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Magnifying elements of some semigroups of partial transformations

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Abstract. Let X be a nonempty set and let $P(X)$ denote the semigroup (under the composition) of partial transformations from a subset of X to X and $E(X)$ denote the subsemigroup of $P(X)$ containing surjective partial transformations on X. For a fixed nonempty subset Y of X, let $\overline{PT}(X,Y) = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$ and $PT_{(X,Y)} = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha = Y\}$. We give necessary and sufficient conditions for elements in semigroups $\overline{PT}(X,Y)$ and $PT_{(X,Y)}$ to be left or right magnifying.

1. Introduction

Let S be a semigroup. An element $a \in S$ is called a *left* (*right*) *magnifying element* if there exist a proper subset M of S such that $S = aM$ ($S = Ma$). Such elements are mentioned in 1963 by E. S. Ljapin [5]. M. Gutan showed in [1] that there exists semigroups containing both strong and non-strong magnifying elements. In [2] he proved that every semigroup containing magnifying elements is factorizable. In [3] he proposed the method of construction of semigroups having good left magnifying elements.

Let $B(X)$ be the set of all binary relations on the set X. Then $P(X)$, where $P(X) = \{\alpha \in B(X) \mid \alpha : A \to B \text{ when } A, B \subseteq X\}$, is a semigroup called the *semigroup of partial transformations* on X. The semigroup of surjective partial transformations on X is denoted by $E(X)$, i.e. $E(X) = \{\alpha \in P(X) \mid \operatorname{ran}\alpha = X\}$. The necessary and sufficient conditions for elements of $P(X)$ to be the left or right magnifying elements were found in [6].

$T(X) = \{\alpha \in P(X) \mid \operatorname{dom}\alpha = X\}$ is a semigroup called the *full transformation semigroup* on X. $ET(X) = E(X) \cap T(X)$ is a *semigroup of surjective full transformations* on X.

For a fixed nonempty subset Y of X, let

$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\} \quad \text{and} \quad T_{(X,Y)} = \{\alpha \in T(X) \mid Y\alpha = Y\},$$

where $Y\alpha = \{y\alpha \mid y \in Y\}$. Then $\overline{T}(X,Y)$ and $T_{(X,Y)}$ are subsemigroups of $T(X)$. $T_{(X,Y)}$ is also a subsemigroup of $\overline{T}(X,Y)$.

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The semigroups $\overline{PT}(X,Y)$ and $PT_{(X,Y)}$ are defined similarly. Namely,

$$\overline{PT}(X,Y) = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$$

and

$$PT_{(X,Y)} = \{\alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha = Y\},$$

where $\operatorname{dom}\alpha$ is the domain of α and $(\operatorname{dom}\alpha \cap Y)\alpha = \{z\alpha \mid z \in \operatorname{dom}\alpha \cap Y\}$. Then $\overline{PT}(X,Y)$ and $PT_{(X,Y)}$ are subsemigroups of $P(X)$. $PT_{(X,Y)}$ also is a subsemigroup of $\overline{PT}(X,Y)$.

The purpose of this paper is providing the necessary and sufficient conditions for elements in semigroups $\overline{PT}(X,Y)$ and $PT_{(X,Y)}$ to be left or right magnifying.

2. Preliminaries

Throughout this paper, the cardinality of a set X is denoted by $|X|$ and $X = A \dot{\cup} B$ means X is a disjoint union of A and B. The proper subset B of a set A is denoted by $B \subset A$.

For $\alpha, \beta \in P(X)$, $\alpha\beta \in P(X)$ is defined by $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in \operatorname{dom}(\alpha\beta)$. The identity map on X, i.e. id_X, is the identity element of $\overline{PT}(X,Y)$ and $PT_{(X,Y)}$. The empty function on X, i.e. $\emptyset_X$ is a zero element of $\overline{PT}(X,Y)$ but $\emptyset_X \notin PT_{(X,Y)}$.

For $\alpha \in P(X)$, we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

where the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i \mid i \in I\}$. Then $\operatorname{ran}\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

For $\alpha \in \overline{PT}(X,Y)$, we write

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$; $B_j, C_k \subseteq X \setminus Y$; $\{a_i\} \subseteq Y, \{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$.

For $\alpha \in PT_{(X,Y)}$, we write

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset$; $B_j \subseteq X \setminus Y$; $\{a_i\} = Y, \{b_j\} \subseteq X \setminus Y$.

If X is finite, then Y is also finite. So we get $\overline{PT}(X,Y)$ and $PT_{(X,Y)}$ are finite semigroups. Since finite semigroups do not contain left and right magnifying elements (cf. [4]), we will consider only the case when X is an infinite set.

3. Left Magnifying Elements in $\overline{PT}(X,Y)$

Lemma 3.1. *If $\alpha \in \overline{PT}(X,Y)$ is a left magnifying element in $\overline{PT}(X,Y)$, then* $\operatorname{dom}\alpha = X$, *$\alpha$ is injective and $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran}\alpha$.*

Proof. Assume that α is a left magnifying element in $\overline{PT}(X,Y)$. Then there exists a proper subset M of $\overline{PT}(X,Y)$ such that $\alpha M = \overline{PT}(X,Y)$. Since $id_X \in \overline{PT}(X,Y)$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Thus $X = \operatorname{dom} id_X \subseteq \operatorname{dom}\alpha$ and hence $\operatorname{dom}\alpha = X$. Since id_X is injective, we also have α is injective. Since α is not an empty function, we have $Y \cap \operatorname{ran}\alpha \neq \emptyset$. Let $y \in Y \cap \operatorname{ran}\alpha$ and let $x \in y\alpha^{-1}$. Then $x\alpha = y$ and so $x = xid_X = x\alpha\beta = y\beta \in Y$. So $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran}\alpha$. □

Lemma 3.2. *If $\alpha \in \overline{PT}(X,Y)$ is a left magnifying element in $\overline{PT}(X,Y)$, then α is not surjective.*

Proof. Assume that α is a left magnifying element in $\overline{PT}(X,Y)$ and α is surjective. Then there exists $M \subset \overline{PT}(X,Y)$ such that $\alpha M = \overline{PT}(X,Y)$. By Lemma 3.1, we get $\operatorname{dom}\alpha = X$, α is injective and $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran}\alpha$. Then

$$\alpha = \begin{pmatrix} a_i & b_j \\ y_i & z_j \end{pmatrix}$$

where $\{a_i\} = Y = \{y_i\}$ and $\{a_i\}\dot\cup\{b_j\} = X = \{y_i\}\dot\cup\{z_j\}$. There is

$$\alpha^{-1} = \begin{pmatrix} y_i & z_j \\ a_i & b_j \end{pmatrix} \in \overline{PT}(X,Y)$$

such that $\alpha^{-1}\alpha = id_X$. Let $\beta \in \overline{PT}(X,Y)$. Then $\alpha\beta \in \overline{PT}(X,Y)$. Since $\overline{PT}(X,Y) = \alpha M$, we get $\alpha\beta = \alpha\gamma$ for some $\gamma \in M$. So $\beta = id_X\beta = \alpha^{-1}(\alpha\beta) = \alpha^{-1}(\alpha\gamma) = id_X\gamma = \gamma \in M$. Thus $\overline{PT}(X,Y) \subseteq M$ that contradicts with M is a proper subset of $\overline{PT}(X,Y)$. Therefore, α is not surjective. □

Theorem 3.3. *$\alpha \in \overline{PT}(X,Y)$ is a left magnifying element in $\overline{PT}(X,Y)$ if and only if the following statements hold:*

1. $\operatorname{dom}\alpha = X$,
2. $y\alpha^{-1} \subseteq Y$ *for all* $y \in Y \cap \operatorname{ran}\alpha$ *and*
3. α *is injective but not surjective.*

Proof. Assume that α is a left magnifying element in $\overline{PT}(X,Y)$. By the above lemmas, we have $\operatorname{dom}\alpha = X$, $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran}\alpha$ and α is injective but not surjective.

Conversely, choose $M = \{\delta \in \overline{PT}(X,Y) \mid \operatorname{dom}\delta \neq X\}$ and assume that the conditions 1-3 hold. Then we get $M \subset \overline{PT}(X,Y)$. Let $\beta \in \overline{PT}(X,Y)$. If $\beta = \emptyset_X$, then there is $\emptyset_X \in M$ such that $\beta = \alpha\emptyset_X$. If $\beta \neq \emptyset_X$, we let $Y = \{a_i\} \,\dot\cup\, \{b_j\}$ when $\operatorname{dom}\beta \cap Y = \{a_i\}$ and $X \setminus Y = \{s_k\} \,\dot\cup\, \{t_l\}$ when dom, $\beta \cap (X \setminus Y) = \{s_k\}$. Then

$$\alpha = \begin{pmatrix} a_i & b_j & s_k & t_l \\ y_i & z_j & u_k & v_l \end{pmatrix}$$

where $\{y_i\}, \{z_j\} \subseteq Y$ and $\{u_k\}, \{v_l\} \subseteq X \setminus Y$. Since α is not surjective, we have $\operatorname{ran}\alpha \neq X$. Define $\gamma : \{y_i\} \cup \{u_k\} \to X$ by

$$\gamma = \begin{pmatrix} y_i & u_k \\ a_i\beta & s_k\beta \end{pmatrix}.$$

Since α is injective, γ is well-defined. Since $(\operatorname{dom}\gamma \cap Y)\gamma = \{y_i\}\gamma = \{a_i\beta\} \subseteq Y$, $\gamma \in \overline{PT}(X,Y)$. But $\operatorname{dom}\gamma = \{y_i\} \cup \{u_k\} \subseteq \operatorname{ran}\alpha \neq X$, so $\gamma \in M$.

Let $x \in \operatorname{dom}\beta = \{a_i\} \cup \{s_k\} = \operatorname{dom}(\alpha\gamma)$.

If $x = a_i$ for some $i \in I$, then $x(\alpha\gamma) = a_i(\alpha\gamma) = (a_i\alpha)\gamma = y_i\gamma = a_i\beta = x\beta$.

If $x = s_k$ for some $k \in K$, then $x(\alpha\gamma) = s_k(\alpha\gamma) = (s_k\alpha)\gamma = u_k\gamma = s_k\beta = x\beta$.

Thus $\beta = \alpha\gamma$. Hence $\overline{PT}(X,Y) = \alpha M$. Therefore, α is a left magnifying element in $\overline{PT}(X,Y)$. □

Taking $Y = X$ in Theorem 3.3 we obtain

Corollary 3.4. *$\alpha \in P(X)$ is a left magnifying element in $P(X)$ if and only if $\operatorname{dom}\alpha = X$ and α is injective but not surjective.*

Example 3.5. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define

$$\alpha = \begin{pmatrix} n \\ n+2 \end{pmatrix}_{n\in\mathbb{N}}.$$

Then $(\operatorname{dom}\alpha \cap Y)\alpha = (2\mathbb{N})\alpha = 2\mathbb{N}\setminus\{2\} \subseteq Y$ and so $\alpha \in \overline{PT}(X,Y)$. Moreover, we get $\operatorname{dom}\alpha = \mathbb{N} = X$, $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{ran}\alpha$ and α is injective but α is not surjective. By Theorem 3.3, α is a left magnifying element in $\overline{PT}(X,Y)$. By the proof of Theorem 3.3, there exists $M = \{\delta \in \overline{PT}(X,Y) \mid \operatorname{dom}\delta \neq \mathbb{N} = X\} \subset \overline{PT}(X,Y)$ such that $\alpha M = \overline{PT}(X,Y)$.

4. Right Magnifying Elements in $\overline{PT}(X,Y)$

Lemma 4.1. *If $\alpha \in \overline{PT}(X,Y)$ is a right magnifying element in $\overline{PT}(X,Y)$, then α is surjective.*

Proof. Assume that α is a right magnifying element in $\overline{PT}(X,Y)$. Then there is a proper subset M of $\overline{PT}(X,Y)$ such that $M\alpha = \overline{PT}(X,Y)$. Since $id_X \in \overline{PT}(X,Y)$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. From id_X is surjective, this implies α is surjective. □

Lemma 4.2. *If $\alpha \in \overline{PT}(X,Y)$ is a right magnifying element in $\overline{PT}(X,Y)$, then $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$.*

Proof. Assume α is a right magnifying element in $\overline{PT}(X,Y)$. Then there exists a proper subset M of $\overline{PT}(X,Y)$ such that $M\alpha = \overline{PT}(X,Y)$. By Lemma 4.1, α is surjective.

Suppose that $y_0\alpha^{-1} \cap Y = \emptyset$ for some $y_0 \in Y$ and define

$$\beta = \begin{pmatrix} Y \\ y_0 \end{pmatrix}.$$

Then $\beta \in \overline{PT}(X,Y)$. Since $M\alpha = \overline{PT}(X,Y)$, there is $\gamma \in M$ such that $\gamma\alpha = \beta$. But α is surjective and $y_0\alpha^{-1} \cap Y = \emptyset$, so $y_0\alpha^{-1} \subseteq X \setminus Y$. Thus for each $y \in Y$,

$y_0 = y\beta = (y\gamma)\alpha$. So $y\gamma \in y_0\alpha^{-1} \subseteq X \setminus Y$ which is a contradiction. Therefore $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. □

Lemma 4.3. *If $\alpha \in \overline{PT}(X, Y)$ is a right magnifying element in $\overline{PT}(X, Y)$, then $\operatorname{dom}\alpha \neq X$ or α is not injective.*

Proof. Assume that α is a right magnifying element in $\overline{PT}(X, Y)$. By Lemmas 4.1 and 4.2, α is surjective and $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Suppose that $\operatorname{dom}\alpha = X$ and α is injective. Let $X = \{a_i\} \cup \{b_j\}$ be such that $Y = \{a_i\}$. Then

$$\alpha = \begin{pmatrix} a_i & b_j \\ y_i & z_j \end{pmatrix}$$

where $\{y_i\} = Y$ and $\{z_j\} = X \setminus Y$. There is $\alpha^{-1} \in \overline{PT}(X, Y)$ such that $\alpha\alpha^{-1} = id_X$. Let $\beta \in \overline{PT}(X, Y)$. Then $\beta\alpha \in \overline{PT}(X, Y)$. Since $\overline{PT}(X, Y) = M\alpha$, we have $\beta\alpha = \delta\alpha$ for some $\delta \in M$. Thus $\beta = (\beta\alpha)\alpha^{-1} = (\delta\alpha)\alpha^{-1} = \delta \in M$. Hence $\overline{PT}(X, Y) \subseteq M$. That yields $M = \overline{PT}(X, Y)$ which contradicts with $M \subset \overline{PT}(X, Y)$. Therefore, $\operatorname{dom}\alpha \neq X$ or α is not injective. □

Theorem 4.4. *$\alpha \in \overline{PT}(X, Y)$ is a right magnifying element in $\overline{PT}(X, Y)$ if and only if the following statements hold:*

1. *α is surjective,*
2. *$y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and*
3. *$\operatorname{dom}\alpha \neq X$ or α is not injective.*

Proof. Assume that α is a right magnifying element in $\overline{PT}(X, Y)$. Conditions 1-3 are a consequence of Lemmas 4.1, 4.2 and 4.3.

Conversely, assume that conditions 1-3 are satisfied. We have two cases.

Case 1: $\operatorname{dom}\alpha \neq X$. Choose $M = \{\delta \in \overline{PT}(X, Y) \mid \delta \text{ is not surjective}\}$. Then $M \subset \overline{PT}(X, Y)$. Let $\beta \in \overline{PT}(X, Y)$. Then

$$\beta = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}.$$

where $A_i \cap Y \neq \emptyset$; $B_j, C_k \subseteq X \setminus Y$; $\{a_i\} \subseteq Y, \{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Since α is surjective, we have $\operatorname{ran}\beta \subseteq X = \operatorname{ran}\alpha$. From $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$, we have $a_i\alpha^{-1} \cap Y \neq \emptyset \neq b_j\alpha^{-1} \cap Y$. Choose $d_{a_i} \in a_i\alpha^{-1} \cap Y$ and $d_{b_j} \in b_j\alpha^{-1} \cap Y$. Then $d_{a_i}\alpha = a_i$ and $d_{b_j}\alpha = b_j$. Since $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, we have $c_k \in \operatorname{ran}\alpha$ and we can choose $c'_k \in \operatorname{dom}\alpha$ such that $c'_k\alpha = c_k$. Define

$$\gamma = \begin{pmatrix} A_i & B_j & C_k \\ d_{a_i} & d_{b_j} & c'_k \end{pmatrix}.$$

Then $\gamma \in \overline{PT}(X, Y)$. Since $\operatorname{ran}\gamma \subseteq \operatorname{dom}\alpha \neq X$, γ is not surjective. Thus $\gamma \in M$.

Let $\operatorname{dom}(\gamma\alpha) = (\operatorname{ran}\gamma \cap \operatorname{dom}\alpha)\gamma^{-1} = (\operatorname{ran}\gamma)\gamma^{-1} = \operatorname{dom}\gamma = \operatorname{dom}\beta$ and $x \in \operatorname{dom}\beta$.

If $x \in A_i$ for some $i \in I$, then $x(\gamma\alpha) = (x\gamma)\alpha = d_{a_i}\alpha = a_i = x\beta$.

If $x \in B_j$ for some $j \in J$, then $x(\gamma\alpha) = (x\gamma)\alpha = d_{b_j}\alpha = b_j = x\beta$.

If $x \in C_k$ for some $k \in K$, then $x(\gamma\alpha) = (x\gamma)\alpha = c'_k\alpha = c_k = x\beta$.
Thus $\gamma\alpha = \beta$ and hence $\overline{PT}(X,Y) \subseteq M\alpha$ which implies that $M\alpha = \overline{PT}(X,Y)$.
Case 2: α is not injective. Choose $M = \{\delta \in \overline{PT}(X,Y) \mid \delta \text{ is not surjective}\}$. Then $M \subset \overline{PT}(X,Y)$. Let $\beta \in \overline{PT}(X,Y)$. Then

$$\beta = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}.$$

where $A_i \cap Y \neq \emptyset$; $B_j, C_k \subseteq X \setminus Y$; $\{a_i\}, \{b_j\} \subseteq Y$ and $\{c_k\} \subseteq X \setminus Y$.

Let $\gamma \in \overline{PT}(X,Y)$ be as in Case 1. Since α is not injective, there is $x_0 \in \operatorname{ran}\alpha$ and distinct elements $x_1, x_2 \in \operatorname{dom}\alpha$ such that $x_1\alpha = x_0 = x_2\alpha$. Note that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. If $x_0 \in \operatorname{ran}\beta$, then there is exactly one (either x_1 or x_2) in $\operatorname{ran}\gamma$. If $x_0 \notin \operatorname{ran}\beta$, then $x_1, x_2 \notin \operatorname{ran}\gamma$. Thus γ is not surjective and so $\gamma \in M$. Analogously as in Case 1, we get $\gamma\alpha = \beta$ and hence $\overline{PT}(X,Y) \subseteq M\alpha$. This means that $M\alpha = \overline{PT}(X,Y)$.

Therefore, α is a right magnifying element in $\overline{PT}(X,Y)$. □

For $Y = X$ we obtain the following corollary.

Corollary 4.5. *$\alpha \in P(X)$ α is a right magnifying element in $P(X)$ if and only if α is surjective and* ($\operatorname{dom}\alpha \neq X$ *or α is not injective*).

Example 4.6. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define

$$\alpha = \begin{pmatrix} 1 & \{2,3\} & 4 & \{5,6\} & n+2 \\ 1 & 2 & 3 & 4 & n \end{pmatrix}_{n\geq 5} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 4 & 5 & 8 & n+4 \\ 1 & 2 & 3 & 4 & n \end{pmatrix}_{n\geq 5}.$$

Then $(\operatorname{dom}\alpha \cap Y)\alpha = (2\mathbb{N})\alpha = 2\mathbb{N} \subseteq Y$ and $(\operatorname{dom}\beta \cap Y)\beta = (2\mathbb{N}\setminus\{2,6\})\beta = 2\mathbb{N} \subseteq Y$. So α, $\beta \in \overline{PT}(X,Y)$. It is clear that α is surjective. Furthermore, $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and α is not injective but $\operatorname{dom}\alpha = \mathbb{N} = X$. We can see that β is a bijection and $y\beta^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ but $\operatorname{dom}\beta = \mathbb{N} \setminus \{2,3,6,7\} \neq X$. By Theorem 4.4, α, β are right magnifying elements in $\overline{PT}(X,Y)$. Then by the proof of Theorem 4.4, there is $M = \{\delta \in \overline{PT}(X,Y) \mid \delta \text{ is not surjective}\} \subset \overline{PT}(X,Y)$ such that $M\alpha = \overline{PT}(X,Y)$ and $M\beta = \overline{PT}(X,Y)$.

5. Left Magnifying Elements in $PT_{(X,Y)}$

Lemma 5.1. *If $\alpha \in PT_{(X,Y)}$ is a left magnifying element in $PT_{(X,Y)}$, then* $\operatorname{dom}\alpha = X$ *and α is injective.*

Proof. Assume that α is a left magnifying element in $PT_{(X,Y)}$. Then there exists a proper subset M of $PT_{(X,Y)}$ such that $\alpha M = PT_{(X,Y)}$. Since $id_X \in PT_{(X,Y)}$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Thus $\operatorname{dom}\alpha = X$ and α is injective. □

Lemma 5.2. *If $\alpha \in PT_{(X,Y)}$, where $Y \neq X$, is a left magnifying element in $PT_{(X,Y)}$, then α is not surjective.*

Proof. Given $Y \neq X$. Assume that α is a left magnifying element in $PT_{(X,Y)}$ and α is surjective. Then there exists $M \subset PT_{(X,Y)}$ such that $\alpha M = PT_{(X,Y)}$. By Lemma 5.1, we get $\operatorname{dom}\alpha = X$ and α is injective. Thus α is a bijection on X. Since $\alpha\beta \in PT_{(X,Y)} = \alpha M$, $\alpha\beta = \alpha\gamma$ for some $\gamma \in M$. So $\beta = \gamma$ and hence $\beta \in M$. Thus $PT_{(X,Y)} \subseteq M$. So $M = PT_{(X,Y)}$ which is a contradiction. Therefore, α is not surjective. □

Theorem 5.3. *If $Y \neq X$, then $\alpha \in PT_{(X,Y)}$ is a left magnifying element in $PT_{(X,Y)}$ if and only if $\operatorname{dom}\alpha = X$ and α is injective but not surjective.*

Proof. Let $Y \neq X$. Assume that α is a left magnifying element in $PT_{(X,Y)}$. By Lemmas 5.1 and 5.2, we have $\operatorname{dom}\alpha = X$ and α is injective but not surjective. Conversely, assume that $\operatorname{dom}\alpha = X$ and α is injective but not surjective. Choose $M = \{\delta \in PT_{(X,Y)} \mid \operatorname{dom}\delta \neq X\}$. Then $M \subset PT_{(X,Y)}$.

We prove that $\alpha M = PT_{(X,Y)}$. Let $\beta \in PT_{(X,Y)}$ and $Y = \{a_i\} \dot\cup \{b_j\}$ where $\operatorname{dom}\beta \cap Y = \{a_i\}$ and $X \setminus Y = \{s_k\}\dot\cup\{t_l\}$ when $\operatorname{dom}\beta \cap (X \setminus Y) = \{s_k\}$. Then

$$\alpha = \begin{pmatrix} a_i & b_j & s_k & t_l \\ y_i & z_j & u_k & v_l \end{pmatrix}$$

where $Y = \{y_i\} \cup \{z_j\}$ and $\{u_k\}, \{v_l\} \subseteq X \setminus Y$. Define $\gamma : \{y_i\} \cup \{u_k\} \to X$ by

$$\gamma = \begin{pmatrix} y_i & u_k \\ a_i\beta & s_k\beta \end{pmatrix}.$$

Since α is injective, γ is well-defined and $(\operatorname{dom}\gamma \cap Y)\gamma = \{y_i\}\gamma = \{a_i\beta\} = (\operatorname{dom}\beta \cap Y)\beta = Y$, hence $\gamma \in PT_{(X,Y)}$. Since α is not surjective, from $\operatorname{dom}\gamma = \{y_i\} \cup \{u_k\} \subseteq \operatorname{ran}\alpha \neq X$ it follows $\gamma \in M$. But $x(\alpha\gamma) = (x\alpha)\gamma = x\beta$ for all $x \in \operatorname{dom}\beta = \{a_i\} \cup \{s_k\} = \operatorname{dom}(\alpha\gamma)$. Hence $\alpha\gamma = \beta$ and so $\alpha M = PT_{(X,Y)}$. So, α is a left magnifying element in $PT_{(X,Y)}$. □

Theorem 5.4. *$E(X)$ has no left magnifying elements.*

Proof. Suppose that α is a left magnifying element in $E(X)$. Then α is a left magnifying element in $PT_{(X,Y)}$ when $Y = X$. By Lemma 5.1, $\operatorname{dom}\alpha = X$ and α is injective. Since $\alpha \in E(X)$, α is surjective. Then there is $\alpha^{-1} \in E(X)$ such that $\alpha^{-1}\alpha = id_X$. Since α is left magnifying, there is $M \subset E(X)$ such that $\alpha M = E(X)$. Let $\beta \in E(X)$. Analogously as in the proof of Lemma 5.2, we obtain $\beta \in M$. Thus $M = E(X)$. That is a contradiction. Hence, $E(X)$ has no left magnifying elements. □

Example 5.5. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define

$$\alpha = \begin{pmatrix} 2n-1 & 2n \\ 2n+1 & 2n \end{pmatrix}_{n\in\mathbb{N}}.$$

Since $(\operatorname{dom}\alpha \cap Y)\alpha = (2\mathbb{N})\alpha = 2\mathbb{N} = Y$, $\alpha \in PT_{(X,Y)}$, $\operatorname{dom}\alpha = \mathbb{N} = X$ and α is injective. But $\operatorname{ran}\alpha = \mathbb{N} \setminus \{1\} \neq X$, then α is not surjective. By Theorem 5.3, α

is a left magnifying element in $PT_{(X,Y)}$. Let $M = \{\delta \in PT_{(X,Y)} \mid \operatorname{dom}\delta \neq \mathbb{N}\}$. Then, analogously as in the proof of Theorem 5.3, for each $\beta \in PT_{(X,Y)}$, there exists $\gamma \in M$ such that $\alpha\gamma = \beta$. Thus $PT_{(X,Y)} = \alpha M$ for some $M \subset PT_{(X,Y)}$.

6. Right Magnifying Elements in $PT_{(X,Y)}$

Lemma 6.1. *If $\alpha \in PT_{(X,Y)}$ is a right magnifying element in $PT_{(X,Y)}$, then α is surjective.*

Proof. Assume that α is a right magnifying element in $PT_{(X,Y)}$. Then $M\alpha = PT_{(X,Y)}$ for some proper subset M of $PT_{(X,Y)}$. Since $id_X \in PT_{(X,Y)}$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. So, α must be surjective. □

Lemma 6.2. *If $\alpha \in PT_{(X,Y)}$ is a right magnifying element in $PT_{(X,Y)}$, then $\operatorname{dom}\alpha \neq X$ or α is not injective.*

Proof. Assume α is a right magnifying element in $PT_{(X,Y)}$. Then $M\alpha = PT_{(X,Y)}$ for some $M \subset PT_{(X,Y)}$. Suppose that $\operatorname{dom}\alpha = X$ and α is injective. By Lemma 6.1, α is surjective. Let $\beta \in PT_{(X,Y)}$. Then $\beta\alpha \in PT_{(X,Y)}$. Since $PT_{(X,Y)} = M\alpha$, we have $\beta\alpha = \delta\alpha$ for some $\delta \in M$. Since α is a bijection on X with $Y\alpha = Y$, we get $\beta = \delta \in M$. Hence $PT_{(X,Y)} \subseteq M$. That yields $M = PT_{(X,Y)}$ which contradicts with $M \subset PT_{(X,Y)}$. Therefore, $\operatorname{dom}\alpha \neq X$ or α is not injective. □

Theorem 6.3. *$\alpha \in PT_{(X,Y)}$ is a right magnifying element in $PT_{(X,Y)}$ if and only if α is surjective and ($\operatorname{dom}\alpha \neq X$ or α is not injective).*

Proof. Assume that α is a right magnifying element in $PT_{(X,Y)}$. By Lemmas 6.1 and 6.2, α is surjective and ($\operatorname{dom}\alpha \neq X$ or α is not injective).

Conversely, assume that α is surjective and ($\operatorname{dom}\alpha \neq X$ or α is not injective). We have two cases:

Case 1: $\operatorname{dom}\alpha \neq X$. Choose $M = \{\delta \in PT_{(X,Y)} \mid \delta \text{ is not surjective}\}$. Then $M \subset PT_{(X,Y)}$. Let $\beta \in PT_{(X,Y)}$. Then

$$\beta = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}.$$

where $A_i \cap Y \neq \emptyset$, $B_j \subseteq X \setminus Y$, $\{a_i\} = Y$ and $\{b_j\} \subseteq X \setminus Y$. $(\operatorname{dom}\alpha \cap Y)\alpha = Y$ implies $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$. Then $a_i\alpha^{-1} \cap Y \neq \emptyset$ and $d_{a_i}\alpha = a_i$ for $d_{a_i} \in a_i\alpha^{-1} \cap Y$. Since $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, $b_j \in \operatorname{ran}\alpha$ and $b'_j\alpha = b_j$ for somee $b'_j \in \operatorname{dom}\alpha$. Define

$$\gamma = \begin{pmatrix} A_i & B_j \\ d_{a_i} & b'_j \end{pmatrix}.$$

Then $\gamma \in PT_{(X,Y)}$. Since $\operatorname{ran}\gamma \subseteq \operatorname{dom}\alpha \neq X$, γ is not surjective. Thus $\gamma \in M$. Consequently, $x(\gamma\alpha) = (x\gamma)\alpha = x\beta$ for all $x \in \operatorname{dom}\beta = \operatorname{dom}(\gamma\alpha)$. Hence $\gamma\alpha = \beta$ and $PT_{(X,Y)} \subseteq M\alpha$ which gives $M\alpha = PT_{(X,Y)}$.

CASE 2: α is not injective. Choose $M = \{\delta \in PT_{(X,Y)} \mid \delta \text{ is not surjective}\}$. Then $M \subset PT_{(X,Y)}$. Let $\beta \in PT_{(X,Y)}$. Then

$$\beta = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}.$$

where $A_i \cap Y \neq \emptyset$; $B_j \subseteq X \setminus Y$; $\{a_i\} = Y$ and $\{b_j\} \subseteq X \setminus Y$. Let $\gamma \in PT_{(X,Y)}$ be as in Case 1. Since α is not injective, there is $x_0 \in \operatorname{ran}\alpha$ and distinct elements $x_1, x_2 \in \operatorname{dom}\alpha$ such that $x_1\alpha = x_0 = x_2\alpha$. Obviously $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. If $x_0 \in \operatorname{ran}\beta$, then there is exactly one (either x_1 or x_2) in $\operatorname{ran}\gamma$. If $x_0 \notin \operatorname{ran}\beta$, then $x_1, x_2 \notin \operatorname{ran}\gamma$. Thus γ is not surjective and so $\gamma \in M$. Analogously as in Case 1, we obtain $\gamma\alpha = \beta$. Hence $PT_{(X,Y)} \subseteq M\alpha$. This means that $M\alpha = PT_{(X,Y)}$. Therefore, α is a right magnifying element in $PT_{(X,Y)}$. □

Corollary 6.4. *$\alpha \in E(X)$ is a right magnifying element in $E(X)$ if and only if $\operatorname{dom}\alpha \neq X$ or α is not injective.*

Example 6.5. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define

$$\alpha = \begin{pmatrix} 2n & 2n+1 \\ 2n & 2n-1 \end{pmatrix}_{n\in\mathbb{N}} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & \{3,4\} & \{5,6\} & n+2 \\ 1 & 2 & 3 & 4 & n \end{pmatrix}_{n\geqslant 5}.$$

Then $(\operatorname{dom}\alpha \cap Y)\alpha = 2\mathbb{N} = (\operatorname{dom}\beta \cap Y)\beta$ and so α, $\beta \in PT_{(X,Y)}$. It is clear that α is injective. Since $\operatorname{ran}\alpha = \mathbb{N} = X$, α is surjective. but $\operatorname{dom}\alpha = \mathbb{N} \setminus \{1\} \neq X$, so $\operatorname{dom}\beta = \mathbb{N} = X$ and β is surjective but not injective. By Theorem 6.3, α, β are right magnifying elements in $PT_{(X,Y)}$. Then there is $M = \{\delta \in PT_{(X,Y)} \mid \delta \text{ is not surjective}\} \subset PT_{(X,Y)}$ such that $M\alpha = PT_{(X,Y)}$ and $M\beta = PT_{(X,Y)}$.

Added in proof (January 5, 2021). One of the Reviewers informed us that the results of our Sections 3 and 4 are similar to results obtained in the paper: R. Chinram, S. Buapradist, N. Yaqoob, P. Petchkaew, *Left and right magnifying elements in some generalized partial transformation semigroups*, submitted to Commun. Algebra, but the proofs are different.

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