Semigroups in which the radical of every quasi-ideal is a subsemigroup

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Abstract. For a non-empty subset A of a semigroup S, $\sqrt{A}$ denotes the radical of A, i.e., $\sqrt{A} = \{x \in S \mid x^n \in A$ for some positive integer $n\}$. This paper characterizes when the radical $\sqrt{Q}$ is a subsemigroup of S for every quasi-ideal Q of S.

1. Introduction and Preliminaries

Let S be a semigroup. For $a, b \in S$, the subsemigroup of S generated by $\{a, b\}$ is denoted by $\langle a, b \rangle$. A non-empty subset A of S is called a *left* (respectively, *right*) *ideal* of S if $SA \subseteq A$ (respectively, $AS \subseteq A$). And, A is called a *two-sided ideal* (or *ideal*) of S if it is both a left and a right ideal of S. A non-empty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$ (cf. [2], [3]).

For a non-empty subset A of a semigroup S, $\sqrt{A}$ denotes the *radical* of S, i.e.,

$$\sqrt{A} = \{a \in S \mid a^n \in A \text{ for some positive integer } n\}.$$

In [1], M. Ćirić and S. Bogdanović characterized when the radical $\sqrt{A}$ is a subsemigroup of S for every ideals A of S. Indeed, the authors studied when the radical of every ideal of S is a subsemigroup of S; and when the radical of every bi-ideal of S is a subsemigroup of S. The notion of quasi-ideals generalizes ideals, and the notion of bi-ideals generalizes quasi-ideals, but quasi-ideals have been widely studied; see [3]. In the line of [1], this paper considers the case of quasi-ideals. Indeed, we characterize when the radical $\sqrt{Q}$ of every quasi-ideal Q of S is a subsemigroup of S.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive integers. Let a, b be any

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elements of a semigroup S with identity. Define

$$\begin{aligned} a \mid b &\iff b = xay \text{ for some } x, y \in S;\\ a \mid_r b &\iff b = ax \text{ for some } x \in S;\\ a \mid_l b &\iff b = ya \text{ for some } y \in S;\\ a \mid_t b &\iff a \mid_r b \wedge a \mid_l b;\\ a \to b &\iff a \mid b^n \text{ for some } n \in \mathbb{N}; \text{ and}\\ a \xrightarrow{h} b &\iff a \mid_h b^n \ \text{ for some } n \in \mathbb{N} \text{ where } h \text{ is } r, l \text{ or } t. \end{aligned}$$

2. Main results

In [3], a non-empty subset Q of a semigroup S is a quasi-ideal of S if and only if it is an intersection of a left and a right ideal of S. We begin the section with the following theorem.

Theorem 2.1. *Let S be a semigroup with identity. Then the radical of every quasi-ideal of S is a subsemigroup of S if and only if*

$$\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}].$$

Proof. Assume that the radical of every quasi-ideal of S is a subsemigroup of S. Let $a, b \in S$, and let $i, j \in \mathbb{N}$. Put

$$Q = \{a^i, b^j\}S \cap S\{a^i, b^j\}.$$

Then Q is a quasi-ideal of S such that $a, b \in \sqrt{Q}$. By assumption, $ab \in \sqrt{Q}$. Hence $(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}$ for some $n \in \mathbb{N}$.

Conversely, assume that for all a, b in S and i, j in $\mathbb{N}$ there exists $n \in \mathbb{N}$ such that $(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}$. Let Q be a quasi-ideal of S, and let $a, b \in \sqrt{Q}$. Then $a^i \in Q$ and $b^j \in Q$ for some $i, j \in \mathbb{N}$. By assumption, there exists $n \in \mathbb{N}$ such that $(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}$. Thus $ab \in \sqrt{Q}$, because

$$(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\} \subseteq QS \cap SQ \subseteq Q.$$

Hence $\sqrt{Q}$ is a subsemigroup of S. □

Let $S = \{a, b, c, d, 1\}$ be a semigroup with the multiplication:

$\cdot$	a	b	c	d	1
a	a	a	a	a	a
b	a	a	a	a	b
c	a	a	b	a	c
d	a	a	b	b	d
1	a	b	c	d	1

The quasi-ideal of S is $\{\{a\},\{a,b\},\{a,b,c\},\{a,b,d\},\{a,b,c,d\},S\}$. Observe that $\sqrt{\{a\}}=\{a,b\}$, $\sqrt{\{a,b\}}=\sqrt{\{a,b,c\}}=\sqrt{\{a,b,d\}}=\sqrt{\{a,b,c,d\}}=\{a,b,c,d\}$ and $\sqrt{S}=S$; then the radical of every quasi-ideal of S is a subsemigroup of S.

In general, the radical of quasi-ideals of a semigroup with identity need not be subsemigroups, as the following example shows:

Let $S=\{a,b,c,d,f,1\}$ be a semigroup with the multiplication:

$\cdot$	a	b	c	d	f	1
a	a	a	a	a	a	a
b	a	b	a	d	a	b
c	a	f	c	c	f	c
d	a	b	d	d	b	d
f	a	f	a	c	a	f
1	a	b	c	d	f	1

The quasi-ideal of S is $\{\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,f\},\{a,b,d\}, \{a,c,d\}, \{a,b,f\},\{a,c,f\},\{a,b,c,d,f\},S\}$. We have $\sqrt{\{a,c,d\}}=\{a,c,d,f\}$ which is not a subsemigroup of S.

Theorem 2.2. *Let S be a semigroup with identity. Then the radical of every right ideal of S is a quasi-ideal of S if and only if*

$$\forall a,b,c\in S\ [a\mid_r c\wedge b\mid_l c\Longrightarrow \forall i,j\in\mathbb{N}\ [a^i\xrightarrow{r} c\vee b^j\xrightarrow{r} c]].$$

Proof. Assume that the radical of every right ideal of S is a quasi-ideal of S. Let $a,b,c\in S$ such that $a\mid_r c$ and $b\mid_l c$. Then $c=au$ and $c=vb$ for some $u,v\in S$. Let $i,j\in\mathbb{N}$. Put $R=\{a^i,b^j\}S$; then R is a right ideal of S and $a,b\in\sqrt{R}$. By assumption, $\sqrt{R}$ is a quasi-ideal of S. Since $c=au$ and $c=vb$,

$$c\in\sqrt{R}S\cap S\sqrt{R}\subseteq\sqrt{R}.$$

Thus $c^n\in R$ for some $n\in\mathbb{N}$, whence $a^i\xrightarrow{r} c$ or $b^j\xrightarrow{r} c$.

Conversely, assume that for all a,b,c in S,

$$a\mid_r c\wedge b\mid_l c\Longrightarrow \forall i,j\in\mathbb{N}\ [a^i\xrightarrow{r} c\vee b^j\xrightarrow{r} c].$$

Let R be a right ideal of S. To show that $\sqrt{R}S\cap S\sqrt{R}\subseteq\sqrt{R}$, let $x\in\sqrt{R}S\cap S\sqrt{R}$. Then $x=au$ and $x=vb$ for some $u,v\in S$ and $a,b\in\sqrt{R}$. Since $a,b\in\sqrt{R}$, there exist $i,j\in\mathbb{N}$ such that $a^i,b^j\in R$. By assumption, there exists $n\in\mathbb{N}$ such that $x^n\in\{a^i,b^j\}S$. Since

$$\{a^i,b^j\}S\subseteq RS\subseteq R,$$

then $x\in\sqrt{R}$. Hence $\sqrt{R}$ is a quasi-ideal of S. □

As Theorem 2.2, we obtain the following.

Theorem 2.3. *Let S be a semigroup with identity. Then the radical of every left ideal of a semigroup S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S\ [a \mid_r c \wedge b \mid_l c \Longrightarrow \forall i, j \in \mathbb{N}\ [a^i \xrightarrow{l} c \vee b^j \xrightarrow{l} c]].$$

Theorem 2.4. *Let S be a semigroup with identity. Then the radical of every quasi-ideal of S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S\ [a \mid_r c \wedge b \mid_l c \Longrightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [c^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}]].$$

Proof. Assume that the radical of every quasi-ideal of S is a quasi-ideal of S. Let $a, b, c \in S$ such that $a \mid_r c$ and $b \mid_l c$. Then $c = au$ and $c = vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. Put

$$Q = \{a^i, b^j\}S \cap S\{a^i, b^j\}.$$

Then Q is a quasi-ideal of S and $a, b \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a quasi-ideal of S. Since $c = au$ and $c = vb$,

$$c \in \sqrt{Q}S \cap S\sqrt{Q} \subseteq \sqrt{Q}.$$

Hence $c^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}$ for some $n \in \mathbb{N}$.

Conversely, assume that for all $a, b, c \in S$,

$$a \mid_r c \wedge b \mid_l c \Longrightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [c^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}].$$

Let Q be a quasi-ideal of S. We need show that $\sqrt{Q}S \cap S\sqrt{Q} \subseteq \sqrt{Q}$. Let $x \in \sqrt{Q}S \cap S\sqrt{Q}$. Then $x = au$ and $x = vb$ for some $a, b \in \sqrt{Q}$ and $u, v \in S$. Since $a, b \in \sqrt{Q}$, there exist $i, j \in \mathbb{N}$ such that $a^i, b^j \in Q$. By assumption, there exists $n \in \mathbb{N}$ such that $x^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}$. Since

$$\{a^i, b^j\}S \cap S\{a^i, b^j\} \subseteq QS \cap SQ \subseteq Q,$$

then $x \in \sqrt{Q}$, whence $\sqrt{Q}$ is a quasi-ideal of S. □

Theorem 2.5. *Let S be a semigroup with identity. The radical of every ideal of S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S\ [a \mid_r c \wedge b \mid_l c \Longrightarrow \forall i, j \in \mathbb{N}\ [a^i \to c \vee b^j \to c]].$$

Proof. Assume that the radical of every ideal of S is a quasi-ideal of S. Let $a, b, c \in S$ such that $a \mid_r c$ and $b \mid_l c$. Then $c = au$ and $c = vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. Put $A = S\{a^i, b^j\}S$, then A is an ideal of S and $a, b \in \sqrt{A}$. By assumption, $\sqrt{A}$ is a quasi-ideal of S. Since $c = au$ and $c = vb$,

$$c \in \sqrt{A}S \cap S\sqrt{A} \subseteq \sqrt{A}.$$

Then there exists $n \in \mathbb{N}$ such that $c^n \in A$. Hence $a^i \to c$ or $b^j \to c$. The opposite direction can be proved similarly to the converse of Theorem 2.2. □

Theorem 2.6. *Let S be a semigroup with identity. The radical of every quasi-ideal of S is a bi-ideal of S if and only if*

$$\forall a,b,c \in S\ \forall i,j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [(abc)^n \in \{a^i, c^j\}S \cap S\{a^i, c^j\}].$$

Proof. Assume that the radical of every quasi-ideal of S is a bi-ideal of S. Let $a, b, c \in S$, and let $i, j \in \mathbb{N}$. Put $Q = \{a^i, c^j\}S \cap S\{a^i, c^j\}$. Observe firstly that Q is a quasi-ideal of S and $a, c \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a bi-ideal of S. Then

$$abc \in \sqrt{Q}S\sqrt{Q} \subseteq \sqrt{Q}.$$

Hence $(abc)^n \in \{a^i, c^j\}S \cap S\{a^i, c^j\}$ for some $n \in \mathbb{N}$.

Conversely, assume that for any $a, b, c \in S$, and $i, j \in \mathbb{N}$,

$$(abc)^n \in \{a^i, c^j\}S \cap S\{a^i, c^j\} \text{ for some } n \in \mathbb{N}.$$

Let Q be a quasi-ideal of S. Let $a, c \in \sqrt{Q}$, and let $b \in S$. Then $a^i, c^j \in Q$ for some $i, j \in \mathbb{N}$. By assumption, $(abc)^n \in \{a^i, c^j\}S \cap S\{a^i, c^j\}$ for some $n \in \mathbb{N}$. Consider

$$(abc)^n \in \{a^i, c^j\}S \cap S\{a^i, c^j\} \subseteq QS \cap SQ \subseteq Q.$$

Thus $abc \in \sqrt{Q}$, and $\sqrt{Q}$ is a bi-ideal of S. □

Theorem 2.7. *Let S be a semigroup with identity. The radical of every quasi-ideal of a semigroup S is a right ideal of S if and only if*

$$a^k \xrightarrow{t} ab \text{ for all } a, b \in S \text{ and } k \in \mathbb{N}.$$

Proof. Assume that the radical of every quasi-ideal of S is a right ideal of S. Let $a, b \in S$ and $k \in \mathbb{N}$. Put $Q = a^kS \cap Sa^k$. Then Q is a quasi-ideal of S and $a \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a right ideal of S. Thus $ab \in \sqrt{Q}S \subseteq \sqrt{Q}$. We then have that there exists $n \in \mathbb{N}$ such that $(ab)^n \in Q$. Hence $a^k \xrightarrow{t} ab$.

Conversely, assume that $a^k \xrightarrow{t} ab$ for all $a, b \in S$ and $k \in \mathbb{N}$. Let Q be a quasi-ideal of S, and let $a \in \sqrt{Q}$ and $b \in S$. Then $a^k \in Q$ for some $k \in \mathbb{N}$. Since

$$a^kS \cap Sa^k \subseteq QS \cap SQ \subseteq Q,$$

$(ab)^n \in Q$ for some $n \in \mathbb{N}$. This implies $ab \in \sqrt{Q}$, and hence $\sqrt{Q}$ is a right ideal of S. □

As Theorem 2.7, we obtain the following theorem.

Theorem 2.8. *Let S be a semigroup with identity. The radical of every quasi-ideal of S is a left ideal of S if and only if $a^k \xrightarrow{t} ba$ for all $a, b \in S$ and $k \in \mathbb{N}$.*

Theorem 2.9. *Let S be a semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every quasi-ideal of S is an ideal of S;*

(2) *$a^k \xrightarrow{t} ab$ and $a^k \xrightarrow{t} ba$ for all $a, b \in S$ and $k \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Let $a, b \in S$, and let $k \in \mathbb{N}$. Put $A = a^k S \cap S a^k$. Clearly, A is a quasi-ideal of S and $a \in \sqrt{A}$. By assumption, $\sqrt{A}$ is an ideal of S. Thus $ab \in \sqrt{A}S \subseteq \sqrt{A}$ and $ba \in S\sqrt{A} \subseteq \sqrt{A}$. This implies $(ab)^m, (ba)^n \in A$ for some $m, n \in \mathbb{N}$. Hence $a^k \xrightarrow{t} ab$ and $a^k \xrightarrow{t} ba$.

(2) $\Rightarrow$ (1): Assume (2). Let Q be a quasi-ideal of S. To show that $\sqrt{Q}$ is an ideal of S, let $a \in \sqrt{Q}$ and $b \in S$. Since $a \in \sqrt{Q}$, $a^k \in Q$ for some $k \in \mathbb{N}$. By assumption, there exist $m, n \in \mathbb{N}$ such that $(ab)^m, (ba)^n \in a^k S \cap S a^k$. Hence $(ab)^m, (ba)^n \in Q$, because

$$a^k S \cap S a^k \subseteq QS \cap SQ \subseteq Q.$$

This implies $ab, ba \in \sqrt{Q}$, and thus $\sqrt{Q}$ is an ideal of S. □

Theorem 2.10. *Let S be a semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every bi-ideal of S is a quasi-ideal of S;*

(2) $\forall a, b, c \in S\ [a \mid_r c \wedge b \mid_l c \Longrightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [c^n \in \{a^i, b^j\} S \{a^i, b^j\}]]$.

Proof. (1) $\Rightarrow$ (2): Assume (1). Let $a, b, c \in S$ such that $c = au$ and $c = vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. It is observed that

$$B = \{a^i, b^j\} S \{a^i, b^j\}$$

is a bi-ideal of S and $a, b \in \sqrt{B}$. By assumption, $\sqrt{B}$ is a quasi-ideal of S. Therefore, $c \in \sqrt{B}S \cap S\sqrt{B} \subseteq \sqrt{B}$. Hence $c^n \in \{a^i, b^j\} S \{a^i, b^j\}$ for some $n \in \mathbb{N}$.

(2) $\Rightarrow$ (1): Assume (2). Let B be a bi-ideal of S. Let $x \in \sqrt{B}S \cap S\sqrt{B}$. Then $x = au$ and $x = vb$ for some $a, b \in \sqrt{B}$ and $u, v \in S$. Hence, there exist $i, j \in \mathbb{N}$ such that $a^i, b^j \in B$. By assumption,

$$x^n \in \{a^i, b^j\} S \{a^i, b^j\} \subseteq BSB \subseteq B.$$

Thus $x \in \sqrt{B}$. Hence $\sqrt{B}$ is a quasi-ideal of S. □

Theorem 2.11. *Let S be a semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every subsemigroup of S is a quasi-ideal of S;*

(2) $\forall a, b, c \in S\ [a \mid_r c \wedge b \mid_l c \Longrightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ [c^n \in \langle a^i, b^j \rangle]]$.

Proof. (1) $\Rightarrow$ (2): Assume (1), and let $a, b, c \in S$, such that $a \mid_r c$ and $b \mid_l c$. Then $c = au$ and $c = vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. Put $A = \langle a^i, b^j \rangle$. By (1), $\sqrt{A}$ is a quasi-ideal of S. Since $c = au$ and $c = vb$, $c \in \sqrt{A}S \cap S\sqrt{A}$. Then

$$c \in \sqrt{A}S \cap S\sqrt{A} \subseteq \sqrt{A}.$$

Hence $c^n \in \langle a^i, b^j \rangle$ for some $n \in \mathbb{N}$.

(2) $\Rightarrow$ (1): Assume (2), and let A be a subsemigroup of S. Let $x \in \sqrt{A}S \cap S\sqrt{A}$; then $x = au$ and $x = vb$ for some $a, b \in \sqrt{A}$ and $u, v \in S$. We then have that $a^i, b^j \in A$ for some $i, j \in \mathbb{N}$. By assumption, $x^n \in \langle a^i, b^j \rangle$. Since $\langle a^i, b^j \rangle \subseteq A$, $x \in \sqrt{A}$. Thus $\sqrt{A}$ is a quasi-ideal of S. □

Finally, we have the following result.

Theorem 2.12. *Let S be a semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every quasi-ideal of S is a quasi-ideal of S;*

(2) $\forall a, b \in S$ $[\sqrt{\{a,b\}S \cap S\{a,b\}}$ *is a quasi-ideal of* $S]$;

(3) $\forall a, b, c \in S$ $[a \mid_r c \wedge b \mid_l c \Longrightarrow \exists n \in \mathbb{N}$ $[c^n \in \{a^2, b^2\}S \cap S\{a^2, b^2\}]]$;

(4) $\forall a, b, c \in S$ $[a \mid_r c \wedge b \mid_l c \Longrightarrow \forall k \in \mathbb{N}$ $\exists n \in \mathbb{N}$ $[c^n \in \{a^k, b^k\}S \cap S\{a^k, b^k\}]]$.

Proof. (1) $\Rightarrow$ (2): Assume (1), and let $a, b \in S$. Since $\{a,b\}S \cap S\{a,b\}$ is a quasi-ideal of S and (1), $\sqrt{\{a,b\}S \cap S\{a,b\}}$ is a quasi-ideal of S.

(2) $\Rightarrow$ (3): Assume (2), and let $a, b, c \in S$ such that $a \mid_r c$ and $b \mid_l c$. Then $c = au$ and $c = vb$ for some $u, v \in S$. Clearly, $a, b \in \sqrt{\{a^2, b^2\}S \cap S\{a^2, b^2\}}$. By (2), $\sqrt{\{a^2, b^2\}S \cap S\{a^2, b^2\}}$ is a quasi-ideal of S. From $c = au$ and $c = vb$, it follows that

$$\begin{aligned} c &\in \sqrt{\{a^2, b^2\}S \cap S\{a^2, b^2\}}S \cap S\sqrt{\{a^2, b^2\}S \cap S\{a^2, b^2\}} \\ &\subseteq \sqrt{\{a^2, b^2\}S \cap S\{a^2, b^2\}}. \end{aligned}$$

Thus $c^n \in \{a^2, b^2\}S \cap S\{a^2, b^2\}$ for some $n \in \mathbb{N}$.

(3) $\Rightarrow$ (4): Assume (3), and let $a, b, c \in S$ such that $a \mid_r c$ and $b \mid_l c$. Then $c = au$ and $c = vb$ for some $u, v \in S$. By (3), $c^n \in \{a^2, b^2\}S \cap S\{a^2, b^2\}$ for some $n \in \mathbb{N}$. It is observed that

$$\{a^2, b^2\}S \cap S\{a^2, b^2\} \subseteq \{a, b\}S \cap S\{a, b\}.$$

Then

$$c^n \in \{a, b\}S \cap S\{a, b\}.$$

Suppose that there exists $m \in \mathbb{N}$ where $k \in \mathbb{N}$ such that

$$c^m \in \{a^k, b^k\}S \cap S\{a^k, b^k\}.$$

By (3), there exists $l \in \mathbb{N}$ such that

$$(c^m)^l \in \{a^{2k}, b^{2k}\}S \cap S\{a^{2k}, b^{2k}\}.$$

Consider

$$\begin{aligned}(c^m)^l &\in \{a^{2k}, b^{2k}\}S \cap S\{a^{2k}, b^{2k}\}\\ &= \{a^{k+1}a^{k-1}, b^{k+1}b^{k-1}\}S \cap S\{a^{k-1}a^{k+1}, b^{k-1}b^{k+1}\}\\ &\subseteq \{a^{k+1}, b^{k+1}\}S \cap S\{a^{k+1}, b^{k+1}\}.\end{aligned}$$

Hence

$$c^{ml} = (c^m)^l \in \{a^{k+1}, b^{k+1}\}S \cap S\{a^{k+1}, b^{k+1}\}.$$

Therefore (4) holds.

(4) $\Rightarrow$ (1): Assume (4), and let Q be a quasi-ideal of S. Let $x \in \sqrt{Q}S \cap S\sqrt{Q}$. Then $x = au$ and $x = vb$ for some $u, v \in S$ and $a, b \in \sqrt{Q}$. Then $a^i, b^j \in Q$ for some $i, j \in \mathbb{N}$. By (4), there exists $n \in \mathbb{N}$ such that

$$x^n \in \{a^{i+j}, b^{i+j}\}S \cap S\{a^{i+j}, b^{i+j}\}.$$

Consider

$$x^n \in \{a^{i+j}, b^{i+j}\}S \cap S\{a^{i+j}, b^{i+j}\} \subseteq \{a^i, b^j\}S \cap S\{a^i, b^j\} \subseteq QS \cap SQ \subseteq Q.$$

Thus $x \in \sqrt{Q}$, and Q is a quasi-ideal of S. □

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