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Applications of $\bigcap$-large pseudo N-injective acts in quasi-Frobenius monoid theory and its relationship with some classes of injectivity

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Abstract. The aim of this paper is to review thoroughly the applications of $\bigcap$-large pseudo N-injective acts in quasi-Frobenius monoid theory, and therefore, the relationship of $\bigcap$-large pseudo N-injective acts with some class of injectivity is studied. Applications of the properties of $\bigcap$-large pseudo injective acts in quasi-Frobenius monoid theory are proven. Also, it's proved that the subsequent parity, every union (direct sum) of the two $\bigcap$-large pseudo injective acts is a $\bigcap$-large pseudo injective act if and only if every $\bigcap$-large pseudo injective act is injective under Noetherian condition for a right monoid S. Additionally, we proved that the category of strongly $\bigcap$-large pseudo N-injective right S-acts are going to be egalitarian to the category of projective right S-acts under monoid conditions. The connections between quasi injective and $\bigcap$-large pseudo injective acts are investigated.

1. Introduction

Acts over semigroups appeared and were utilized in a spread of applications like graph theory, combinatorial problems, algebraic automata theory, mathematical linguistics, the theory of machines, and theoretical computing. In a semigroup theory, it represents semigroups as semigroups of functions from a set to itself such it's almost like to Cayley's theorem [10]. This suggests that a semigroup action consists of a semigroup S, a set A, and a mapping of the elements of the semigroup S to functions from the set A to itself. Thereby in any mathematical structure on a set, the collection gathering of structure-preserving maps of the set to itself is an example of an abstract algebraic object called a semigroup. On the opposite hand,

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if you're given an abstractly defined semigroup, when can it's represented as a semigroup of maps of a mathematical structure? One can say that it's represented by actions. As for the monoid, the action is a functor from that category to an arbitrary category. It's well-known that a really natural concept and important tool within the study of monoids is that the idea of getting monoids working on certain (finite) sets. This provides how to show any monoid into a (finite) transformation monoid. Additionally, a monoid action is in contrast to group operations where a monoid action often comes with a natural grading which will be wont to perform certain calculations more efficiently. Due to the importance of the unit, it's rather thought of a semigroup as a "monoid apart from unit," instead of the standard way of a monoid as a "semigroup with unit" [10].

Now, during this paper, S means a monoid with zero elements 0 and each right S-act M is unitary with zero elements Θ which is denoted by $\mathrm{M_S}$. It's possible to seek out the S-act in many names mentioned as S-acts, S-sets, S-operands, S-polygons, transition acts, S-automata [10]. Note that we'll use terminology and notations from [1, 2, 3, 4, 5, 8, 13, 15] freely. For more information about S-act we refer the reader for [9, 10, 12, 17].

A *right S-act* $\mathrm{M_S}$ with zero is a non-empty set with a function f from $M \times S$ into M such $(m, s) \mapsto ms$ and therefore, the following conditions hold

(1) $m(st) = (ms)t$ for all $m \in M$ and $s, t \in S$,

(2) $m1 = m$, where 1 is that the identity element of S,

(3) $m.0 = 0$, where 0 is the zero element of S.

A *subact* N of an S-act $\mathrm{M_S}$ is a non-empty subset of M such that $xs \in N$ for all $x \in N$ and $s \in S$. A subact N of $\mathrm{M_S}$ is called large (or essential) in $\mathrm{M_S}$ if and only if any homomorphism $f : M_{\mathrm{S}} \longrightarrow H_{\mathrm{S}}$, where $\mathrm{H_S}$ is any S-act with restriction to N is one to at least one, then f is itself one to at least one. During this case, we are saying that $\mathrm{M_S}$ is an important extension of N. A non-zero subact N of $\mathrm{M_S}$ is *intersection large* if for all non-zero subact A of $\mathrm{M_S}$, $A \bigcap N \neq \Theta$, and will be denoted by N is $\bigcap$-large in $\mathrm{M_S}$ [7]. A non-zero S-act $\mathrm{M_S}$ over a monoid S is called $\bigcap$-*reversible* if every non-zero subact of $\mathrm{M_S}$ is $\bigcap$-large. A monoid S is called $\bigcap$-*reversible* if $\mathrm{S_S}$ is $\bigcap$-reversible S-act [19]. An equivalence relation ρ on a right S-act $\mathrm{M_S}$ is a congruence relation if and only if aρb implies that $as\rho bs$ for all $a, b \in M_{\mathrm{S}}$ and $s \in S$.

An S-act $\mathrm{A_S}$ is called *injective* if for every monomorphism $\alpha : \mathrm{C_S} \longrightarrow \mathrm{B_S}$ and every S-homomorphism $\beta : \mathrm{C_S} \longrightarrow \mathrm{A_S}$, there exists an S-homomorphism $\sigma : \mathrm{B_S} \longrightarrow \mathrm{A_S}$ such $\sigma\alpha = \beta$ [11]. Let $\mathrm{M_S}$, $\mathrm{N_S}$ be S-acts. $\mathrm{N_S}$ is *pseudo M-*

injective if for each S-subact A of M_S, each S-monomorphism $f : A \longrightarrow N_S$ is often extended to an S-homomorphism $g : M_S \longrightarrow N_S$. An S-act N_S is *pseudo injective* if it's pseudo N-injective. An S-act M_S is $\bigcap$*-large pseudo N-injective* if for any $\bigcap$-large subact X of N, any monomorphism $f : X \longrightarrow M_S$ is often extended to some $g : N \longrightarrow M_S$. M_S is $\bigcap$*-large pseudo injective* if M is $\bigcap$-large pseudo M–injective.

2. Applications of $\bigcap$-large pseudo injective acts

Let N be a simple subact of an S-act M_S. Then $\mathrm{Soc_N}(M_S)$ is called a *homogeneous component* of $\mathrm{Soc}(M_S)$ containing N. Thus

$$\mathrm{Soc}_N\left(M_S\right) := \bigcup \left\{X \ be \ subact \ of \ M_S : X \cong N\right\}.$$

Definition 2.1. Let S be a moniod and A be a class of S-acts, A is called *socle fine* whenever for any $M_S, N_S \in A$, we have $\mathrm{Soc}(M_S) \cong \mathrm{Soc}(N_S)$ if and only if $M_S \cong N_S$.

An S-act M_S is *Noetherian* if every subact of M_S is finitely generated. A monoid S is a *right Noetherian* if S_S is Noetherian. Equivalently, S is a right Noetherian if and only if S satisfies the ascending chain condition for right ideals.

An S-act M_S is *projective* if for every S-epimorphism g from S-act A_S into S-act B_S and each homomorphism h from M_S into B_S, there's a homomorphism f from M_S into A_S such that $g \circ h = f$.

Definition 2.2. A monoid S is quasi-Frobenius if and only if S satisfies any of the following equivalent conditions:

1. S is Noetherian on one side and self-injective on one side.
2. S is Artinian on a side and self-injective on a side.
3. All right (or all left) S-acts that are projective are also injective.
4. All right (or all left) S-acts that are injective are also projective.

For example, every semisimple monoid is quasi-Frobenius, since all acts are projective and injective. We denote by SL the category of strongly $\bigcap$-large pseudo N-injective right S-acts, by PR the category of projective right S-acts and E is the injective hull.

Theorem 2.3. *The following conditions are equivalent for a monoid S.*

(1) *S is quasi-Frobenius.*

(2) *The class PR $\bigcup$ SL is socle fine.*

Proof. (1)⇒(2). If S is quasi-Frobenius, then projective S-acts are injective. Thus, PR $\bigcup$ SL=SL.

Let M_S, $N_S \in$ SL with Soc(M_S)≅Soc(N_S).Then E(Soc(M_S))≅E(Soc(N_S)). Since S is right Artinian, Soc(M_S) is a $\bigcap$-large subact of M_S and Soc(N_S) is a $\bigcap$-large subact of N_S. Hence, E(M) ≅E(N). Then, by Proposition 2.7 in [6], we obtain $M_S \cong N_S$. Thus the class PR $\bigcup$ SL is socle fine.

(2)⇒(1). Let P be a projective right S-act. Then P ∈ PR, E(P) ∈ SL and Soc(P)=Soc(E(P)). By (2), we get P≅ E(P) and hence, P is injective. It follows that S is quasi-Frobenius. □

Theorem 2.4. *The following conditions are equivalent for a monoid S.*

(1) *S is semisimple.*

(2) *The class of all $\bigcap$-large pseudo injective acts is socle fine.*

(3) *The class SE is the socle fine.*

Proof. (1)⇒(2) since over every semisimple monoid S in the class of all S-acts is socle fine.

(2)⇒(3). It is clear.

(3)⇒(1). Clearly Soc (E(S_S))=Soc(Soc(S_S)). Since E (S_S) and Soc(S_S) are $\bigcap$-large pseudo injective, we obtain E(S_S)=Soc(S_S) by (3). It implies that E(S_S) is semisimple and so S is semisimple. □

3. $\bigcap$-large pseudo N-injective and injective acts

Recall that Soc(M_S) represents all $\bigcap$-large subacts of M_S and it's mentioned as $Soc(M_S\ ) := \bigcap\{X \mid X$ is $\bigcap-$large subact of $M_S\}$. Also, $\mathrm{Soc}_N\ (M_S)$ represents the homogeneous component of Soc(M_S) containing N where N is a simple subact of an S-act M_S. Thus,

$$\mathrm{Soc}_N\ (M_S)\ := \bigcup\{X \text{ be subact of } M_S : X \cong N\ \}.$$

Definition 3.1. An S-act M_S is referred to as *strongly $\bigcap$-large pseudo injective* if, M_S is $\bigcap$-large pseudo N-injective for all right S-act N_S.

Recall that an ordered groupoid S is called *Artinian* if S satisfies the descending chain condition for ideals.

Lemma 3.2. *Let M_S and N_S be S-acts and M_S be $\bigcap$-reversible. Then, N_S is an injective S-act if and only if N_S is $\bigcap$-large pseudo M-injective for all M_S.*

Proof. By Proposition 2.10(3) in [13], if N_S is $\bigcap$-large pseudo M-injective for all M_S, then every S-monomorphism $f : N_S \to M_S$ is essential since $f(N)$ is $\bigcap$-large in M_S. So, it's split for all S-acts M_S, thus N_S is injective. □

Recall that a proper subact N of an S-act M_S is *maximal* if for every subact K of M_S with $N \subseteq K \subseteq M_S$ implies either K = N or K = M_S.

Definition 3.3. M_S is called a *V-act* (or *cosemisimple*) if every proper subact of M_S is an intersection of maximal subacts. S is called a *V-monoid* if the right act S_S is a V-act.

It is well known that M_S is a V-act if and only if every simple act is M-injective.

Theorem 3.4. *Let S is a right Noetherian monoid. Then*

(1) *Every direct sum of two $\bigcap$-large pseudo injective acts is $\bigcap$-large pseudo injective if and only if every $\bigcap$-large pseudo injective is injective.*

(2) *Essential extensions of semisimple right S-acts are $\bigcap$-large pseudo injective if and only if S is right V-monoid.*

Proof. (1). Assume that the act M_S is $\bigcap$-large pseudo injective(this means that M_S is $\bigcap$-large pseudo M-injective), S is a right Noetherian monoid and E(M) is injective envelope of M_S. Then, since E(M) is injective, so, it's $\bigcap$-large pseudo injective and by assumption (Every direct sum of two $\bigcap$-large pseudo injective acts is $\bigcap$-large pseudo injective) N_S=$M_S \bigoplus$E(M) is $\bigcap$-large pseudo injective. Consider the injection maps j_1:$M_S \longrightarrow$E(M), j_2 : E(M)$\longrightarrow M_S \bigoplus$E(M), j_3 : $M_S \longrightarrow M_S \bigoplus$E(M) and the identity map I_M : $M_S \longrightarrow M_S$. Let π_M: $M_S \bigoplus$E(M)$\longrightarrow M_S$ be the projection map such that $\pi_M \circ j_3$=I_M. Now $M_S \bigoplus$E(M) is $\bigcap$-large pseudo injective, so this implies there exists an S-homomorphism g : $M_s \bigoplus$E(M) $\longrightarrow M_s \bigoplus$E(M) such that $g \circ j_2 \circ j_1 = j_3 \circ I_M$, then $\pi_M \circ g \circ j_2 \circ j_1$= $\pi_M \circ j_3 \circ I_M$.

Thus, I_M=$\pi_M \circ g \circ j_2 \circ j_1$, so that f =$\pi_M \circ g \circ j_2$ and then I_M=f$\circ j_1$. Therefore, M_S is a retract of E(M) and then it is injective. For the converse, let M_S and N_S be two $\bigcap$-large pseudo injective S-act. By hypothesis M_S and N_S are injective which implies that the direct sum of any two injective S-acts is injective whence S is Noetherian monoid [15] and then every injective act is $\bigcap$-large pseudo injective. Therefore, the direct sum of two $\bigcap$-large pseudo injective is $\bigcap$-large pseudo injective.

(2). Let M_S be a semisimple act. Then $M_S \oplus E(M)$ is an essential extension of a semisimple act. It follows that $M_S \oplus E(M)$ is an $\bigcap$-large pseudo injective act and so by (1) M_S is injective. Thus, S is a right V-monoid and a right Noetherian monoid. The converse is obvious because every semisimple right S-act is injective. □

Proposition 3.5. *Let M_S be an S-act and $\{N_i \mid i \in I\}$ be a family of S-acts. Then $\prod_{i\in I} N_i$ is $\bigcap$-large pseudo M-injective if and only if N_i is $\bigcap$-large pseudo M-injective for every $i \in I$.*

Proof. ($\Rightarrow$). Assume that $N_S = \prod_{i\in I} N_i$ is $\bigcap$-large pseudo M-injective, where M_S is an S-act. Let X be a $\bigcap$-large subact of M_S and f be S-monomorphism from X to N_i. Since N_S is a $\bigcap$-large pseudo M-injective act then there exists an S-homomorphism $g : M_S \longrightarrow N_S$ such that $g \circ i = j_i \circ f$, where i is the inclusion map of X into M_S and j_i is the injection map of N_i into N_S. Define $h : M_S \longrightarrow N_i$ such that $h = \pi_i \circ g$ where π_i is the projection map from N_S onto N_i. Then $h \circ i = \pi_i \circ g \circ i = \pi_i \circ j_i \circ f = f$. That is for all $x \in X$, $h(x) = h(i(x)) = \pi_i(g(x)) = \pi_i(g(i(x))) = \pi_i(j_i(f(x))) = (\pi_i \circ j_i)(f(x)) = f(x)$. Figure 1 illustrates this:

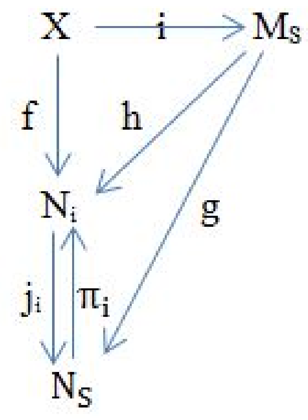

Figure 1: Illustrate that N_S is a $\bigcap$-large pseudo M-injective.

($\Leftarrow$). Assume that N_i is $\bigcap$-large pseudo M-injective for each $i \in I$, where M_S is an S-act. Let A be a $\bigcap$-large subact of M_S, f be S-monomorphism from A to $N_S = \prod_{i\in I} N_i$. Since N_i is $\bigcap$-large pseudo M-injective, there exists an S-homomorphism $\beta_i : M_S \longrightarrow N_i$, such that $\beta_i \circ i = \pi_i \circ f$. Then, we claim that there exists an S-homomorphism β : $\mathrm{M_S}\longrightarrow\mathrm{N_S}$ such that $\beta_i{=}\pi_i \circ \beta$. We claim that $\beta \circ i = f$. Since $\beta_i \circ i = \pi_i \circ \beta \circ i$, then $\pi_i \circ f = \pi_i \circ \beta \circ i$, so we obtain $f = \beta \circ i$. Figure 2 explains this:

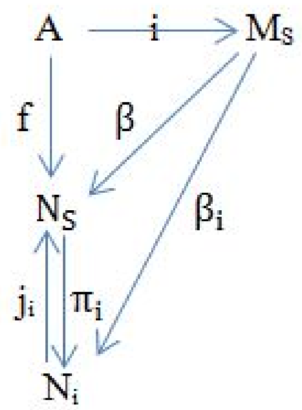

Figure 2: Clarifies that $\mathrm{N_i}$ is $\bigcap$-large pseudo M-injective act.

Therefore $\mathrm{N_S}$ is $\bigcap$-large pseudo M-injective. □

Corollary 3.6. *Let M_S and N_i be S-acts, where $i \in I$ and I is a finite index set. Then, for every i, N_i is $\bigcap$-large M-pseudo injective if and only if $\bigoplus_{i=1}^{n} N_i$ is $\bigcap$-large M-pseudo injective.*

Proposition 3.7. *Let M_S be a $\bigcap$-large subact of M_s^n. M_S^n is $\bigcap$-large pseudo injective for any finite integer n, if and only if M_S is $\bigcap$-large pseudo M-injective (this means M_S is $\bigcap$-large pseudo injective).*

Proof. Let M_S^n be $\bigcap$-large pseudo injective. Since $\mathrm{M_S}$ is a $\bigcap$-large subact of M_S^n, so by Proposition 3.4 in [15], M_S^n is $\bigcap$-large pseudo M-injective. As $\mathrm{M_S}$ is retract of M_S^n, for this reason, and by Lemma 3.3 in [15], $\mathrm{M_S}$ is $\bigcap$-large pseudo M-injective. Conversely, if $\mathrm{M_S}$ is $\bigcap$-large pseudo M-injective, then by Proposition 3.5, M_S^n is $\bigcap$-large pseudo M-injective. □

Every pseudo N-injective act is $\bigcap$-large pseudo N-injective. The subsequent proposition answers the question: When the converse is true?

Proposition 3.8. *For a $\bigcap$-reversible act N_S the following conditions are equivalent:*

(a) *M_S is pseudo N-injective;*

(b) *M_S is $\bigcap$-large pseudo N-injective.*

Proof. $(a) \Rightarrow (b)$. Follows from the definition.

$(b) \Rightarrow (a)$. Let $\mathrm{M_S}$ be $\bigcap$-large pseudo N-injective, A be any subact of $\mathrm{N_S}$. Let $f : A \longrightarrow M_{\mathrm{S}}$ be any monomorphism and α : $\mathrm{A} \longrightarrow \mathrm{N_S}$ be the inclusion map. $\mathrm{N_S}$ being $\bigcap$-reversible implies α is an essential monomorphism. Since $\mathrm{M_S}$ is $\bigcap$-large pseudo N-injective, there exists $h \in Hom(N, M)$ such that $f = h \circ \alpha$. Hence $\mathrm{M_S}$ is pseudo N-injective. □

Each quasi injective act is $\bigcap$-large pseudo injective, but the converse is not true in general. The subsequent theorem gives the condition for the converse to be correct.

An S-act $\mathrm{H_S}$ is called *cog-reversible* if each congruence ρ on $\mathrm{H_S}$ with $\rho \neq I_H$ is large on $\mathrm{H_S}$.

Theorem 3.9. *Let M_S be a cog-reversible nonsingular S-act with $\updownarrow_M(s) = \Theta$ for each $s \in S$ and M_S be $\bigcap$-reversible. Then M_S is $\bigcap$-large pseudo injective if and only if M_S is quasi injective act.*

Proof. Let A be a subact of an S-act $\mathrm{M_S}$ and f be a nonzero S-homomorphism from A into $\mathrm{M_S}$. Since $\mathrm{M_S}$ is $\bigcap$-reversible, so A is $\bigcap$-large subact of $\mathrm{M_S}$. If f is an S-monomorphism, then there is nothing to prove. So assume f is not an S-monomorphism. Since E(M) is injective, then E(M) is an M (respectively E(M))-injective. Thus, there is an S-homomorphism $h : M_{\mathrm{S}} \longrightarrow E(M)$ such that $h \circ \omega_A = \omega_M \circ f$, where ω_A (respectively ω_M) is the inclusion mapping of A (respectively $\mathrm{M_S}$) into $\mathrm{M_S}$ (respectively E(M)). Again there is an S-homomorphism $g : E(M) \longrightarrow E(M)$ such that $g \circ \omega_M = h$. Then either $\ker(h) = I_M$ or $\ker(h) \neq I_M$. If $\ker(h) = I_M$, then h is an S-monomorphism. The Largeness of $\mathrm{M_S}$ in E(M) implies that g is an S-monomorphism, so $g(M_{\mathrm{S}}) \subseteq M_{\mathrm{S}}$ by Theorem 3.6 in [13]. Thus, $h(M_{\mathrm{S}}) \subseteq M_{\mathrm{S}}$ which is extension of f, since $h(A) = h \circ \omega_A(A) = \omega_M \circ f(A) = f(A)$. If $\ker(h) \neq I_M$, then $\ker(h)$ is large on $\mathrm{M_S}$, so $\mathrm{M_S}/\ker(h)$ is singular. But $\mathrm{M_S}/\ker(h) \cong h(M) \subseteq M_{\mathrm{S}}$, so $\mathrm{M_S}/\ker(h)$ is nonsingular. These two cases imply that $\ker(h) = M \times M$. This implies that h (and hence f) is a zero map. □

The subsequent theorem illustrates that $\mathrm{M_1}$ and $\mathrm{M_2}$ are quasi injective acts whence the direct sum is $\bigcap$-large pseudo injective by using some conditions.

Theorem 3.10. *Let M_1 and M_2 be $\bigcap$-reversible S-acts such that M_i is $\bigcap$-large in $M_1 \bigoplus M_2$ for each $i = 1, 2$. If $M_1 \bigoplus M_2$ is $\bigcap$-large pseudo injective, then M_1 and M_2 are quasi injective acts.*

Proof. Let A be a subact of M_1 and $f : A \longrightarrow M_1$ be an S-homomorphism. Then, by assumption M_1 is $\bigcap$-reversible S-act, so A is $\bigcap$-large in M_1 and then in $\mathrm{M}_1 \bigoplus \mathrm{M}_2$. Define $\alpha : A \longrightarrow M_1 \bigoplus M_2$ by $\alpha(a) = (f(a), a)$, $\forall a \in A$. Then α is an S-monomorphism. By Theorem 3.4, $\mathrm{M}_1 \bigoplus \mathrm{M}_2$ is $\bigcap$-large M_1-pseudo injective, so there exists an S-homomorphism β : $\mathrm{M}_1 \longrightarrow \mathrm{M}_1 \bigoplus \mathrm{M}_2$ such that $\beta \circ \mathrm{i} = \alpha$. Now, let j_1 and π_1 be the injection and projection map of M_1 into $\mathrm{M}_1 \bigoplus \mathrm{M}_2$ and $\mathrm{M}_1 \bigoplus \mathrm{M}_2$ onto M_1. Then, $\sigma = \pi_1 \beta L)$ $\mathrm{M}_1 \longrightarrow \mathrm{M}_1$ be an S-homomorphism that extends f, this means $\sigma i = \pi_1\ \beta i = \pi_1\ j_1 f\ = I_{M_1} f = f$, which implies $\sigma i = f$. □

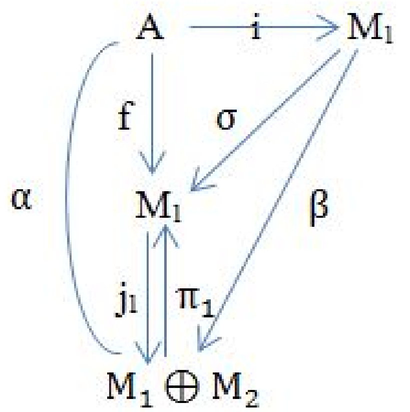

Figure 3: Explains that $\mathrm{M}_1 \bigoplus \mathbf{M_2}$ **is** $\bigcap$**-large** $\mathbf{M_1}$**-pseudo injective act.**

Proposition 3.11. (cf. [6]) *Let M_S be an S-act and $\{N_i \,|\, i \in I\}$ be a family of S-acts. Then $\prod_{i \in I} N_i$ is M-injective if and only if N_i is M-injective for all $i \in I$.*

Corollary 3.12. *For any integer $n \geqslant 2$, let M_S be $\bigcap$-reversible and a cog-reversible nonsingular S-act with $\updownarrow_M(s) = \Theta$ for each $s \in S$. Then M_S^n is $\bigcap$-large pseudo M-injective if and only if M_S is quasi injective.*

Proof. If M_S^n is $\bigcap$-large pseudo injective, then by Theorem 3.4 M_S^n is $\bigcap$-large pseudo M-injective. Then, by Lemma 2.3 in [13], $\mathrm{M_S}$ is $\bigcap$-large pseudo

M-injective. Since $\mathrm{M_S}$ is a cog-reversible nonsingular S-act, so by Theorem 3.10, $\mathrm{M_S}$ is quasi injective act. Conversely, if $\mathrm{M_S}$ is quasi injective act, then by Proposition 3.11, M_S^n is quasi injective and in particular, is $\bigcap$-large pseudo M-injective. □

Proposition 3.13. *Let $M_S=\bigoplus_{i\in I} M_i$ be a direct sum of a cog-reversible non-singular with $\updownarrow_M(s) = \Theta$ for each $s \in S$ and $\bigcap$-reversible S-acts M_i. An S-act M_S is quasi injective if and only if it is $\bigcap$-large pseudo injective.*

Proof. Let $\mathrm{M_S}$ be a $\bigcap$-large pseudo injective S-act. This means that $\mathrm{M_S}$ is $\bigcap$-large pseudo M-injective, so by Proposition 2.4 in [13], $\mathrm{M_j}$ is $\bigcap$-large pseudo M-injective, Now, each $\mathrm{M_j}$ is $\bigcap$-large pseudo M-injective act, so by Theorem 3.9, each $\mathrm{M_j}$ is quasi injective. Therefore, by Proposition 3.11 $\mathrm{M_S}$ is quasi injective. The rest is obvious. □

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