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Generalized Green's relations and GV-ordered semigroups

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Abstract In this paper an extensive study of the concepts of generalized Green's relations and GV-semigroups without order to ordered semigroups have been given. Our approach allows one to see the nature of generalized Green's relations in the class of GV-ordered semigroups. Moreover we show that an ordered semigroup S is a GV-ordered semigroup if and only if S is a complete semilattice of completely π-regular and Archimedean ordered semigroups.

1. Introduction and preliminaries

An *ordered semigroup* S is a partially ordered set $(S, \leqslant)$ and at the same time a semigroup $(S, \cdot)$ such that for all a, b and $c \in S$, $a \leqslant b$ implies $ac \leqslant bc$ and $ca \leqslant cb$. It is denoted by $(S, \cdot, \leqslant)$. For an ordered semigroup S and $H \subseteq S$, denote the downward closure of H by $(H] = \{t \in S : t \leqslant h, \text{ for some } h \in H\}$. Throughout this paper S will stand for an ordered semigroup unless otherwise stated.

An ordered semigroup S is said to be *Archimedean* if for every $a, b \in S$ there exists $n \in \mathbb{N}$ such that $a^n \in (SbS]$. A nonempty subset I of S is said to be a *left* (resp. *right*) *ideal* of S, if $SI \subseteq I$ (resp. $IS \subseteq I$) and $(I] \subseteq I$. If I is both a left and right ideal, then it is called an ideal of S. We call S a (resp. left, right) simple ordered semigroup if it does not contain any proper (resp. left, right) ideal. We denote by $R(x)$, $L(x)$, $I(x)$ the right ideal, left ideal, ideal of S, respectively, generated by x $(x \in S)$, where $R(x) = (x \cup xS]$, $L(x) = (x \cup Sx]$, $I(x) = (x \cup xS \cup Sx \cup SxS]$ for all $x \in S$. For an ordered semigroup $(S, \cdot, \leqslant)$, we denote $S^1 = S \cup \{1\}$, where 1 is a symbol, such that $1a = a$, $a1 = a$ for each $a \in S$ and $1 \cdot 1 = 1$. An ordered semigroup S is said to be *regular* (resp. *completely regular*) ordered semigroup if for every $a \in S$, $a \in (aSa]$ (resp. $a \in (a^2Sa^2]$). An ordered semigroup S is called *π-regular* (resp. *completely π-regular*) if for every $a \in S$ there is $m \in \mathbb{N}$ such that $a^m \in (a^mSa^m]$ (resp. $a^m \in (a^{2m}Sa^{2m}]$). The set of all regular, completely regular and π-regular elements in an ordered semigroup S are denoted by $Reg_{\leqslant}(S)$, $Gr_{\leqslant}(S)$ and $\pi Reg_{\leqslant}(S)$ respectively.

The class of completely regular ordered semigroups is a subclass of the class of regular ordered semigroups. Galbiati and Veronesi [3] studied class of semigroups

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(without order), where these two notion coincides. These semigroups are named after them as GV-semigroups. In this paper we extend the notion of GV-semigroups to ordered semigroups.

In a semigroup (without a partial order) Green's relations use to play a significant role to study regular semigroups. Following L. Marki, O. Steinfeld [10] and J.L. Galbiati, M.L. Veronesi [3] we generalized Green-Kehayopulu relations [7], to study π-regular, completely π-regular and GV-ordered semigroups.

Due to Kehayopulu [7] Green's relations on a regular ordered semigroup given as follows: $a\mathcal{L}b$ if $L(a) = L(b)$, $a\mathcal{R}b$ if $R(a) = R(b)$, $a\mathcal{J}b$ if $I(a) = I(b)$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

These four relations $\mathcal{L}, \mathcal{R}, \mathcal{J}$, and $\mathcal{H}$ are equivalence relations.

A congruence ρ on S is called a*semilattice congruence* if for every $a, b \in S$, $a \ \rho \ a^2$ and $ab \ \rho \ ba$. By a *complete semilattice congruence* we mean a semilattice congruence σ on S such that for $a, b \in S$, $a \leqslant b$ implies that $a \ \sigma \ ab$. If σ is a semilattice congruence on S, then $(x)_\sigma$ is a subsemigroup for any $x \in S$. An ordered semigroup S is called a *complete semilattice of subsemigroups* of type τ if there exists a complete semilattice congruence ρ such that each ρ-congruence class $(x)_\rho$ is a type τ subsemigroup of S. Equivalently [9], there exist a semilattice Y and a family of subsemigroups $\{S_\alpha\}_{\alpha \in Y}$ of type τ of S such that:

1. $S_\alpha \cap S_\beta = \phi$ for any $\alpha, \ \beta \in Y \ with \ \alpha \neq \beta$,
2. $S = \bigcup_{\alpha \in Y} S_\alpha$,
3. $S_\alpha S_\beta \subseteq S_{\alpha \beta}$ for any $\alpha, \ \beta \in Y$,
4. $S_\beta \cap (S_\alpha] \neq \phi$ implies $\beta \preceq \alpha$, where $\preceq$ is the order of the semilattice Y defined by $\preceq := \{(\alpha, \ \beta) \mid \beta = \alpha \ \beta \ (\beta \ \alpha)\}$.

An element $e \in S$ is called an *ordered idempotent* [5] if $e \leqslant e^2$. We denote the set of all ordered idempotents of an ordered semigroup S by $E_\leqslant(S)$. An element $b \in S$ is *inverse* of $a \in S$ if $a \leqslant aba$ and $b \leqslant bab$. We denote the set of all ordered inverses of an element a of an ordered semigroup S by $V_\leqslant(a)$.

The *zero* of an ordered semigroup $(S, \cdot, \leqslant)$ is an element of S, usually denoted by 0, such that $0 \leqslant x$ and $0.x = x.0 = 0$ for all $x \in S$. An ordered semigroup S with 0 is called *nil* if for every $a \in S$ there is $n \in \mathbb{N}$ such that $a^n = 0$.

Cao and Xu [4] defined a nil-extension of an ordered semigroup as follows:

Let I be an ideal of an ordered semigroup S. Then $(S/I, \cdot, \preceq)$ is called the *Rees factor* ordered semigroup of S modulo I, and S is called an *ideal extension* of I by the ordered semigroup S/I. Moreover S is said to be a *nil-extension* of I if $(S/I, \cdot, \preceq)$ is a nil ordered semigroup.

2. Main results

Let S be a π-regular ordered semigroup. Following Galbiati and Veronesi [3], let us define the relations $\mathcal{L}^*, \mathcal{R}^*, \mathcal{J}^*, \mathcal{H}^*$ by: For $a, b \in S$,

$$a\mathcal{L}^*b \text{ if and only if } a^m\mathcal{L}b^n$$
$$a\mathcal{R}^*b \text{ if and only if } a^m\mathcal{R}b^n$$
$$a\mathcal{J}^*b \text{ if and only if } a^m\mathcal{J}b^n \text{ and } \mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$$

where m, n are the smallest positive integers such that $a^m, b^n \in Reg_{\leqslant}(S)$. These four relations are equivalence relations on S.

We denote $\mathcal{L}^*(a), \mathcal{R}^*(a), \mathcal{H}^*(a)$, and $\mathcal{J}^*(a)$ respectively the $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*$, and $\mathcal{J}^*$-classes containing an element a of S.

Lemma 3.1. *Let S be a π-regular ordered semigroup. Every $\mathcal{L}^*$ ($\mathcal{R}^*$, $\mathcal{J}^*$)-class contains at least one ordered idempotent.*

Proof. Let L be a $\mathcal{L}^*$-class and $a \in L$. Let m be the smallest positive integer such that $a^m \leqslant a^m x a^m$, for some $x \in S$. This implies $xa^m \leqslant (xa^m)^2$. Therefore $xa^m \in E_{\leqslant}(S)$. We have to show that $a\mathcal{L}^*xa^m$. Let $y = xa^m$. Now $a^m \leqslant a^m x a^m \leqslant a^m x a^m x a^m \leqslant a^m (xa^m)^2$, so that for every $r \in \mathbb{N}$, $a \leqslant a^m (xa^m)^r$. Let $a^m \leqslant a^m (xa^m)^{r_1}$, where r_1 is the smallest positive integer such that $(xa^m)^{r_1} \in Reg_{\leqslant}(S)$. Now $y^{r_1} = xa^m \dots xa^m xa^m (r_1 \text{ times}) = (xa^m \dots x)a^m = pa^m$ where $p = xa^m \dots x \in S$. Therefore $a^m \mathcal{L} y^{r_1}$. Therefore $y\mathcal{L}^*(a)$. This implies $xa^m \in L$. Therefore L contains an ordered idempotent. □

Proposition 3.2. *Let S be a π-regular ordered semigroup and $a, b \in S$. Then the following statements hold in S:*

(1) *$a\mathcal{L}^*b$ if and only if there exists $a' \in V_{\leqslant}(a^p)$ and $b' \in V_{\leqslant}(b^q)$ such that $a'a^p\mathcal{L}b'b^q$ where p, q are the smallest positive integers such that $a^p, b^q \in Reg_{\leqslant}(S)$.*

(2) *$a\mathcal{R}^*b$ if and only if there exists $a' \in V_{\leqslant}(a^p)$ and $b' \in V_{\leqslant}(b^q)$ such that $a^pa'\mathcal{R}b^qb'$ where p, q are the smallest positive integers such that $a^p, b^q \in Reg_{\leqslant}(S)$.*

(3) *$a\mathcal{H}^*b$ if and only if there exists $a'' \in V_{\leqslant}(a^m)$ and $b'' \in V_{\leqslant}(b^n)$ such that $a''a^m\mathcal{L}b''b^n$ and $a^ma''\mathcal{R}b^nb''$ where m, n are the smallest positive integers such that $a^m, b^n \in Reg_{\leqslant}(S)$.*

Proof. We proof only the last condition. Two first conditions follows similarly.

(3): Let $a\mathcal{H}^*b$. Then $a^m\mathcal{H}b^n$ where m, n are the smallest positive integer such that a^m, $b^n \in Reg_{\leqslant}(S)$. Since $a^m \in Reg_{\leqslant}(S)$ there exists $a' \in S$ such that $a^m \leqslant a^m a' a^m$. Clearly $a^m a', a' a^m \in E_{\leqslant}(S)$. Let $e = a'a^m$ and $f = a^m a'$. Then $e\mathcal{L}a^m$ and $f\mathcal{R}a^m$. So that $e\mathcal{L}^*a\mathcal{L}^*b$ and $f\mathcal{R}^*a\mathcal{R}^*b$. Since $b^n \in Reg_{\leqslant}(S)$ then there exists $b' \in S$ such that $b^n \leqslant b^n b' b^n$. Let $e_1 = b'b^n$ and $f_1 = b^n b'$. Then $e_1, f_1 \in E_{\leqslant}(S)$. Clearly $e_1 \mathcal{L}^* b \mathcal{L}^* a$ and $f_1 \mathcal{R}^*_b \mathcal{R}^*_a$. Since $e\mathcal{L}^* a$ we have $e \leqslant x_1 a^m$ for some $x_1 \in S^1$. Also $a^m \leqslant a^m e$ and $a^m \leqslant f a^m$. Say $a'' = e x_1 f$. Then $a^m \leqslant a^m e \leqslant a^m x_1 a^m \leqslant a^m e x_1 a^m \leqslant a^m e x_1 f a^m \leqslant a^m a'' a^m$ and $a'' = e x_1 f \leqslant e(e x_1 f) \leqslant e(e)a'' \leqslant e(x_1 a^m)a'' \leqslant (e x_1)(f a^m)a^{''} \leqslant (e x_1 f)a^m a'' = a'' a^m a''$. Therefore $a'' \in V_{\leqslant}(a^m)$. Therefore $e\mathcal{L} a'' a^m$.

Also $e_1 \mathcal{L}^* b$ gives $e_1 \leqslant x_2 b$ for some $x_2 \in S^1$. Also $b^n \leqslant b^n e_1$ and $b^n \leqslant f_1 b^n$. Take $b'' = e_1 x_2 f_1$. Now $b^n \leqslant b^n e_1 \leqslant b^n x_2 b^n \leqslant (b^n e_1) x_2 b^n \leqslant b^n e_1 x_2 (f_1 b^n) \leqslant b^n (e_1 x_2 f_1) b^n \leqslant b^n b'' b^n$ and $b'' = e_1 x_2 f_1 \leqslant e_1(e_1 x_2 f_1) \leqslant e_1 e_1 b'' \leqslant e_1(x_2 b^n) b'' \leqslant e_1 x_2 f_1 b^n b'' \leqslant b'' b^n b''$. Therefore $b'' \in V_{\leqslant}(b^n)$. Therefore $e_1 \mathcal{L} b'' b^n$. Thus $a^{''} a^m \mathcal{L} e \mathcal{L} a^m \mathcal{L} b^n \mathcal{L} e_1 \mathcal{L} b^{''} b^n$. Similarly $a^m a^{''} \mathcal{R} b^n b^{''}$. Hence the proof.

Conversely assume that the given conditions hold in S. Since $a^m a'' \mathcal{R} b^n b''$ and $a'' a^m \mathcal{L} b'' b^n$ for some $a'' \in V_{\leqslant}(a^m)$, $b'' \in V_{\leqslant}(b^n)$, then there are $x, y, z, w \in S^1$ such that $a^m a'' \leqslant (b^n b'')x$, $b^n b'' \leqslant (a^m a'')y$, $a'' a^m \leqslant z(b'' b^n)$ and $b'' b^n \leqslant w(a'' a^m)$. Since $a'' \in V_{\leqslant}(a^m)$, we have $a^m \leqslant a^m a'' a^m \leqslant (b^n b'' x)a^m \leqslant b^n u$, where $u = b'' x a^m \in S$. Again $a^m \leqslant a^m a'' a^m \leqslant a^m(z b'' b^n) \leqslant w_1 b^n$ where $w_1 = a^m z b'' \in S$. Similarly taking $b'' \in V_{\leqslant}(b^n)$ it can shown that $b^n \leqslant a^m w_2$ and $b^n \leqslant w_3 a^m$ for some $w_2, w_3 \in S$. Therefore $a^m \mathcal{H} b^n$ and hence $a\mathcal{H}^* b$. □

We now generalize the concept of GV-semigroups (without order) to ordered semigroups. Some interesting interplays between GV-ordered semigroups and generalized Green's relations have been given here.

Definition 3.3. An ordered semigroup S is said to be a *GV-ordered semigroup* if S is π-regular and $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$.

Example 3.4. The set $S = \{a, b, c, d\}$ with respect to the multiplication $'\cdot'$ and the order $'\leqslant'$ defined below forms a GV-ordered semigroup.

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	b	b	c	b
d	a	b	b	d

$\leqslant_s = \{(a, a), (a, b), (b, b), (c, b), (c, c), (d, b), (d, d)\}$.

For an $e \in E_{\leqslant}(S)$, Bhuniya and Hansda [1] introduced the set

$$G_e = \{a \in S : \ a \leqslant ea, \ a \leqslant ae \text{ and } e \leqslant za, \ e \leqslant az \text{ for some } z \in S\}.$$

They showed that G_e is a t-simple subsemigroup in a completely regular ordered semigroup.

Lemma 3.5. *Let S be a GV-ordered semigroup. Then for every $a \in S$ there exists $e \in E_{\leqslant}(S)$ and $z \in G_e$ such that $a^m \leqslant a^m e$, $a^m \leqslant e a^m$, $e \leqslant z a^m$, and $e \leqslant a^m z$.*

Proof. Let S be a GV-ordered semigroup, then S is π-regular and $Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$. Let $a \in S$. Then $a^m \in Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$ for some $m \in \mathbb{N}$. Therefore S is a completely π-regular ordered semigroup. Therefore by [[11], Lemma 3.7] the result follows. □

Theorem 3.6. *Let S be a GV-ordered semigroup. Then $G_e \subseteq \mathcal{H}^*(e) \subseteq \mathcal{J}^*(e)$ for every $e \in E_{\leqslant}(S)$.*

Proof. Let S be a GV-ordered semigroup and $e \in E_{\leqslant}(S)$. Consider the subsemigroup G_e and $a \in G_e$, $y \in V_{\leqslant}(a)$ in G_e. Therefore $a \leqslant ue,\ a \leqslant ev,\ e \leqslant aw,\ e \leqslant za$ for some $u, v, w, z \in G_e$ and $a \leqslant aya$. Now $ya \leqslant (yu)e,\ ay \leqslant e(vy),\ e \leqslant za \leqslant (za)ya,\ e \leqslant aw \leqslant ay(aw)$. Therefore we have $ya\mathcal{L}e$ and $ay\mathcal{R}e$. Hence $ya\mathcal{L}ee$ and $ay\mathcal{R}ee$. Therefore we have $a\mathcal{H}^*e$, by Proposition 3.2. Hence $a \in \mathcal{H}^*(e)$. Therefore $G_e \subseteq \mathcal{H}^*(e)$.

Next, let $a \in \mathcal{H}^*(e)$. Then $a^n\mathcal{H}e$ where n is the smallest positive integer such that $a^n \in Reg_{\leqslant}(S)$. Therefore $\mathcal{H}^*(e) \subseteq \mathcal{J}^*(e)$. Hence the proof. □

Corollary 3.7. *Let S be a GV-ordered semigroup. Then for every $a \in S$ there is $e \in E_{\leqslant}(S)$ such that $a^m \in G_e \subseteq \mathcal{H}^*(e) \subseteq \mathcal{J}^*(e)$ for some $m \in \mathbb{N}$.*

Proof. This follows from Lemma 3.5 and Lemma 3.6. □

Corollary 3.8. *Let S be a GV-ordered semigroup. Then for every $a \in S$ there exists $e \in E_{\leqslant}(S)$ such that $\mathcal{J}^*(a) = \mathcal{J}^*(e)$.*

Proof. This follows from Corollary 3.7. □

Lemma 3.9. *Let S be a GV-ordered semigroup. Then for all $a \in S$, $\mathcal{J}^*(a) = \mathcal{J}^*(a^2)$.*

Proof. Let S be a GV-ordered semigroup and $a \in S$. Let m be the smallest positive integer such that $a^m \in Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$. Then there is $x \in S$ such that $a^m \leqslant a^{2m}xa^{2m}$. Let k be the smallest positive integer such that $a^{2k} \in Reg_{\leqslant}(S)$. Then $k \leqslant m$, as $a^{2m} \in Reg_{\leqslant}(S)$. Let $m = k + t$ for some $t \in \mathbb{N}$. So $a^m \leqslant a^{2m}xa^{2m} \leqslant a^{2m}xa^m a^m \leqslant a^{2m}xa^{2m}xa^{3m} \leqslant a^{2m}xa^{2k}a^{2t}xa^{3m}$. Also $a^{2k} \leqslant a^{4k}za^{4k}$ for some $z \in S$. This implies $a^{2k} \leqslant a^{4k}za^{4k} \leqslant a^{2k}a^{2k}za^{4k} \leqslant a^{2k}a^{4k}za^{4k}za^{4k} \leqslant \ldots \leqslant wa^{mk}u = wa^{mk-m}a^m u = wa^{m(k-1)}a^m u$ for some $w, u \in S$. Thus $a^{2k} \in (Sa^mS]$. Therefore $a\mathcal{J}^*a^2$. □

Corollary 3.10. *Let S be a GV-ordered semigroup. Then for all $a \in S$, $\mathcal{J}^*(a) = \mathcal{J}^*(a^m)$ for all $m \in \mathbb{N}$.*

Lemma 3.11. *Let S be a GV-ordered semigroup. Then for all $a, b \in S$, $\mathcal{J}^*(ab) = \mathcal{J}^*(ba)$.*

Proof. Let S be a GV-ordered semigroup and $a, b \in S$. Let m, t be the smallest positive integers such that $(ba)^t$, $(ab)^m \in Reg_{\leqslant}(S)$. Now $(S(ba)^tS] = (S(ba)^{2t}S]$ as S is a GV-ordered semigroup. Also $(S(ba)^{2t}S] \subseteq (S(ab)^tS]$. If $t \geq m$, then $(S(ba)^tS] \subseteq (S(ab)^tS] \subseteq (S(ab)^mS]$. If $t \leqslant m$, then $(ba)^t \leqslant (ba)^{2t}x(ba)^{2t} \leqslant (ba)^{2t}x(ba)^{3t}x(ba)^tx(ba)^{3t}$. Proceed on we get $(ba)^t \in (S(ba)^{rt}S]$ for all $r \in \mathbb{N}$. In particular $r = m+1$. $(S(ba)^tS] \subseteq (S(ba)^{(m+1)t}S] \subseteq (S(ba)^{(m+1)t-(m+1)}(ba)^{(m+1)}S]$ $\subseteq (S(ba)^{(m+1)(t-1)}(ba)^{m+1}S] \subseteq (S(ba)^{m+1}S] \subseteq (S(ab)^mS]$. Similarly we can prove that $(S(ab)^mS] \subseteq (S(ba)^tS]$. Therefore $ab\mathcal{J}^*ba$. □

Lemma 3.12. *Let S be a GV-ordered semigroup. Then $a\mathcal{H}^*a^n$ where n is the smallest positive integer such that $a^n \in Reg_{\leqslant}(S)$.*

Proof. Let S be a GV-ordered semigroup and $a \in S$. Let n be the smallest positive integer such that $a^n \in Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$, as S is a GV-ordered semigroup. Then there exists $a' \in S$ such that $a^n \leqslant a^{2n}a'a^{2n} \leqslant \ldots \leqslant a^{kn}xa^{kn}$, for some $x \in S$ and for all $k \in \mathbb{N}$. Let r be the smallest positive integer such that $(a^n)^r \in Reg_{\leqslant}(S)$. Then there exists $a'' \in S$ such that $(a^n)^r \leqslant (a^n)^r a'' (a^n)^r \leqslant y_1a^n$, where $y_1 = (a^n)^r a'' a^{nr-n} \in S$. Similarly $(a^n)^r \leqslant a^ny_2$. Also we have $a^n \leqslant a^{rn}y_3$, $a^n \leqslant y_4a^{rn}$ for some y_3, $y_4 \in S$. Therefore $a^n\mathcal{H}(a^n)^r$, that is, $a\mathcal{H}^*a^n$. □

In the following theorem the class of GV-ordered semigroups have been characterized by their subsemigroups which are both Archimedean and completely π-regular.

Theorem 3.13. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is a GV-ordered semigroup,*

(2) *S is completely π-regular and every $\mathcal{H}^*$-class of S contains an ordered idempotent,*

(3) *S is a complete semilattice of completely π-regular and Archimedean ordered semigroups,*

(4) *For all $a, b \in S$, there exist $n \in \mathbb{N}$ such that $(ab)^n \in ((ab)^{n+1}Sa(ab)^{n+1}]$,*

(5) *For all $a, b \in S$, there exist $n \in \mathbb{N}$ such that $(ab)^n \in ((ab)^{n+1}bS(ab)^{n+1}]$.*

Proof. (1) ⇒ (2): Let S be a GV-ordered semigroup. Consider a $\mathcal{H}^*$-class H^* and $a \in H^*$. Let m be the smallest positive integer such that $a^m \in Reg_{\leqslant}(S) = Gr_{\leqslant}(S)$. Then there exists $x \in S$ such that $a^m \leqslant a^{2m}xa^{2m}$. Let $e = a^{2m}xa^{2m}xa^{2m}$. Then $e = a^{2m}xa^{2m}xa^{2m} \leqslant a^{2m}xa^ma^mxa^{2m} \leqslant (a^{2m}xa^{2m}xa^{2m})$, $a^mxa^{2m} \leqslant e(a^{2m}xa^{2m}xa^{2m}) = e^2$. Thus $e \in E_{\leqslant}(S)$.

Now $a^m \leqslant a^{2m}xa^{2m} \leqslant a^m(a^{2m}xa^{2m}xa^{2m}) = a^me$ and $a^m \leqslant a^{2m}xa^{2m} \leqslant (a^{2m}xa^{2m}xa^{2m})a^m = ea^m$. Also $e = a^{2m}xa^{2m}xa^{2m} = (a^{2m}xa^{2m}xa^m)a^m = ya^m$ and $e = a^m(a^mxa^{2m}xa^{2m}) = a^mz$ for some $y = a^{2m}xa^{2m}xa^m$ *and* $z =$

$a^m x a^{2m} x a^{2m} \in S$. Therefore $a^m \leqslant a^m e \leqslant a^m e^n$,$a^m \leqslant ea^m \leqslant e^n a^m$ for some $n \in \mathbb{N}$. And $e^n = e \ldots e(\text{n times}) = (ya^m \ldots y)a^m$, $e^n = e \ldots e(\text{n times}) \leqslant a^m(z \ldots a^m z)$. Therefore $e^n \mathcal{H} a^m$. Hence $e\mathcal{H}^* a$ and therefore $e \in H^*(a)$. Therefore $\mathcal{H}^*$-class contains an ordered idempotent. Since S is a GV-ordered semigroup, therefore it is completely π-regular.

(2) $\Rightarrow$ (3): Let $a \in S$. Consider an $\mathcal{H}^*$-class $H^*(a)$. Then there exists an ordered idempotent $e \in H^*(a)$. Therefore $e^n \leqslant xa^m$, $e^n \leqslant a^m y$, $a^m \leqslant ue^n$, $a^m \leqslant e^n v$ for some $x, y, u, v \in S^1$ and m is the smallest positive integer such that $a^m \in Reg_{\leqslant}(S)$. Since S is completely π-regular, $(a^m)^k \leqslant ((a^m)^{k+p} y_2 (a^m)^{k+p})$ for all $p \in \mathbb{N}$ and for some $y_2 \in S$, $k \in \mathbb{N}$. Now $e \leqslant e^n \leqslant xa^m \leqslant xaa^{m-1}$. So $a \mid e$. Therefore $e \leqslant x_1 a y_1$ for some $x_1, y_1 \in S^1$. $e \leqslant e^2 \leqslant x_1 a y_1 e \leqslant x_1 a y_1 e^n e^n \leqslant x_1 a y_1 e^n a^m y \leqslant \ldots \leqslant x_1 a y_1 (a^m)^k y_5$ for some $y_5 \in S$. Therefore we have $e \leqslant x_1 a y_1 a^{mk+pm} y_2 a^{mk+pm} y_5 \leqslant x_1 a y_1 a^{mk+pm-2} a^2 y_2 a^{mk+pm} y_5$. Therefore $a^2 \mid e$. Thus S is a π-regular ordered semigroup and for all $a \in S$, $e \in E_{\leqslant}(S)$, $a \mid e$ implies $a^2 \mid e$. Therefore by [[2], Theorem 4.1], S is a complete semilattice Y of ordered semigroups $\{S_\alpha\}_{\alpha \in Y}$, S_α is a nil-extension of simple and π-regular ordered semigroups $\{K_\alpha\}_{\alpha \in Y}$. Hence S is a complete semilattice Y of Archimedean and π-regular ordered semigroups $\{S_\alpha\}_{\alpha \in Y}$ by [[4], Theorem 3.8]. Since S is completely π-regular, therefore S_α is also completely π-regular by [[4], Theorem 2.4]. Hence S is a complete semilattice of completely π-regular and Archimedean ordered semigroup.

(3) $\Rightarrow$ (1): Let S is a complete semilattice Y of completely π-regular and Archimedean ordered semigroups S_α, $S_\alpha \in Y$. Then S is π-regular. Let $a \in Reg_{\leqslant}(S)$. Then $a \in S_\alpha$ for some $\alpha \in Y$. Now $a \leqslant axa$ for some $x \in S_\beta$. Therefore $axa \in S_\alpha S_\beta S_\alpha \subseteq S_{\alpha\beta}$. Therefore $a \in (S_{\alpha\beta}]$. Hence $S_\alpha \cap (S_{\alpha\beta}] \neq \phi$, that is $\alpha \leqslant \alpha\beta$. Therefore $\alpha = \alpha(\alpha\beta) = \alpha^2\beta = \alpha\beta$. Therefore $S_\alpha = S_{\alpha\beta}$. Again $a \leqslant axa \leqslant a(xax)a$. Now $y = xax \in S_\beta S_\alpha S_\beta \subseteq S_{\beta\alpha} = S_{\alpha\beta} = S_\alpha$. Therefore $a \in Reg_{\leqslant}(S_\alpha)$. Now let σ be the semilattice congruence. Then $a \in (a]_\sigma = ((aya)_\sigma] = ((a^2 y)_\sigma] = ((a^2)_\sigma (y)_\sigma] \subseteq LReg_{\leqslant}(a)_\sigma \subseteq LReg_{\leqslant}(S)$. Similarly $a \in RReg_{\leqslant}(S)$. Therefore $a \in Gr_{\leqslant}(S)$. Hence S is a GV-ordered semigroup.

(3) $\Rightarrow$ (4): Let S be a complete semilattice Y of completely π-regular and Archimedean ordered semigroups S_α, $\alpha \in Y$. Now each S_α is a nil-extension of simple and completely π-regular ordered semigroup K_α, $\alpha \in Y$. Let $a, b \in S$. Then $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$. Therefore $ab, ba \in S_{\alpha\beta}$. Hence $(ab)^n, (ba)^m \in K_{\alpha\beta}$ for some $n, m \in \mathbb{N}$. Since $K_{\alpha\beta}$ is ideal, therefore $(ab)^n$, $(ab)^{n+1}(ba)^m(ab)^{n+1} \in K_{\alpha\beta}$. Again since $K_{\alpha\beta}$ is simple, $(ab)^n \leqslant (ab)^{n+1}(ba)^m(ab)^{n+1} x (ab)^{n+1}(ba)^m(ab)^{n+1}$ for some $x \in K_{\alpha\beta}$. Therefore $(ab)^n \in ((ab)^{n+1} S (ab)^{n+1}]$.

(4) $\Rightarrow$ (3): Clearly S a is completely π-regular ordered semigroup by the given condition. Assume $a, b \in S$. Then $(ab)^n \in ((ab)^{n+1} S a (ab)^{n+1}]$, that is, $(ab)^n \in (Sa^2 S]$ for some $n \in \mathbb{N}$. Therefore by [[12], Lemma 3.5], S is a complete semilattice of Archimedean ordered semigroup. Hence S is a complete semilattice of completely π-regular and Archimedean ordered semigroup.

(3) $\Leftrightarrow$ (5) : This is similar to the proof of (3) $\Leftrightarrow$ (4). □

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