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On orthogonal systems of ternary quasigroups admitting nontrivial paratopies

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Abstract. In the present work we describe all orthogonal systems consisting of three ternary quasigroup operations and of all (three) ternary selectors, admitting at least one nontrivial paratopy. In [11] we proved that there exist precisely 48 orthogonal systems of the considered form, admitting at least one paratopy, which components are three quasigroup operations, or two quasigroup operations and a selector. Now we show that there exist precisely 105 such systems, admitting at least one nontrivial paratopy which components are two selectors and a quasigroup operation, or three selectors.

1. Introduction

An n-ary groupoid (Q, A) is called an *n-ary quasigroup* if in the equality

$$A(x_1, x_2, \dots, x_n) = x_{n+1}$$

any element of the set $\{x_1, x_2, \dots, x_{n+1}\}$ is uniquely determined by the other n elements. If (Q, A) is an n-ary quasigroup and $\sigma \in S_n$, then the operation ${}^{\sigma}A$ defined by the equivalence:

$${}^{\sigma}A(x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1, x_2, \dots, x_n) = x_{n+1},$$

for every $x_1, x_2, \dots, x_n, x_{n+1} \in Q$, is called a *$\sigma$-parastrophe* (or, simply, a *parastrophe*) of (Q, A). The operation ${}^{\sigma}A$ is called a *principal parastrophe* of A if $\sigma(n+1) = n+1$. The main notions and general properties of n-ary quasigroups may be found in [3]. Following [3], we will denote by π_i the transposition $(i, n+1)$, where $i \in \{1, 2, \dots, n\}$, so ${}^{(i,n+1)}A = {}^{\pi_i}A$.

The n-ary operations $A_1, A_2, \dots, A_n$, defined on a set Q, are called *orthogonal* if, for every $a_1, a_2, \dots, a_n \in Q$, the system of equations

$$\{A_i(x_1, x_2, \dots, x_n) = a_i\}_{i=\overline{1,n}}$$

has a unique solution in Q. A system of n-ary operations $A_1, A_2, \dots, A_s$, defined on a set Q, where $s \geqslant n$, is called *orthogonal* if every n operations of this

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system are orthogonal. For every mapping $\theta : Q^n \to Q^n$ there exist, and are unique, n n-ary operations $A_1, A_2, \ldots, A_n$, defined on Q, such that $\theta((x_1^n)) = (A_1(x_1^n), A_2(x_1^n), \ldots, A_n(x_1^n))$, for every $(x_1^n) \in Q^n$, where by (x_1^n) we denote $(x_1, \ldots, x_n)$. Moreover, the mapping θ is a bijection if and only if the operations $A_1, A_2, \ldots, A_n$ are orthogonal. The operations $E_1, E_2, \ldots, E_n$, defined on Q, where $E_i(x_1^n) = x_i$, for every $x_1, x_2, \ldots, x_n \in Q$, $i = 1, 2, \ldots, n$, are called *n-ary selectors*. An n-ary operation A is a quasigroup operation if and only if the system $\{A, E_1, E_2, \ldots, E_n\}$ is orthogonal. Orthogonal systems of n-ary operations (quasigroups) are considered in [1], [5], [7], [10]. Algebraic transformations of orthogonal systems of operations, that keep the orthogonality, have been defined and considered in [2] and [6].

If $\Sigma = \{A_1, A_2, \ldots, A_n, E_1, E_2, \ldots E_n\}$ is an orthogonal system, then we will denote the system $\{A_1\theta, A_2\theta, \ldots, A_n\theta, E_1\theta, E_2\theta, \ldots, E_n\theta\}$ by $\Sigma\theta$. Any bijection $\theta : Q^n \to Q^n$ is called a *paratopy* of Σ if $\Sigma\theta = \Sigma$ (cf. [2]).

V. Belousov proved in [2] that there exist precisely nine orthogonal systems of the form $\Sigma = \{A, B, F, E\}$, where A and B are binary quasigroups defined on a set Q and F, E are the binary selectors on Q, which admit at least one nontrivial paratopy. He also shown that many paratopies of Σ imply identities of length five with two variables (called minimal identities) in one of two quasigroups of Σ. Later, (see [4]) V. Belousov obtained a classification of such identities. It is known that minimal identities in quasigroups imply the orthogonality of some pairs of their parastrophes.

It is shown in [11] and in the present paper that there exists precisely 153 orthogonal systems, consisting of three ternary quasigroups and the ternary selectors, which admit at least one nontrivial paratopy. Moreover, the paratopies of these systems imply 67 identities. In [8] each of these identities is reduced to one of the following four types:

I. ${}^\alpha A({}^\beta A, {}^\gamma A, {}^\delta A) = E_1$,

II. ${}^\alpha A({}^\beta A, {}^\gamma A, E_1) = E_2$,

III. ${}^\alpha A({}^\beta A, E_1, E_2) = {}^\gamma A({}^\delta A, E_1, E_3)$,

IV. ${}^\alpha A({}^\beta A, E_1, E_2) = {}^\gamma A({}^\delta A, E_1, E_2)$,

where A is a ternary quasigroup operation and $\alpha, \beta, \gamma, \delta \in S_4$. It is known that some of the obtained identities imply the orthogonality of parastrophes of the corresponding quasigroups ([3], [10], [11]).

Let $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$, where A_1, A_2, A_3 are ternary quasigroups defined on a set Q and E_1, E_2, E_3 are the ternary selectors on Q, be an orthogonal system and let $\theta : Q^3 \to Q^3$, $\theta = (B_1, B_2, B_3)$, be a mapping, where B_1, B_2, B_3 are ternary operations on Q. If θ is a paratopy of Σ, then $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_1\theta, E_2\theta, E_3\theta\} = \{A_1\theta, A_2\theta, A_3\theta, B_1, B_2, B_3\} = \Sigma$, which imply $\{B_1, B_2, B_3\} \subset \Sigma$, i.e. all paratopies of Σ are triplets of operations from Σ. We study the necessary and sufficient conditions when a triplet of operations from Σ defines a paratopy of Σ. As the ternary selectors E_1, E_2, E_3 are fixed, we consider the tuples containing all possible distributions of the ternary selectors in their positions. In [11] we examined the paratopies which components are three quasigroup operations, or two quasigroup operations and a ternary selector.

In the present article we continue the investigation of the paratopies of Σ, and prove that there exist 105 such orthogonal systems, that admit at least one nontrivial paratopy consisting of a ternary quasigroup and two ternary selectors, or of three ternary selectors.

2. Paratopies consisting of two ternary selectors and a ternary quasigroup operation

It is proved in this section that there exist precisely 87 orthogonal systems $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$, consisting of three ternary quasigroup operations A_1, A_2, A_3 and three ternary selectors E_1, E_2, E_3, admitting at least one paratopy, which components are two ternary selectors and a ternary quasigroup operation.

Lemma 2.1. *The triplet* (E_1, E_2, A_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(E_1, E_2, A_1), A_3 = {}^{\pi_3}A_1$ *and* $A_1(E_1, E_2, A_1(E_1, E_2, A_1)) = {}^{\pi_3}A_1$;
2. $A_3 = A_1(E_1, E_2, A_1), A_2 = {}^{\pi_3}A_1$ *and* $A_1(E_1, E_2, A_1(E_1, E_2, A_1)) = {}^{\pi_3}A_1$;
3. $A_1 = {}^{\pi_3}A_2(E_1, E_2, A_3) = {}^{\pi_3}A_3(E_1, E_2, A_2)$.

Proof. Let the triplet (E_1, E_2, A_1) be a paratopy of the system Σ. As $E_1\theta = E_1, E_2\theta = E_2, E_3\theta = A_1$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_1, E_2, A_1\}$, that is $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_3, A_2, A_3\}$.

1. If $A_1\theta = A_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_1, E_2, A_2), \theta^3 = (E_1, E_2, A_3)$, $\theta^4 = \varepsilon$. From $A_1\theta = A_2$ it follows

$$A_2 = A_1(E_1, E_2, A_1). \tag{1}$$

Also, $A_1\theta = A_2$ implies $A_1\theta^3 = E_3$, i.e. $A_1(E_1, E_2, A_3) = E_3$, so

$$A_3 = {}^{\pi_3}A_1. \tag{2}$$

Moreover, from $A_1\theta = A_2$ it follows $A_1\theta^2 = A_3$, i.e. $A_1(E_1, E_2, A_2) = A_3$. Using (1) and (2) in the last equality, we get

$$A_1(E_1, E_2, A_1(E_1, E_2, A_1)) = {}^{\pi_3}A_1. \tag{3}$$

Conversely, if (1), (2) and (3) hold, then (1) implies $A_1\theta = A_2$. From (2) it follows $A_3\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, hence $A_3\theta = E_3$. Using (1) and (2) in (3), we get $A_1(E_1, E_2, A_2) = A_3$, hence $A_2 = {}^{\pi_3}A_1(E_1, E_2, A_3)$, which implies $A_2\theta = {}^{\pi_3}A_1$. Using (2) in the last equality, we obtain $A_2\theta = A_3$.

2. If $A_1\theta = A_2, A_2\theta = E_3, A_3\theta = A_3$, then $A_3\theta = A_3$, i.e. $A_3(E_1, E_2, A_1) = A_3$, implies $A_1 = E_3$, which is a contradiction as A_1 is a quasigroup.

3. If $A_1\theta = A_3, A_2\theta = A_2, A_3\theta = E_3$, then $A_2\theta = A_2$, i.e. $A_2(E_1, E_2, A_1) = A_2$, implies $A_1 = E_3$, which is a contradiction as A_1 is a quasigroup.

4. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = A_2$, then $\theta^2 = (E_1, E_2, A_3), \theta^3 = (E_1, E_2, A_2)$, $\theta^4 = \varepsilon$. From $A_1\theta = A_3$ it follows

$$A_3 = A_1(E_1, E_2, A_1). \tag{4}$$

Also, $A_1\theta = A_3$ implies $A_1\theta^3 = E_3$, i.e. $A_1(E_1, E_2, A_2) = E_3$, so

$$A_2 = {}^{\pi_3}A_1. \tag{5}$$

Moreover, from $A_1\theta = A_3$ it follows $A_1\theta^2 = A_2$, i.e. $A_1(E_1, E_2, A_3) = A_2$. Using (4) and (5) in the last equality, we get

$$A_1(E_1, E_2, A_1(E_1, E_2, A_1)) = {}^{\pi_3}A_1. \tag{6}$$

Conversely, if (4), (5) and (6) hold, then (4) implies $A_1\theta = A_3$. From (5) it follows $A_2\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, hence $A_2\theta = E_3$. Using (4) and (5) in (6), we get $A_1(E_1, E_2, A_3) = A_2$, hence $A_3 = {}^{\pi_3}A_1(E_1, E_2, A_2)$, which implies $A_3\theta = {}^{\pi_3}A_1$. Using (5) in the last equality, we obtain $A_3\theta = A_2$.

5. If $A_1\theta = E_3, A_2\theta = A_2, A_3\theta = A_3$, then $A_2\theta = A_2$, i.e. $A_2(E_1, E_2, A_1) = A_2$, implies $A_1 = E_3$, which is a contradiction as A_1 is a quasigroup.

6. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = A_2$, then $A_2\theta = A_3$ implies

$$A_1 = {}^{\pi_3}A_2(E_1, E_2, A_3). \tag{7}$$

From $A_3\theta = A_2$ it follows

$$A_1 = {}^{\pi_3}A_3(E_1, E_2, A_2). \tag{8}$$

Conversely, if (7) and (8) hold, then (8) implies $A_3\theta = A_2$. From (7) it follows $A_2\theta = A_3$ and $A_1\theta = {}^{\pi_3}A_2(E_1, E_2, A_2)$, so $A_1\theta = E_3$. □

Lemma 2.2. *The triplet* (E_2, E_1, A_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(E_2, E_1, A_1), A_3 = {}^{(12)\pi_3}A_1$ *and* $A_1(E_1, E_2, A_1(E_2, E_1, A_1)) = {}^{(12)\pi_3}A_1$;
2. $A_3 = A_1(E_2, E_1, A_1), A_2 = {}^{(12)\pi_3} A_1$ *and* $A_1(E_1, E_2, A_1(E_2, E_1, A_1)) = {}^{(12)\pi_3}A_1$;
3. $A_1 = {}^{\pi_3} A_2(E_2, E_1, A_2) = {}^{\pi_3} A_3(E_2, E_1, A_3)$;
4. $A_1(E_2, E_1, A_1) = E_3, A_2(E_2, E_1, A_1) = A_3$.

Proof. Let the triplet (E_2, E_1, A_1) be a paratopy of the system Σ. As $E_1\theta = E_2, E_2\theta = E_1, E_3\theta = A_1$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_2, E_1, A_1\}$, so there are six possible cases:

1. If $A_1\theta = A_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_1, E_2, A_2), \theta^3 = (E_2, E_1, A_3)$, $\theta^4 = \varepsilon$. From $A_1\theta = A_2$ it follows

$$A_2 = A_1(E_2, E_1, A_1). \tag{9}$$

Also, $A_1\theta = A_2$ implies $A_1\theta^3 = E_3$, that is $A_1(E_2, E_1, A_3) = E_3$, so

$$A_3 = {}^{(12)\pi_3} A_1. \tag{10}$$

Moreover, $A_1\theta = A_2$ implies $A_1\theta^2 = A_3$, i.e. $A_1(E_1, E_2, A_2) = A_3$. Using (9) and (10) in the last equality we get

$$A_1(E_1, E_2, A_1(E_2, E_1, A_1)) =^{(12)\pi_3} A_1. \tag{11}$$

Conversely, if (9), (10) and (11) hold, then from (9) it follows $A_1\theta = A_2$ and (10) implies $A_3\theta =^{\pi_3} A_1(E_1, E_2, A_1)$, so $A_3\theta = E_3$. Using (9) and (10) in (11) we get $A_1(E_1, E_2, A_2) = A_3$, which implies $A_2 =^{\pi_3} A_1(E_1, E_2, A_3)$, hence $A_2\theta = {}^{\pi_3}A_1(E_2, E_1, E_3)$. Using (10) in the last equality, we obtain $A_2\theta = A_3$.

2. If $A_1\theta = A_2, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (E_1, E_2, A_2)$. From $A_3\theta = A_3$ it follows $A_3\theta^2 = A_3$, i.e. $A_3(E_1, E_2, A_2) = A_3$, so $A_2 = E_3$, which is a contradiction, as A_2 is a quasigroup operation.

3. If $A_1\theta = A_3, A_2\theta = A_2, A_3\theta = E_3$, then $\theta^2 = (E_1, E_2, A_3)$. From $A_2\theta = A_2$ it follows $A_2\theta^2 = A_2$, i.e. $A_2(E_1, E_2, A_3) = A_2$, so $A_3 = E_3$, which is a contradiction, as A_3 is a quasigroup operation.

4. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = A_2$, then $\theta^2 = (E_1, E_2, A_3), \theta^3 = (E_2, E_1, A_2)$, $\theta^4 = \varepsilon$. From $A_1\theta = A_3$ it follows

$$A_3 = A_1(E_2, E_1, A_1). \tag{12}$$

Also, $A_1\theta = A_3$ implies $A_1\theta^3 = E_3$, i.e. $A_1(E_2, E_1, A_2) = E_3$, so

$$A_2 =^{(12)\pi_3} A_1. \tag{13}$$

Moreover, $A_1\theta = A_3$ implies $A_1\theta^2 = A_2$, i.e. $A_1(E_1, E_2, A_3) = A_2$. Using (12) and (13) in the last equality we get

$$A_1(E_1, E_2, A_1(E_2, E_1, A_1)) =^{(12)\pi_3} A_1. \tag{14}$$

Conversely, if (12), (13) and (14) hold, then from (12) it follows $A_1\theta = A_3$ and (13) implies $A_2\theta =^{\pi_3} A_1(E_1, E_2, A_1)$, so $A_2\theta = E_3$. Using (12) and (13) in (14) we get $A_1(E_1, E_2, A_3) = A_2$, which implies $A_3 =^{\pi_3} A_1(E_1, E_2, A_2)$, therefore $A_3\theta = {}^{\pi_3}A_1(E_2, E_1, E_3)$. Using (13) in the last equality, we obtain $A_3\theta = A_2$.

5. If $A_1\theta = E_3, A_2\theta = A_2, A_3\theta = A_3$, then $\theta^2 = \varepsilon$. From $A_2\theta = A_2$ it follows that

$$A_1 =^{\pi_3} A_2(E_2, E_1, A_2). \tag{15}$$

From $A_3\theta = A_3$ it follows

$$A_1 =^{\pi_3} A_3(E_2, E_1, A_3). \tag{16}$$

Conversely, if (15) and (16) hold, then (15) implies $A_2\theta = A_2$ and (16) implies $A_3\theta = A_3$. Also, from (16) we get $A_1\theta = {}^{\pi_3}A_3(E_1, E_2, A_3)$, so $A_1\theta = E_3$.

6. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = A_2$, then $\theta^2 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_1(E_2, E_1, A_1) = E_3, \tag{17}$$

and $A_2\theta = A_3$ can be written in the form

$$A_3 = A_2(E_2, E_1, A_1). \tag{18}$$

Conversely, if (17) and (18) hold, then (17) implies $A_1\theta = E_3$ and (18) implies $A_2\theta = A_3$. Also, from (18) it follows $A_3\theta = A_2(E_1, E_2, E_3)$, i.e. $A_3\theta = A_2$. □

Lemma 2.3. *The triplet (E_1, A_1, E_2) is a paratopy of the system Σ if and only if one of the following conditions holds:*

1. $A_2 = A_1(E_1, A_1, E_2), A_3 =^{(23)\pi_3} A_1$ *and*
$A_1(E_1, {}^{(23)\pi_3} A_1, A_1(E_1, A_1, E_2)) = E_3$;

2. $A_1 =^{\pi_2} A_3(E_1, A_3, E_2), A_2 =^{\pi_3} A_3(E_1, E_3, A_3)$ *and*
$A_3(E_1, {}^{\pi_3} A_3(E_1, E_3, A_3), {}^{\pi_2} A_3(E_1, A_3, E_2)) = A_3$;

3. $A_1 =^{\pi_2} A_2(E_1, A_2, E_2), A_3 =^{\pi_3} A_2(E_1, E_3, A_2)$ *and*
$A_2(E_1, {}^{\pi_3} A_2(E_1, E_3, A_2), {}^{\pi_2} A_2(E_1, A_2, E_2)) = A_2$;

4. $A_2 =^{(23)\pi_3} A_1, A_3 = A_1(E_1, A_1, E_2)$ *and*
$A_1(E_1, {}^{(23)\pi_3} A_1, A_1(E_1, A_1, E_2)) = E_3$;

5. $A_1 =^{(23)\pi_2} A_1 =^{\pi_2} A_2(E_1, A_2, E_2) =^{\pi_2} A_3(E_1, A_3, E_2)$.

Proof. Let the triplet (E_1, A_1, E_2) be a paratopy of the system Σ. As $E_1\theta = E_1, E_2\theta = A_1, E_3\theta = E_2$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_3, A_2, A_3\}$, that is $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_3, A_2, A_3\}$.

1. If $A_1\theta = A_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (E_1, A_2, A_1), \theta^3 = (E_1, A_3, A_2)$, $\theta^4 = (E_1, E_3, A_3)$, $\theta^5 = \varepsilon$. From $A_1\theta = A_2$ it follows

$$A_2 = A_1(E_1, A_1, E_2). \tag{19}$$

Also, $A_1\theta = A_2$ implies $A_1\theta^4 = E_2$, i.e. $A_1(E_1, E_3, A_3) = E_2$, so

$$A_3 = {}^{(23)\pi_3}A_1. \tag{20}$$

Moreover, $A_1\theta = A_2$ implies $A_1\theta^3 = E_3$, i.e. $A_1(E_1, A_3, A_2) = E_3$. Using (19) and (20) in the last equality, we get

$$A_1(E_1, {}^{(23)\pi_3}A_1, A_1(E_1, A_1, E_2)) = E_3. \tag{21}$$

Conversely, if (19), (20) and (21) hold, then (19) implies $A_1\theta = A_2$ and (20) implies $A_3\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, so $A_3\theta = E_3$. Using (19) and (20) in (21), we get $A_1(E_1, A_3, A_2) = E_3$, which implies $A_2 = {}^{\pi_3}A_1(E_1, A_3, E_3)$, hence $A_2\theta = {}^{\pi_3}A_1(E_1, E_3, E_2)$. Using (20) in the last equality, we obtain $A_2\theta = A_3$.

2. If $A_1\theta = A_2, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (E_1, A_2, A_1), \theta^3 = (E_1, E_3, A_2)$, $\theta^4 = \varepsilon$. From $A_3\theta = A_3$ it follows

$$A_1 = {}^{\pi_2}A_3(E_1, A_3, E_2). \tag{22}$$

Also, from $A_3\theta = A_3$ it follows $A_3\theta^3 = A_3$, i.e. $A_3(E_1, E_3, A_2) = A_3$, so

$$A_2 = {}^{\pi_3}A_3(E_1, E_3, A_3). \tag{23}$$

Moreover, $A_3\theta = A_3$ implies $A_3\theta^2 = A_3$, i.e. $A_3(E_1, A_2, A_1) = A_3$, Using (22) and (23) in the last equality, we obtain

$$A_3(E_1, {}^{\pi_3}A_3(E_1, E_3, A_3), {}^{\pi_2}A_3(E_1, A_3, E_2)) = A_3. \tag{24}$$

Conversely, if (22), (23) and (24) hold, then (22) implies $A_3\theta = A_3$ and (23) implies $A_2\theta = {}^{\pi_3}A_3(E_1, E_2, A_3)$, so $A_2\theta = E_3$. Using (22) and (23) in (24), we get $A_3(E_1, A_2, A_1) = A_3$, which implies $A_1 = {}^{\pi_3}A_3(E_1, A_2, A_3)$, hence $A_1\theta = {}^{\pi_3}A_3(E_1, E_3, A_3)$. Using (23) in the last equality, we get $A_1\theta = A_2$.

3. If $A_1\theta = A_3, A_2\theta = A_2, A_3\theta = E_3$, then $\theta^2 = (E_1, A_3, A_1), \theta^3 = (E_1, E_3, A_3)$, $\theta^4 = \varepsilon$. From $A_2\theta = A_2$ it follows

$$A_1 = {}^{\pi_2}A_2(E_1, A_2, E_2). \tag{25}$$

Also, from $A_2\theta = A_2$ it follows $A_2\theta^3 = A_2$, i.e. $A_2(E_1, E_3, A_3) = A_2$, so

$$A_3 = {}^{\pi_3}A_2(E_1, E_3, A_2). \tag{26}$$

Moreover, $A_2\theta = A_2$ implies $A_2\theta^2 = A_2$, i.e. $A_3(E_1, A_3, A_1) = A_2$. Using (25) and (26) in the last equality, we obtain

$$A_2(E_1, {}^{\pi_3}A_2(E_1, E_3, A_2), {}^{\pi_2}A_2(E_1, A_2, E_2)) = A_2. \tag{27}$$

Conversely, if (25), (26) and (27) hold, then (25) implies $A_2\theta = A_2$ and (26) implies $A_3\theta = {}^{\pi_3}A_2(E_1, E_2, A_2)$, so $A_3\theta = E_3$. Using (25) and (26) in (27), we get $A_2(E_1, A_3, A_1) = A_2$, which implies $A_1 = {}^{\pi_3}A_2(E_1, A_3, A_2)$, hence $A_1\theta = {}^{\pi_3}A_2(E_1, E_3, A_2)$. Using (26) in the last equality, we get $A_1\theta = A_3$.

4. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = A_2$, then $\theta^2 = (E_1, A_3, A_1), \theta^3 = (E_1, A_2, A_3)$, $\theta^4 = (E_1, E_3, A_2)$, $\theta^5 = \varepsilon$. From $A_1\theta = A_3$ it follows

$$A_3 = A_1(E_1, A_1, E_2). \tag{28}$$

Also, $A_1\theta = A_3$ implies $A_1\theta^4 = E_2$, i.e. $A_1(E_1, E_3, A_2) = E_2$, so

$$A_2 = {}^{(23)\pi_3}A_1. \tag{29}$$

Moreover, $A_1\theta = A_3$ implies $A_1\theta^3 = E_3$, i.e. $A_1(E_1, A_2, A_3) = E_3$. Using (28) and (29) in the last equality, we obtain

$$A_1(E_1, {}^{(23)\pi_3}A_1, A_1(E_1, A_1, E_2)) = E_3. \tag{30}$$

Conversely, if (28), (29) and (30) hold, then (28) implies $A_1\theta = A_3$ and (29) implies $A_2\theta = {}^{\pi_3}A_1(E_1, E_2, A_1)$, so $A_2\theta = E_3$. Using (28) and (29) in (30), we get $A_1(E_1, A_2, A_3) = E_3$, which implies $A_3 = {}^{\pi_3}A_1(E_1, A_2, E_3)$, hence $A_3\theta = {}^{\pi_3}A_1(E_1, E_3, E_2)$. Using (29) in the last equality, we get $A_3\theta = A_2$.

5. If $A_1\theta = E_3, A_2\theta = A_2, A_3\theta = A_3$, from $A_1\theta = E_3$ it follows

$$A_1 = {}^{(23)\pi_1}A_1. \tag{31}$$

The equality $A_2\theta = A_2$ implies

$$A_1 = {}^{\pi_2}A_2(E_1, A_2, E_2). \tag{32}$$

From $A_3\theta = A_3$ it follows

$$A_1 = {}^{\pi_2}A_3(E_1, A_3, E_2). \tag{33}$$

Conversely, if (31), (32) and (33) hold, then (31) implies $A_1\theta = E_3$, from (32) it follows $A_2\theta = A_2$ and (33) implies $A_3\theta = A_3$.

6. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = A_2$, then $\theta^2 = (E_1, E_3, A_1), \theta^3 = \varepsilon$. Remark that $A_2 = A_2\theta^3 = A_3\theta^2 = A_2\theta = A_3$, which is a contradiction, as Σ is an orthogonal system. □

Lemma 2.4. *The triplet (E_2, A_1, E_1) is a paratopy of the system Σ if and only if one of the following conditions holds:*

1. $A_1 =^{(132)\pi_2} A_3, A_2 = A_3(E_3, E_1, A_3)$ *and*
${}^{(132)\pi_2}A_3(E_2, {}^{(132)\pi_2} A_3, E_1) = A_3(E_3, E_1, A_3)$;

2. $A_1 =^{\pi_2} A_3(E_2, A_3, E_1), A_2 =^{\pi_3} A_3(E_3, E_1, A_3)$ *and*
${}^{\pi_2}A_3({}^{\pi_3}A_3(E_3, E_1, A_3), A_3, {}^{\pi_2} A_3(E_2, A_3, E_1)) = E_3$;

3. $A_1 =^{\pi_2} A_2(E_2, A_2, E_1), A_3 =^{\pi_3} A_2(E_3, E_1, A_2)$ *and*
${}^{\pi_2}A_2({}^{\pi_3}A_2(E_3, E_1, A_2), A_2, {}^{\pi_2} A_2(E_2, A_2, E_1)) = E_3$;

4. $A_1 =^{(132)\pi_2} A_2, A_3 = A_2(E_3, E_1, A_2)$ *and*
$A_2(A_2, E_3, A_2(E_3, E_1, A_2)) = {}^{(132)\pi_2}A_2$;

5. $A_1 =^{\pi_2} A_2(E_2, A_2, E_1) =^{\pi_2} A_3(E_2, A_3, E_1)$ *and* $A_1 = {}^{(132)\pi_2}A_1$;

6. $A_1 =^{\pi_1} A_2(A_2, E_3, E_2), A_3 = A_2(E_3, E_1, {}^{\pi_1} A_2(A_2, E_3, E_2))$ *and*
$A_1 = {}^{(132)\pi_2}A_1$.

Proof. Let the triplet (E_2, A_1, E_1) be a paratopy of the system Σ. As $E_1\theta = E_2, E_2\theta = A_1, E_3\theta = E_1$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_2, A_1, E_1\}$, that is $\{A_1\theta, A_2\theta, A_3\theta\} = \{E_3, A_2, A_3\}$.

1. If $A_1\theta = A_2, A_2\theta = A_3, A_3\theta = E_3$, then $\theta^2 = (A_1, A_2, E_2), \theta^3 = (A_2, A_3, A_1)$, $\theta^4 = (A_3, E_3, A_2)$, $\theta^5 = (E_3, E_1, A_3), \theta^6 = \varepsilon$. From $A_3\theta = E_3$ it follows

$$A_1 =^{(132)\pi_2} A_3. \tag{34}$$

The equality $A_2\theta = A_3$ implies $A_2\theta^6 = A_3\theta^5$, so

$$A_2 = A_3(E_3, E_1, A_3). \tag{35}$$

Using (34) and (35) in $A_1\theta = A_2$, we get

$${}^{(132)\pi_2}A_3(E_2, {}^{(132)\pi_2} A_3, E_1) = A_3(E_3, E_1, A_3). \tag{36}$$

Conversely, if (34), (35) and (36) hold, then from (34) it follows $A_3\theta = E_3$. The equality (35) implies $A_2\theta = A_3$. Using (34) and (35) in (36), we obtain $A_1(E_2, A_1, E_1) = A_2$, which implies $A_1\theta = A_2$.

2. If $A_1\theta = A_2, A_2\theta = E_3, A_3\theta = A_3$, then $\theta^2 = (A_1, A_2, E_2), \theta^3 = (A_2, E_3, A_1)$, $\theta^4 = (E_3, E_1, A_2)$, $\theta^5 = \varepsilon$. From $A_3\theta = A_3$ it follows

$$A_1 = {}^{\pi_2}A_3(E_2, A_3, E_1). \tag{37}$$

Also, $A_3\theta = A_3$ implies $A_3\theta^4 = A_3$, i.e. $A_3(E_3, E_1, A_2) = A_3$, so

$$A_2 =^{\pi_3} A_3(E_3, E_1, A_3). \tag{38}$$

The equality $A_1\theta = A_2$ implies $A_1\theta^2 = E_3$. From (37) and $A_1\theta^2 = E_3$, we get ${}^{\pi_2}A_3(A_2, A_3, A_1) = E_3$ so, using (37) and (38) in the last equality, we obtain

$$^{\pi_2}A_3(^{\pi_3}A_3(E_3, E_1, A_3), A_3, ^{\pi_2}A_3(E_2, A_3, E_1)) = E_3. \tag{39}$$

Conversely, if (37), (38) and (39) hold, then from (37) it follows $A_3\theta = A_3$. The equality (38) implies $A_2\theta = {}^{\pi_3}A_3(E_1, E_2, A_3)$, so $A_2\theta = E_3$. Using (37) and (38) in (39), we get ${}^{\pi_2}A_3(A_2, A_3, A_1) = E_3$, which implies $A_1 = {}^{\pi_3}A_3(A_2, E_3, A_3)$, so $A_1\theta =^{\pi_3} A_3(E_3, E_1, A_3)$. Using (38) in the last equality, we obtain $A_1\theta = A_2$.

3. If $A_1\theta = A_3, A_2\theta = A_2, A_3\theta = E_3$, then $\theta^2 = (A_1, A_3, E_2), \theta^3 = (A_3, E_3, A_1)$, $\theta^4 = (E_3, E_1, A_3)$, $\theta^5 = \varepsilon$. From $A_2\theta = A_2$ it follows

$$A_1 =^{\pi_2} A_2(E_2, A_2, E_1). \tag{40}$$

Also, $A_2\theta = A_2$ implies $A_2\theta^4 = A_2$, i.e. $A_2(E_3, E_1, A_3) = A_2$, so

$$A_3 =^{\pi_3} A_2(E_3, E_1, A_2). \tag{41}$$

The equality $A_1\theta = A_3$ implies $A_1\theta^2 = E_3$ so, using (40) in $A_1\theta^2 = E_3$, we get ${}^{\pi_2}A_2(A_3, A_2, A_1) = E_3$. Now, from (40), (41) and the last equality, we obtain

$$^{\pi_2}A_2(^{\pi_3}A_2(E_3, E_1, A_2), A_2, ^{\pi_2}A_2(E_2, A_2, E_1)) = E_3. \tag{42}$$

Conversely, if (40), (41) and (42) hold, then from (40) it follows $A_2\theta = A_2$. The equality (41) implies $A_3\theta =^{\pi_3} A_2(E_1, E_2, A_2)$, so $A_3\theta = E_3$. Using (40) and (41) in (42), we get ${}^{\pi_2}A_2(A_3, A_2, A_1) = E_3$, which implies $A_1 = {}^{\pi_3}A_2(A_3, E_3, A_2)$, so $A_1\theta =^{\pi_3} A_2(E_3, E_1, A_2)$. Using (41) in the last equality, we obtain $A_1\theta = A_3$.

4. If $A_1\theta = A_3, A_2\theta = E_3, A_3\theta = A_2$, then $\theta^2 = (A_1, A_3, E_2), \theta^3 = (A_3, A_2, A_1)$, $\theta^4 = (A_2, E_3, A_3)$, $\theta^5 = (E_3, E_1, A_2)$, $\theta^6 = \varepsilon$. From $A_2\theta = E_3$ it follows

$$A_1 =^{(132)\pi_2} A_2. \tag{43}$$

The equality $A_3\theta = A_2$ implies $A_3\theta^6 = A_2\theta^5$, so

$$A_3 = A_2(E_3, E_1, A_2). \tag{44}$$

From $A_1\theta = A_3$ it follows $A_1\theta^6 = A_2\theta^4$, i.e. $A_1 = A_2(A_2, E_3, A_3)$, using (43) and (44) in the last equality, we get

$$A_2(A_2, E_3, A_2(E_3, E_1, A_2)) =^{(132)\pi_2} A_2. \tag{45}$$

Conversely, if (43), (44) and (45) hold, then from (43) it follows $A_2\theta = E_3$. The equality (44) implies $A_3\theta = A_2$. Using (43) and (44) in (45), we obtain $A_1 = A_2(A_2, E_3, A_3)$, which implies $A_1\theta = A_2(E_3, E_1, A_2)$ and using (44) in the last equality, we get $A_1\theta = A_3$.

5. If $A_1\theta = E_3, A_2\theta = A_2, A_3\theta = A_3$, then from $A_1\theta = E_2$ it follows

$$A_1 = {}^{(132)\pi_2}A_1. \tag{46}$$

The equality $A_2\theta = A_2$ implies

$$A_1 ={}^{\pi_2} A_2(E_2, A_2, E_1). \tag{47}$$

From $A_3\theta = A_3$ it follows

$$A_1 ={}^{\pi_2} A_3(E_2, A_3, E_1). \tag{48}$$

Conversely, if (46), (47) and (48) hold, then from (46) and (47) it follows $A_1\theta = E_3$ and $A_2\theta = A_2$, respectively, and (48) implies $A_3\theta = A_3$.

6. If $A_1\theta = E_3, A_2\theta = A_3, A_3\theta = A_2$, then $\theta^2 = (A_1, E_3, E_2), \theta^3 = (E_3, E_1, A_1)$, $\theta^4 = \varepsilon$. From $A_1\theta = E_3$ it follows

$$A_1 = {}^{(132)\pi_2}A_1. \tag{49}$$

From $A_2\theta = A_3$ it follows $A_2\theta^2 = A_2$, i.e. $A_2(A_1, E_3, E_2) = A_2$, so

$$A_1 ={}^{\pi_1} A_2(A_2, E_3, E_2). \tag{50}$$

The equality $A_3\theta = A_2$ implies $A_3\theta^4 = A_2\theta^3$, so $A_3 = A_2(E_3, E_1, A_1)$. Using (50) in the last equality, we get

$$A_3 = A_2(E_3, E_1, {}^{\pi_1} A_2(A_2, E_3, E_2)). \tag{51}$$

Conversely, if (49), (50) and (51) hold, then (49) implies $A_1\theta = E_3$, therefore $\theta^2 = (A_1, E_3, E_2)$, $\theta^3 = (E_3, E_1, A_1)$ and $\theta^4 = \varepsilon$. From (50) it follows $A_2(A_1, E_3, E_2) = A_2$, so $A_2\theta^2 = A_2$, which implies $A_2\theta^3 = A_2\theta$. Using (50) in (51), we obtain $A_3 = A_2(E_3, E_1, A_1)$, so $A_3 = A_2\theta^3$. From $A_2\theta^3 = A_2\theta$. and $A_3 = A_2\theta^3$ it follows $A_2\theta = A_3$. The equality $A_3 = A_2\theta^3$ also implies $A_3\theta = A_2$. □

Lemma 2.5. *The triplet (A_1, E_1, E_2) is a paratopy of the system Σ if and only if one of the following conditions holds:*

1. $A_3 ={}^{(132)\pi_1} A_1, A_2 = A_1(A_1, E_1, E_2)$ *and*
$A_1(E_3, {}^{(132)\pi_1} A_1, A_1(A_1, E_1, E_2)) = E_2$;

2. $A_1 ={}^{\pi_1} A_3(A_3, E_1, E_2), A_2 ={}^{\pi_3} A_3(E_2, E_3, A_3)$ *and*
$A_3(E_3, {}^{\pi_3} A_3(E_2, E_3, A_3), {}^{\pi_1} A_3(A_3, E_1, E_2)) = A_3$;

3. $A_1 ={}^{\pi_1} A_2(A_2, E_1, E_2), A_3 ={}^{\pi_3} A_2(E_2, E_3, A_2)$ *and*
$A_2(E_3, {}^{\pi_3} A_2(E_2, E_3, A_2), {}^{\pi_1} A_2(A_2, E_1, E_2)) = A_2$;

4. $A_2 ={}^{(132)\pi_3} A_1, A_3 = A_1(A_1, E_1, E_2)$ *and*
$A_1(E_3, {}^{(132)\pi_3} A_1, A_1(A_1, E_1, E_2)) = E_2$;

5. $A_1 ={}^{\pi_1} A_2(A_2, E_1, E_2) ={}^{\pi_1} A_3(A_3, E_1, E_2)$ *and* $A_1 = {}^{(123)\pi_1}A_1$;

6. $A_1 = {}^{\pi_2}A_2(E_3, A_2, E_1), A_3 = A_2({}^{\pi_2}A_2(E_3, A_2, E_1), E_1, E_2)$ *and* $A_1 = {}^{(123)\pi_1}A_1$.

The proof is analogous to that of Lemma 2.4.

Lemma 2.6. *The triplet* (A_1, E_2, E_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*

1. $A_2 = A_1(A_1, E_2, E_1), A_3 =^{(13)\pi_3} A_1$ *and*
$A_1(^{(13)\pi_3}A_1, E_2, A_1(A_1, E_2, E_1)) = E_1$;

2. $A_1 =^{\pi_1} A_3(A_3, E_2, E_1), A_2 =^{\pi_3} A_3(E_3, E_2, A_3)$ *and*
$A_3(^{\pi_3}A_3(E_3, E_2, A_3), E_2, ^{\pi_1}A_3(A_3, E_2, E_1)) = A_3$;

3. $A_1 =^{\pi_1} A_2(A_2, E_2, E_1), A_3 =^{\pi_3} A_2(E_3, E_2, A_2)$ *and*
$A_2(^{\pi_3}A_2(E_3, E_2, A_2), E_2, ^{\pi_1}A_2(A_2, E_2, E_1)) = A_2$;

4. $A_1 =^{(13)\pi_1} A_1 =^{\pi_1} A_2(A_2, E_2, E_1) =^{\pi_1} A_3(A_3, E_2, E_1)$;

5. $A_3 = A_1(A_1, E_2, E_1), A_2 =^{(13)\pi_3} A_1$ *and*
$A_1(^{(13)\pi_3}A_1, E_2, A_1(A_1, E_2, E_1)) = E_3$.

The proof is analogous to that of Lemma 2.3.

Lemma 2.7. *The triplet* (E_1, E_3, A_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*

1. $A_2 = A_1(E_1, E_3, A_1), A_3 =^{(23)\pi_2} A_1$ *and*
$A_1(E_1, A_1(E_1, E_3, A_1), ^{(23)\pi_2}A_1) = E_2$;

2. $A_1 = {}^{\pi_3}A_3(E_1, E_3, A_3), A_2 = {}^{\pi_2}A_3(E_1, A_3, E_2)$ *and*
$A_3(E_1, ^{\pi_3}A_3(E_1, E_3, A_3), ^{\pi_2}A_3(E_1, A_3, E_2)) = A_3$;

3. $A_1 = {}^{\pi_3}A_2(E_1, E_3, A_2), A_3 = {}^{\pi_2}A_2(E_1, A_2, E_2)$ *and*
$A_2(E_1, ^{\pi_3}A_2(E_1, E_3, A_2), ^{\pi_2}A_2(E_1, A_2, E_2)) = A_2$;

4. $A_3 = A_1(E_1, E_3, A_1), A_2 =^{(23)\pi_2} A_1$ *and*
$A_1(E_1, A_1(E_1, E_3, A_1), ^{(23)\pi_2}A_1) = E_2$;

5. $A_1 = {}^{\pi_3}A_2(E_1, E_3, A_2) = {}^{\pi_3}A_3(E_1, E_3, A_3)$ *and* $A_1 = {}^{(23)\pi_3}A_1$.

The proof is analogous to that of Lemma 2.3.

Lemma 2.8. *The triplet* (E_3, E_1, A_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*

1. $A_1 =^{(123)\pi_3} A_3, A_2 = A_3(E_2, A_3, E_1)$ *and*
$A_3(A_3, A_3(E_2, A_3, E_1), E_2) =^{(123)\pi_3} A_3$;

2. $A_1 = {}^{\pi_3}A_3(E_3, E_1, A_3), A_2 = {}^{\pi_2}A_3(E_2, A_3, E_1)$ *and*
$A_3(^{\pi_2}A_3(E_2, A_3, E_1), ^{\pi_3}A_3(E_3, E_1, A_3), E_2) = A_3$;

3. $A_1 =^{\pi_3} A_2(E_3, E_1, A_2), A_3 =^{\pi_2} A_2(E_2, A_2, E_1)$ *and*
$A_2(^{\pi_2}A_2(E_2, A_2, E_1), ^{\pi_3}A_2(E_3, E_1, A_2), E_2) = A_2$;

4. $A_1 =^{(123)\pi_3} A_2, A_3 = A_2(E_2, A_2, E_1)$ *and*
$A_2(A_2, A_2(E_2, A_2, E_1), E_2) = {}^{(123)\pi_3}A_2$;

5. $A_1(E_3, E_1, A_1) = E_2, A_1 =^{\pi_3} A_2(E_3, E_1, A_2) =^{\pi_3} A_3(E_3, E_1, A_3)$;

6. $A_1 =^{\pi_1} A_2(A_2, E_3, E_2), A_3 = A_2(E_2, ^{\pi_1}A_2(A_2, E_3, E_2), E_1)$ *and*
$A_1 = {}^{(123)\pi_3}A_1$.

The proof is analogous to that of Lemma 2.4.

Lemma 2.9. *The triplet* (E_1, A_1, E_3) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(E_1, A_1, E_3), A_3 = {}^{\pi_2}A_1$ *and*
$A_1(E_1, A_1(E_1, A_1, E_3), E_3) = {}^{\pi_2}A_1$;
2. $A_3 = A_1(E_1, A_1, E_3), A_2 = {}^{\pi_2}A_1$ *and*
$A_1(E_1, A_1(E_1, A_1, E_3), E_3) = {}^{\pi_2}A_1$;
3. $A_1 = {}^{\pi_2}A_2(E_1, A_3, E_3) = {}^{\pi_2}A_3(E_1, A_2, E_3)$.

The proof is analogous to that of Lemma 2.1.

Lemma 2.10. *The triplet* (E_3, A_1, E_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_3 = {}^{(13)\pi_2} A_1, A_2 = A_1(E_3, A_1, E_1)$ *and*
$A_1(E_1, A_1(E_3, A_1, E_1), E_3) = {}^{(13)\pi_2}A_1$;
2. $A_3 = A_1(E_3, A_1, E_1), A_2 = {}^{(13)\pi_2} A_1$ *and*
$A_1(E_1, A_1(E_3, A_1, E_1), E_3) = {}^{(13)\pi_2}A_1$;
3. $A_1 = {}^{\pi_2} A_2(E_3, A_2, E_1) = {}^{\pi_2}A_3(E_3, A_3, E_1)$;
4. $A_1(E_3, A_1, E_1) = E_2, A_3 = A_2(E_3, A_1, E_1)$.

The proof is analogous to that of Lemma 2.2.

Lemma 2.11. *The triplet* (A_1, E_1, E_3) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(A_1, E_1, E_3), A_3 = {}^{(12)\pi_2}A_1$ *and*
$A_1({}^{(12)\pi_2}A_1, A_1(A_1, E_1, E_3), E_3) = E_2$;
2. $A_1 = {}^{\pi_1}A_3(A_3, E_1, E_3), A_2 = {}^{\pi_2}A_3(E_2, A_3, E_3)$ *and*
$A_3({}^{\pi_2}A_3(E_2, A_3, E_3), {}^{\pi_1}A_3(A_3, E_1, E_3), E_3) = A_3$;
3. $A_1 = {}^{\pi_1}A_2(A_2, E_1, E_3), A_3 = {}^{\pi_2}A_2(E_2, A_2, E_3)$ *and*
$A_2({}^{\pi_2}A_2(E_2, A_2, E_3), {}^{\pi_1}A_2(A_2, E_1, E_3), E_3) = A_2$;
4. $A_3 = A_1(A_1, E_1, E_3), A_2 = {}^{(12)\pi_2}A_1$ *and*
$A_1({}^{(12)\pi_2}A_1, A_1(A_1, E_1, E_3), E_3) = E_2$;
5. $A_1 = {}^{\pi_1}A_2(A_2, E_1, E_3) = {}^{\pi_1}A_3(A_3, E_1, E_3)$ *and* $A_1 = {}^{(12)\pi_1}A_1$.

The proof is analogous to that of Lemma 2.3.

Lemma 2.12. *The triplet* (A_1, E_3, E_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(A_1, E_3, E_1), A_3 = {}^{(123)\pi_2} A_1$ *and*
$A_1(E_2, A_1(A_1, E_3, E_1), {}^{(123)\pi_2} A_1) = E_3$;
2. $A_1 = {}^{\pi_1} A_3(A_3, E_3, E_1), A_2 = {}^{\pi_2} A_3(E_3, A_3, E_2)$ *and*
$A_3(E_2, {}^{\pi_1} A_3(A_3, E_3, E_1), {}^{\pi_2} A_3(E_3, A_3, E_2)) = A_3$;
3. $A_1 = {}^{\pi_1} A_2(A_2, E_3, E_1), A_3 = {}^{\pi_2} A_2(E_3, A_2, E_2)$ *and*
$A_2(E_2, {}^{\pi_1} A_2(A_2, E_3, E_1), {}^{\pi_2} A_2(E_3, A_2, E_2)) = A_2$;
4. $A_2 = {}^{(123)\pi_2} A_1, A_3 = A_1(A_1, E_3, E_1)$ *and*
$A_1(E_2, A_1(A_1, E_3, E_1), {}^{(123)\pi_2} A_1) = E_3$;
5. $A_1 = {}^{\pi_1} A_2(A_2, E_3, E_1) = {}^{\pi_1} A_3(A_3, E_3, E_1)$ *and* $A_1 = {}^{(132)\pi_1}A_1$;
6. $A_1 = {}^{\pi_3} A_2(E_2, E_1, A_2), A_3 = A_2(E_3, {}^{\pi_3} A_2(E_2, E_1, A_2), E_2)$ *and*

$A_1 = {}^{(132)\pi_1}A_1$.

The proof is similar to the proof of Lemma 2.4.

Lemma 2.13. *The triplet* (E_2, E_3, A_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(E_2, E_3, A_1), A_3 = {}^{(123)\pi_1}A_1$ *and*
$A_1(A_1(E_2, E_3, A_1), {}^{(123)\pi_1}A_1, E_1) = E_2$;
2. $A_1 = {}^{\pi_3}A_3(E_2, E_3, A_3), A_2 = {}^{\pi_1}A_3(A_3, E_1, E_2)$ *and*
$A_3({}^{\pi_3}A_3(E_2, E_3, A_3), {}^{\pi_1}A_3(A_3, E_1, E_2), E_1) = A_3$;
3. $A_1 = {}^{\pi_3}A_2(E_2, E_3, A_2), A_3 = {}^{\pi_1}A_2(A_2, E_1, E_2)$ *and*
$A_2({}^{\pi_3}A_2(E_2, E_3, A_2), {}^{\pi_1}A_2(A_2, E_1, E_2), E_1) = A_2$;
4. $A_3 = A_1(E_2, E_3, A_1), A_2 = {}^{(123)\pi_1}A_1$ *and*
$A_1(A_1(E_2, E_3, A_1), {}^{(123)\pi_1}A_1, E_1) = E_2$;
5. $A_1 = {}^{\pi_3}A_2(E_2, E_3, A_2) = {}^{\pi_3}A_3(E_2, E_3, A_3)$ *and* $A_1 = {}^{(132)\pi_3}A_1$;
6. $A_1 = {}^{\pi_2}A_2(E_3, A_2, E_1), A_3 = A_2(E_2, E_3, {}^{\pi_2}A_2(E_3, A_2, E_1))$ *and*
$A_1 = {}^{(132)\pi_3}A_1$.

The proof is similar to the proof of Lemma **??**.

Lemma 2.14. *The triplet* (E_3, E_2, A_1) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(E_3, E_2, A_1), A_3 = {}^{(123)\pi_1} A_1$ *and*
$A_1(A_1(E_3, E_2, A_1), E_2, {}^{(123)\pi_1} A_1) = E_1$;
2. $A_1 = {}^{\pi_3} A_3(E_3, E_2, A_3), A_2 = {}^{\pi_1} A_3(A_3, E_2, E_1)$ *and*
$A_3({}^{\pi_3}A_3(E_3, E_2, A_3), E_2, {}^{\pi_1} A_3(A_3, E_2, E_1)) = A_3$;
3. $A_1 = {}^{\pi_3} A_2(E_3, E_2, A_2), A_3 = {}^{\pi_1} A_2(A_2, E_2, E_1)$ *and*
$A_2({}^{\pi_3}A_2(E_3, E_2, A_2), E_2, {}^{\pi_1} A_2(A_2, E_2, E_1)) = A_2$;
4. $A_1 = {}^{(123)\pi_3} A_2, A_3 = A_2(A_2, E_2, E_1)$ *and*
$A_2(A_2(A_2, E_2, E_1)) = {}^{(123)\pi_3} A_2$;
5. $A_1(E_3, E_2, A_1) = E_1, A_1 = {}^{\pi_3} A_2(E_3, E_2, A_2) = {}^{\pi_3} A_3(E_3, E_2, A_3)$.

The proof is similar to the proof of Lemma 2.3.

Lemma 2.15. *The triplet* (E_2, A_1, E_3) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(E_2, A_1, E_3), A_3 = {}^{(12)\pi_1}A_1$ *and*
$A_1(A_1(E_2, A_1, E_3), {}^{(12)\pi_1}A_1, E_3) = E_1$;
2. $A_1 = {}^{\pi_2}A_3(E_2, A_3, E_3), A_2 = {}^{\pi_1}A_3(A_3, E_1, E_3)$ *and*
$A_3({}^{\pi_2}A_3(E_2, A_3, E_3), {}^{\pi_1}A_3(A_3, E_1, E_3), E_3) = A_3$;
3. $A_1 = {}^{\pi_1}A_2(E_2, A_2, E_3), A_3 = {}^{\pi_1}A_2(A_2, E_1, E_3)$ *and*
$A_2({}^{\pi_2}A_2(E_2, A_2, E_3), {}^{\pi_1}A_2(A_2, E_1, E_3), E_3) = A_2$;
4. $A_3 = A_1(E_2, A_1, E_3), A_2 = {}^{(12)\pi_1}A_1$ *and*
$A_1(A_1(E_2, A_1, E_3), {}^{(12)\pi_1}A_1, E_3) = E_1$;
5. $A_1 = {}^{\pi_2}A_2(E_2, A_2, E_3) = {}^{\pi_2}A_3(E_2, A_3, E_3)$ *and* $A_1 = {}^{(12)\pi_2}A_1$.

The proof is similar to the proof of Lemma 2.3.

Lemma 2.16. *The triplet* (E_3, A_1, E_2) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_3 =^{(132)\pi_1} A_1, A_2 = A_1(E_3, A_1, E_2)$ *and*
$A_1(A_1(E_3, A_1, E_2), E_1,^{(132)\pi_1} A_1) = E_3$;
2. $A_1 =^{\pi_2} A_3(E_3, A_3, E_2), A_2 =^{\pi_1} A_3(A_3, E_3, E_1)$ *and*
$A_3(^{\pi_2}A_3(E_3, A_3, E_2), E_1,^{\pi_1} A_3(A_3, E_3, E_1)) = A_3$;
3. $A_1 =^{\pi_2} A_2(E_3, A_2, E_2), A_3 =^{\pi_1} A_2(A_2, E_3, E_1)$ *and*
$A_2(^{\pi_2}A_2(E_3, A_2, E_2), E_1,^{\pi_1} A_2(A_2, E_3, E_1)) = A_2$;
4. $A_2 =^{(132)\pi_1} A_1, A_3 = A_1(E_3, A_1, E_2)$ *and*
$A_1(A_1(E_3, A_1, E_2), E_1,^{(132)\pi_1} A_1) = E_3$;
5. $A_1 =^{\pi_2} A_2(E_3, A_2, E_2) =^{\pi_2} A_3(E_3, A_3, E_2)$ *and* $A_1 = {}^{(132)\pi_2}A_1$;
6. $A_1 =^{\pi_3} A_2(E_2, E_1, A_2)$, $A_3 = A_2(^{\pi_3}A_2(E_2, E_1, A_2), E_3, E_1)$ *and*
$A_1 = {}^{(132)\pi_2}A_1$.

The proof is similar to the proof of Lemma 2.4.

Lemma 2.17. *The triplet* (A_1, E_2, E_3) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_2 = A_1(A_1, E_2, E_3), A_3 = {}^{\pi_1}A_1$ *and*
$A_1(A_1(A_1, E_2, E_3), E_2, E_3) = {}^{\pi_1}A_1$;
2. $A_3 = A_1(A_1, E_2, E_3), A_2 = {}^{\pi_1}A_1$ *and*
$A_1(A_1(A_1, E_2, E_3), E_2, E_3) = {}^{\pi_1}A_1$;
3. $A_1 = {}^{\pi_1}A_2(A_3, E_2, E_3) = {}^{\pi_1}A_3(A_2, E_2, E_3)$.

The proof is similar to the proof of Lemma 2.1.

Lemma 2.18. *The triplet* (A_1, E_3, E_2) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_3 =^{(23)\pi_1} A_1, A_2 = A_1(A_1, E_3, E_2)$ *and*
$A_1(A_1(A_1, E_3, E_2), E_2, E_3) = {}^{(23)\pi_1}A_1$;
2. $A_3 = A_1(A_1, E_3, E_2), A_2 =^{(23)\pi_1} A_1$ *and*
$A_1(A_1(A_1, E_3, E_2), E_2, E_3) = {}^{(23)\pi_1}A_1$;
3. $A_1 =^{\pi_1} A_2(A_2, E_3, E_2) =^{\pi_1} A_3(A_3, E_3, E_2)$;
4. $A_3 = A_2(A_1, E_3, E_2)$ *and* $A_1(A_1, E_3, E_2) = E_1$.

The proof is similar to the proof of Lemma 2.2.

From Lemmas $2.1 - 2.18$ we get the following theorem.

Theorem 1. *There exist precisely* 87 *orthogonal systems consisting of three ternary quasigroup operations and the ternary selectors, that admit at least one nontrivial paratopy, which components are two ternary selectors and a ternary quasigroup operation.*

3. Paratopies consisting of three ternary selectors

In the third section it is shown that there exist precisely 18 orthogonal systems $\Sigma = \{A_1, A_2, A_3, E_1, E_2, E_3\}$, consisting of three ternary quasigroup operations A_1, A_2, A_3 and the ternary selectors, which admit at least one nontrivial paratopy, which components are three ternary selectors.

Lemma 3.1. *The triplet* (E_1, E_3, E_2) *is a paratopy of the system* Σ *if and only if one of the following conditions holds:*
1. $A_1 = {}^{(23)}A_1,\ A_2 = {}^{(23)}A_2,\ A_3 = {}^{(23)}A_3$;
2. $A_3 = {}^{(23)}A_2,\ A_1 = {}^{(23)}A_1$;
3. $A_2 = {}^{(23)}A_1,\ A_3 = {}^{(23)}A_3$;
4. $A_3 = {}^{(23)}A_1,\ A_2 = {}^{(23)}A_2$.

Proof. Let the triplet (E_1, E_3, E_2) be a paratopy of the system Σ. As $E_1\theta = E_1, E_2\theta = E_3, E_3\theta = E_2$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_1, E_2, E_3\}$, that is $\{A_1\theta, A_2\theta, A_3\theta\} = \{A_1, A_2, A_3\}$.

1. If $A_1\theta = A_1, A_2\theta = A_2, A_3\theta = A_3$, then $A_1\theta = A_1$ implies

$$A_1 = {}^{(23)}A_1. \tag{52}$$

From $A_2\theta = A_2$ it follows

$$A_2 = {}^{(23)}A_2. \tag{53}$$

The equality $A_3\theta = A_3$ implies

$$A_3 = {}^{(23)}A_3. \tag{54}$$

Conversely, if (52), (53) and (54) hold, then (52) implies $A_1\theta = A_1$, from (53) it follows $A_2\theta = A_2$ and (54) implies $A_3\theta = A_3$.

2. If $A_1\theta = A_1, A_2\theta = A_3, A_3\theta = A_2$, then $A_1\theta = A_1$ implies

$$A_1 = {}^{(23)}A_1. \tag{55}$$

From $A_2\theta = A_3$ it follows

$$A_3 = {}^{(23)}A_2. \tag{56}$$

Conversely, if (55) and (56) hold, then (55) implies $A_1\theta = A_1$ and from (56) it follows $A_2\theta = A_3$. Also, (56) implies $A_3\theta = A_2$.

3. If $A_1\theta = A_2, A_2\theta = A_1, A_3\theta = A_3$, then $A_1\theta = A_2$ implies

$$A_2 = {}^{(23)}A_1. \tag{57}$$

From $A_3\theta = A_3$ it follows

$$A_3 = {}^{(23)}A_3. \tag{58}$$

Conversely, if (57) and (58) hold, then (57) implies $A_1\theta = A_2$ and from (58) it follows $A_3\theta = A_3$. Also, (57) implies $A_2\theta = A_1$.

4. If $A_1\theta = A_2, A_2\theta = A_3, A_3\theta = A_1$, then $\theta^2 = \varepsilon$. The equalities $A_1\theta = A_2$ and $A_2\theta = A_3$ imply $A_1 = A_1\theta^2 = A_2\theta = A_3$, which is a contradiction, as Σ is an orthogonal system of quasigroups.

5. If $A_1\theta = A_3, A_2\theta = A_2, A_3\theta = A_1$, then $A_2\theta = A_2$ implies

$$A_2 = {}^{(23)}A_2. \tag{59}$$

From $A_1\theta = A_3$ it follows

$$A_3 = {}^{(23)}A_1. \tag{60}$$

Conversely, if (59) and (60) hold, then (59) implies $A_2\theta = A_2$ and from (60) it follows $A_1\theta = A_3$. Also, (59) implies $A_3\theta = A_1$.

6. If $A_1\theta = A_3, A_2\theta = A_1, A_3\theta = A_2$, then $\theta^2 = \varepsilon$. The equalities $A_1\theta = A_3$ and $A_3\theta = A_2$ imply $A_1 = A_1\theta^2 = A_3\theta = A_2$, which is a contradiction, as Σ is an orthogonal system of quasigroups. □

Lemma 3.2. *The triplet (E_2, E_1, E_3) is a paratopy of the system Σ if and only if one of the following conditions holds:*
1. $A_1 = {}^{(12)}A_1,\ A_2 = {}^{(12)}A_2,\ A_3 = {}^{(12)}A_3$;
2. $A_3 = {}^{(12)}A_2,\ A_1 = {}^{(12)}A_1$;
3. $A_2 = {}^{(12)}A_1,\ A_3 = {}^{(12)}A_3$;
4. $A_3 = {}^{(12)}A_1,\ A_2 = {}^{(12)}A_2$.

The proof is similar to the proof of Lemma 3.1.

Lemma 3.3. *The triplet (E_3, E_2, E_1) is a paratopy of the system Σ if and only if one of the following conditions holds:*
1. $A_1 = {}^{(13)}A_1,\ A_2 = {}^{(13)}A_2,\ A_3 = {}^{(13)}A_3$;
2. $A_3 = {}^{(13)}A_2,\ A_1 = {}^{(13)}A_1$;
3. $A_2 = {}^{(13)}A_1,\ A_3 = {}^{(13)}A_3$;
4. $A_3 = {}^{(13)}A_1,\ A_2 = {}^{(13)}A_2$.

The proof is similar to the proof of Lemma 3.1.

Lemma 3.4. *The triplet (E_2, E_3, E_1) is a paratopy of the system Σ if and only if one of the following conditions holds:*
1. $A_1 = {}^{(132)}A_1,\ A_2 = {}^{(132)}A_2,\ A_3 = {}^{(132)}A_3$;
2. $A_2 = {}^{(132)}A_1,\ A_3 = {}^{(123)}A_1$;
3. $A_3 = {}^{(132)}A_1,\ A_2 = {}^{(123)}A_1$.

Proof. Let the triplet (E_2, E_3, E_1) be a paratopy of the system Σ. As $E_1\theta = E_2, E_2\theta = E_3, E_3\theta = E_1$, we obtain $\Sigma\theta = \{A_1\theta, A_2\theta, A_3\theta, E_1, E_2, E_3\}$, that is $\{A_1\theta, A_2\theta, A_3\theta\} = \{A_1, A_2, A_3\}$.

1. If $A_1\theta = A_1, A_2\theta = A_2, A_3\theta = A_3$, then then $A_1\theta = A_1$ implies

$$A_1 = {}^{(132)}A_1. \tag{61}$$

From $A_2\theta = A_2$ it follows

$$A_2 = {}^{(132)}A_2. \tag{62}$$

The equality $A_3\theta = A_3$ implies

$$A_3 = {}^{(132)}A_3. \tag{63}$$

Conversely, if (61), (62) and (63) hold, then (61) implies $A_1\theta = A_1$, from (61) it follows $A_2\theta = A_2$ and (61) implies $A_3\theta = A_3$.

2. If $A_1\theta = A_1, A_2\theta = A_3, A_3\theta = A_2$, then $\theta^2 = (E_3, E_1, E_2), \theta^3 = \varepsilon$. The equalities $A_2\theta = A_3$ and $A_3\theta = A_2$ imply $A_2 = A_2\theta^3 = A_3\theta^2 = A_2\theta = A_3$, which is a contradiction, as Σ is an orthogonal system of quasigroup.

3. If $A_1\theta = A_2, A_2\theta = A_1, A_3\theta = A_3$, then $\theta^2 = (E_3, E_1, E_2), \theta^3 = \varepsilon$. The equalities $A_1\theta = A_2$ and $A_2\theta = A_1$ imply $A_1 = A_1\theta^3 = A_2\theta^2 = A_1\theta = A_2$, which is a contradiction, as Σ is an orthogonal system of quasigroup.

4. If $A_1\theta = A_2, A_2\theta = A_3, A_3\theta = A_1$, then $A_1\theta = A_2$ implies

$$A_2 = {}^{(132)}A_1. \tag{64}$$

From $A_2\theta = A_3$ it follows $A_3 = {}^{(132)}A_2$. Using (64) in the last equality, we get

$$A_3 = {}^{(123)}A_1. \tag{65}$$

Conversely, if (64) and (65) hold, then (64) implies $A_1\theta = A_2$ and from (65) it follows $A_3\theta = A_1$. Also, (64) implies $A_2\theta = {}^{(123)}A_1$. Using (65) in the last equality, we get $A_2\theta = A_3$.

5. If $A_1\theta = A_3, A_2\theta = A_1, A_3\theta = A_2$, then $A_1\theta = A_3$ implies

$$A_3 = {}^{(132)}A_1. \tag{66}$$

From $A_3\theta = A_2$ it follows $A_2 = {}^{(132)}A_3$. Using (66) in the last equality, we get

$$A_2 = {}^{(123)}A_1. \tag{67}$$

Conversely, if (66) and (67) hold, then (66) implies $A_1\theta = A_3$ and from (67) it follows $A_2\theta = A_1$. Also, (66) implies $A_3\theta = {}^{(123)}A_1$. Using (67) in the last equality, we get $A_3\theta = A_2$.

6. If $A_1\theta = A_3, A_2\theta = A_2, A_3\theta = A_1$, then $\theta^2 = (E_3, E_1, E_2), \theta^3 = \varepsilon$. The equalities $A_1\theta = A_3$ and $A_3\theta = A_1$ imply $A_1 = A_1\theta^3 = A_3\theta^2 = A_1\theta = A_3$, which is a contradiction, as Σ is an orthogonal system of quasigroup. □

Lemma 3.5. *The triplet (E_3, E_1, E_2) is a paratopy of the system Σ if and only if one of the following conditions holds:*
1. $A_1 = {}^{(123)}A_1,\ A_2 = {}^{(123)}A_2,\ A_3 = {}^{(123)}A_3$;
2. $A_2 = {}^{(123)}A_1,\ A_3 = {}^{(132)}A_1$;
3. $A_3 = {}^{(123)}A_1,\ A_2 = {}^{(132)}A_1$.

The proof is similar to the proof of Lemma 3.4.

From Lemmas 3.1 − 3.5 we obtain the following theorem.

Theorem 2. *There exist precisely* 18 *orthogonal systems, consisting of three ternary quasigroup operations and the ternary selectors, that admit at least one nontrivial paratopy, which components are three ternary selectors.*

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