Q,*,h,i'=i,\cdot,k,c,d,t,a_1,a_2,\ldots,a_n).$$

Corollary 4.5. *Suppose that* $(Q,*) = (Q,*,h,i'=i,\cdot,k,c,d,t,a_1,a_2,\ldots,a_n)$, $(Q,\bullet) = (Q,\bullet,h,i',\cdot,k,e,f,u,a_1,a_2,\ldots,a_n)$ *and* $\Phi : (Q,*) \to (Q,\bullet)$ *is defined by* $\Psi i = (s_i)'$, $i \in \overline{\{1,n\}}$. *Then* Ψ *is an isomorphism if and only if*

(1) $s_i = [s_n + iv]_n$ *for all* $i \in \overline{\{1,n\}}$, *where* $v = [h(s_1 - s_{[1-\tilde{h}]_n})]_n$ *and* $(v,n) = 1$, *and*

(2) $\Psi\, a_{[x+(j-d)\tilde{h}kt]_n} = a_{[y+(r+jv-f)\tilde{h}ku]_n}$ *for all* $j \in \overline{\{1,n\}}$, *where* $x = [k+kct-kt]_n$, $y = [k+keu-ku]_n$ *and* $r = [h(1-s_v-v)+s_n]_n$.

Proof. $(\Rightarrow)$. By Theorem 4.1, (i) of Corollary 4.5 is valid. By Corollary 3.4, the j^{th} entry in the h-translatable sequences of $(Q,*)$ and $(Q,\bullet)$ is $a_{[x+(j-d)\tilde{h}kt]_n}$ and $a_{[y+(j-f)\tilde{h}ku]_n}$ respectively, where $x = [k+kct-kt]_n$ and $y = [k+keu-ku)]_n$. Note that the h-translatable sequence of $(Q,\bullet)$, $b'_1, b'_2, \ldots, b'_n$, satisfies $b'_j = a_{[y+(j-f)\tilde{h}ku]_n}$. Then, Theorem 4.1 (ii) implies $(s_{a_{[x+(j-d)\tilde{h}kt]_n}})' = (b_{[r+jn]_n})'$, where $r = [h(1-s_v-v)+s_v]_n$. Therefore, (ii) of Corollary 4.5 is valid.

$(\Leftarrow)$. This follows from $(\Leftarrow)$ of Theorem 4.1. $\square$

Example 4.6. Let $(Q,\cdot)$ be the quasigroup determined by the following Cayley table.

$\cdot$	1	2	3	4	5
1	3	1	4	5	2
2	4	5	2	3	1
3	2	3	1	4	5
4	1	4	5	2	3
5	5	2	3	1	4

Then $(Q,\cdot)$ is 3-translatable with respect to the natural ordering, with 3-translatable sequence $a_1 = 3$, $a_2 = 1$, $a_3 = 4$, $a_3 = 5$ and $a_5 = 2$. Using Corollary 3.4, we now construct a quasigroup $(Q,*)$ that is 3-translatable with respect to the natural ordering, is an isotope of $(Q,\cdot)$ and is not isomorphic to $(Q,\cdot)$.

Firstly, we want $5*5 = 5$. This ensures that $(Q,*)$ and $(Q,\cdot)$ are not isomorphic, because $(Q,\cdot)$ has no idempotent elements. Now, since we want $(Q,*)$ to be 3-translatable with respect to the natural ordering, in Corollary 3.4 we have $q' = q$ for all $q \in \overline{\{1,5\}}$.

Then, since we want $h = k$, $1 * q' = 1 * q = a_{x_q}$, where $x_q = [r + (q - 1)\tilde{k}kt]_5 = [r + (q-1)t]_5$ and $r = [k - kct - kt + (1-d)\tilde{k}kt]_5$. Choosing $c = 5$ and $d = 1$ we get $r = [3 - 3t]_5$ and $x_q = [3 - 3t + (q-1)t] - 5$. But, since $(Q, *)$ is 3-translatable and $5 * 5 = 5$, by Corollary 3.4, we have $1 * 3' = 1 * 3 = a_{x_3} = 5 = a_4$. Hence, $x_3 = 4 = [3 - 3t + (3-1)t]_5 = [3-t]_5$ and $t = [-1]_5 = 4$ and $r = [3 - 3t]_5 = 1$.

This gives $x_q = [1 + (q-1)4]_5$ and so, $x_1 = 1$, $x_2 = 5$, $x_4 = 3$ and $x_5 = 2$. Therefore, by Corollary 3.4 again, the 3-translatable sequence of $Q, *)$ is a_1, a_5, a_4, a_3, a_2 or $3, 2, 5, 4, 1$. This gives the following Cayley table for $(Q, *)$.

$*$	1	2	3	4	5
1	3	2	5	4	1
2	5	4	1	3	2
3	1	3	2	5	4
4	2	5	4	1	3
5	4	1	3	2	5

Using $(\Leftarrow)$ of Corollary 3.4, we see that $\alpha([c+i]'_5) = \alpha i' = \alpha i = [it]_5 = [4i]_5 = \beta([1+i]'_5) = \beta[i+1]_5$. This gives $\alpha 1 = 4 = \beta 2$, $\alpha 2 = 3 = \beta 3$, $\alpha 3 = 2 = \beta 4$, $\alpha 4 = 1 = \beta 5$ and $\alpha 5 = 5 = \beta 1$. One easily checks that $i * j = \alpha i \cdot \beta j$ for all $i, j \in \overline{\{1, 5\}}$. Therefore, $(Q, *)$ is the required 3-translatable isotope of $(Q, \cdot)$.

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