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D-loops

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Abstract. D-loops are loops with the antiautomorphic inverse property. The class of such loops is larger than the class of IP-loops. The smallest D-loops which is not an IP-loop has six elements. We prove several basic properties of such loops and present methods of constructions of D-loops from IP-loops. Unfortunately, a loop isotopic to a D-loop may not be a D-loop.

1. Introduction

A loop is a quasigroup $Q(\circ)$ with an identity element always denoted by 1. A loop $Q(\circ)$ has the *inverse property*, i.e., it is an *IP-loop*, if for each its element a there exists in Q a uniquely determined *inverse element* a' such that $a'\circ(a\circ b) = (b\circ a)\circ a'$. This means that in an *IP*-loop for *right* and *left translations*, i.e., for $R_a(x) = x\circ a$, $L_a(x) = a\circ x$, we have

$$R_a^{-1} = R_{a'}, \qquad L_a^{-1} = L_{a'}. \tag{1}$$

It is not difficult to shown that in an IP-loop $Q(\circ)$ for all $a, b \in Q$ hold

$$a\circ a' = a'\circ a = 1, \quad (a')' = a \tag{2}$$

and

$$(a\circ b)' = b'\circ a' \,. \tag{3}$$

On the other hand, in any loop $Q(\circ)$ for each $a \in Q$ there are uniquely determined *left* and *right loop-inverse* elements $a_L^{-1}, a_R^{-1} \in Q$ for which we have $a_L^{-1}\circ a = a\circ a_R^{-1} = 1$. A two-sided loop-inverse element to $a \in Q$ is denoted by a^{-1}. Clearly, $(a^{-1})^{-1} = a$. Hence, an element $a^{-1} \in Q$ is loop-inverse to $a \in Q$ if and only if $a \in Q$ is loop-inverse to a^{-1}. In a loop each inverse element is loop-inverse but a loop-inverse element may not be inverse.

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Example 1.1. Consider the following loop $Q(\circ)$:

$\circ$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	6	7	5	4
3	3	1	2	7	6	4	5
4	4	7	6	5	1	3	2
5	5	6	7	1	4	2	3
6	6	4	5	2	3	7	1
7	7	5	4	3	2	1	6

In this loop we have $a_L^{-1} = a_R^{-1}$ for each $a \in Q$. Hence, each element of this loop is loop-inverse. But $a = 5$ is not an inverse element since $4 \circ (5 \circ 6) \neq 6$. The map $h(x) = x_R^{-1}$ is an antiautomorphism of this loop, i.e., it satisfies the identity $h(x \circ y) = h(y) \circ h(x)$. □

Recall that a loop $Q(\circ)$ satisfies the *antiautomorphic inverse property* if for each $x \in Q$ there exists a two-sided loop-inverse element x^{-1} such that $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$ holds for all $x, y \in Q$. A simple example of such loop is an IP-loop. The above example proves that there are also loops with this property which are not IP-loops. Thus, the class of loops with this property is much larger than the class of IP-loops.

This enables us to introduce the following definition.

Definition 1.2. A loop $Q(\circ)$ is called a *D-loop* if it satisfies the *dual automorphic property* for $\varphi(x) = x_R^{-1}$, i.e., if

$$(x \circ y)_R^{-1} = y_R^{-1} \circ x_R^{-1} \tag{4}$$

holds for all $x, y \in Q$.

Theorem 1.3. *A loop $Q(\circ)$ is a D-loop if and only if it satisfies the identity*

$$(x \circ y)_L^{-1} = y_L^{-1} \circ x_L^{-1}. \tag{5}$$

Proof. Suppose that $Q(\circ)$ is a D-loop. Since $x_L^{-1} \circ x = 1$, from (4) it follows

$$1 = 1_R^{-1} = (x_L^{-1} \circ x)_R^{-1} = x_R^{-1} \circ (x_L^{-1})_R^{-1},$$

which together with $1 = x_L^{-1} \circ (x_L^{-1})_R^{-1}$ gives $x_L^{-1} = x_R^{-1}$. Thus (4) implies (5).

Analogously, using $x \circ x_R^{-1} = 1$ and $(x_R^{-1})_L^{-1} \circ x_R = 1$ we can prove that (5) implies (4). □

Corollary 1.4. *For all elements of D-loops we have*

$$a_L^{-1} = a_R^{-1} \quad \textit{and} \quad (a^{-1})^{-1} = a. \qquad \square$$

This means that in the multiplication table of a D-loop $Q(\circ)$ its neutral element is located symmetrically with respect to the main diagonal and the class of all D-loops coincides with the class of loops with the antiautomorphic inverse property but we'll keep the term *D-loop* since it is shorter and more convenient to use.

2. Constructions of D-loops

Below we present several methods of verification when a given loop is a D-loop. To describe these methods we must reminder some definitions from [2], [5] and [6].

Definition 2.1. Let $Q(\cdot)$ be a loop. A permutation φ_a of Q, where $a \in Q$, is called a *right middle translation* or a *right track* (shortly: *track*) of $Q(\cdot)$ if

$$x \cdot \varphi_a(x) = a \tag{6}$$

holds for all $x \in Q$. By a *left middle translation* or a *left track* we mean a permutation λ_a such that

$$\lambda_a(x) \cdot x = a. \tag{7}$$

It is clear that $\lambda_a = \varphi_a^{-1}$ and $\varphi_1(x) = x_R^{-1}$ for all $a, x \in Q$.

The permutation φ_a selects in the multiplication table of a given loop the number of columns in an element a appears. For the loop defined in Example 1.1

$$\varphi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & 1 & 5 & 2 & 3 \end{pmatrix} = (1\,4)(2\,7\,3\,6)(5).$$

Further, permutations will be written in the form of cycles, cycles will be separated by dots. For example, the above permutation φ_4 will be written as $\varphi_4 = (1\,4.\,2\,7\,3\,6.\,5.)$.

It is clear that a loop $Q(\cdot)$, where $Q = \{1, 2, \ldots, n\}$, can be identified with the set $\{\varphi_1, \varphi_2, \ldots, \varphi_n, \}$ of its tracks.

Theorem 2.2. *A loop $Q(\cdot)$ is a D-loop if and only if*

$$\varphi_1 \varphi_a \varphi_1 = \varphi_{a^{-1}}^{-1} \tag{8}$$

for every $a \in Q$, where a^{-1} is (right) inverse to a.

Proof. Let $Q(\cdot)$ be a D-loop. Then $x_R^{-1} = x^{-1}$ for every $x \in Q$ and, according to (6), for all $a, x \in Q$ we have $\varphi_a^{-1}(x) \cdot x = a$. Hence

$$a^{-1} = (\varphi_a^{-1}(x) \cdot x)^{-1} = x^{-1} \cdot (\varphi_a^{-1}(x))^{-1} = \varphi_1(x) \cdot \varphi_1 \varphi_a^{-1}(x).$$

Since also $a^{-1} = \varphi_1(x) \cdot \varphi_{a^{-1}} \varphi_1(x)$, from the above we obtain $\varphi_1 \varphi_a^{-1} = \varphi_{a^{-1}} \varphi_1$, which implies (8).

Conversely, let $x \cdot y = a$. Then $y = \varphi_a(x)$. Hence

$$\begin{aligned} y_R^{-1} \cdot x_R^{-1} &= \varphi_1 \varphi_a(x) \cdot \varphi_1(x) = \varphi_1 \varphi_a \varphi_1(x^{-1}) \cdot x^{-1} \\ &\overset{(8)}{=} \varphi_{a^{-1}}^{-1}(x^{-1}) \cdot x^{-1} = \lambda_{a^{-1}}(x^{-1}) \cdot x^{-1} \overset{(7)}{=} a_R^{-1} = (x \cdot y)_R^{-1}. \end{aligned}$$

This completes the proof. □

Corollary 2.3. *A loop $Q(\cdot)$ is a D-loop if and only if it satisfies one of the following identities:*

(*a*) $\varphi_1\varphi_a^{-1}\varphi_1 = \varphi_{a^{-1}}$,

(*b*) $\varphi_1 R_a\varphi_1 = L_{a^{-1}}$,

(*c*) $\varphi_1 L_a\varphi_1 = R_{a^{-1}}$.

Proof. Indeed, (8) can be written in the form $\varphi_1\varphi_{a^{-1}}\varphi_1 = \varphi_a^{-1}$, which, in view of $\varphi_1\varphi_1 = id_Q$, is equivalent to (a). Moreover, $(x\cdot a)^{-1} = a^{-1}\cdot x^{-1}$ means that $\varphi_1 R_a = L_{a^{-1}}\varphi_1$. The last is equivalent to (b) and (c). □

Example 2.4. Consider the loop $Q(\cdot)$:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	4	2
4	4	6	5	2	1	3
5	5	3	6	1	2	4
6	6	4	2	5	3	1

We will use Theorem 2.2 to verify that this loop is a D-loop.

We have

$$\varphi_1 = (1.\,2.\,3.\,4\,5.\,6.) \qquad \varphi_4 = (1\,4.\,2\,3\,5\,6.)$$
$$\varphi_2 = (1\,2.\,3\,6.\,4.\,5.) \qquad \varphi_5 = (1\,5.\,2\,6\,4\,3.)$$
$$\varphi_3 = (1\,3.\,2\,4\,6\,5.) \qquad \varphi_6 = (1\,6.\,2\,5\,3\,4.)$$

We have to check the condition (8) for $a = 2, 3, 4, 5, 6$ because $\varphi_1\varphi_1\varphi_1 = \varphi_1^{-1}$ holds in each loop. Permutations φ_1 and φ_2 have disjoint cycles hence $\varphi_1\varphi_2\varphi_1 = \varphi_2 = \varphi_2^{-1}$. In other cases we obtain:

$$\varphi_1\varphi_3\varphi_1 = (1\,3.\,2\,5\,6\,4.) = \varphi_3^{-1} \qquad \varphi_1\varphi_5\varphi_1 = (1\,4.\,2\,6\,5\,3.) = \varphi_4^{-1}$$
$$\varphi_1\varphi_4\varphi_1 = (1\,5.\,2\,3\,4\,6.) = \varphi_5^{-1} \qquad \varphi_1\varphi_6\varphi_1 = (1\,6.\,2\,4\,3\,5.) = \varphi_6^{-1}$$

This shows that $Q(\circ)$ is a D-loop. □

Note that in general loops isotopic to D-loops are not D-loops.

Example 2.5. The following loop

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	3	4	5
3	3	6	2	5	1	4
4	4	5	1	6	2	3
5	5	3	4	1	6	2
6	6	4	5	2	3	1

is isotopic to a D-loop $Q(\circ)$ from the previous example. This isotopy has the form $\gamma(x\circ y)=\alpha(x)\cdot\beta(y)$, where $\alpha=(1\,4\,2.\,3.\,5.\,6.)$, $\beta=(1\,2\,5\,4\,6.\,3.)$, $\gamma=(1\,6\,4\,3\,5\,2.)$. The loop $Q(\circ)$ is not a D-loop since $3_L^{-1}\neq 3_R^{-1}$. □

Theorem 2.6. *Let $Q(\cdot)$ be an IP-loop and let $a\in Q$ be fixed. If an element $a'\in Q$ is inverse to a in $Q(\cdot)$, then $Q(\circ)$ with the operation*

$$x\circ y=R_{a'}(x)\cdot L_a(y) \tag{9}$$

is a D-loop with the same identity as in $Q(\cdot)$.

Proof. It is clear that $Q(\circ)$ is a quasigroup. Let an element $a'\in Q$ be inverse to a in $Q(\cdot)$. Then

$$x\circ 1=R_{a'}(x)\cdot L_a(1)=(x\cdot a')\cdot a=x.$$

Similarly $1\circ x=x$. Hence $Q(\circ)$ is a loop with the same identity as in $Q(\cdot)$.

Moreover, for every $x\in Q$ there exists $\overline{x}\in Q$ such that

$$1=x\circ\overline{x}=R_{a'}(x)\cdot L_a(\overline{x})=(x\cdot a')\cdot(a\cdot\overline{x}),$$

which gives $a\cdot\overline{x}=(x\cdot a')^{-1}=(a')^{-1}\cdot x^{-1}=a\cdot x^{-1}$. Thus $\overline{x}=x^{-1}$ for every $x\in Q$. Hence

$$\begin{aligned}(x\circ y)^{-1}&=(R_{a'}(x)\cdot L_a(y))^{-1}=((x\cdot a')\cdot(a\cdot y))^{-1}\\&=(a\cdot y)^{-1}\cdot(x\cdot a')^{-1}=(y^{-1}\cdot a^{-1})\cdot((a')^{-1}\cdot x^{-1})\\&=(y^{-1}\cdot a')\cdot(a\cdot x^{-1})=R_{a'}(y^{-1})\cdot L_a(x^{-1})=y^{-1}\circ x^{-1}.\end{aligned}$$

Therefore $Q(\circ)$ is a D-loop. □

Corollary 2.7. *Any IP-loop of order n determines $n-1$ isotopic D-loops.*

Example 2.8. Starting from the following *IP*-loop:

$\cdot$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	6	7	5	4
3	3	1	2	7	6	4	5
4	4	7	6	5	1	2	3
5	5	6	7	1	4	3	2
6	6	4	5	3	2	7	1
7	7	5	4	2	3	1	6

and using (9) with $a=2$ we obtain a D-loop:

$\circ$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	6	7	5	4
3	3	1	2	5	4	7	6
4	4	5	6	7	1	2	3
5	5	4	7	1	6	3	2
6	6	7	5	3	2	4	1
7	7	6	4	2	3	1	5

which is not an *IP*-loop because $3 \circ (2 \circ 5) \neq 5$. Hence $a = 2$ is not an inverse element in $Q(\circ)$. Putting $x * y = R_3(x) \circ L_2(y)$ we obtain a quasigroup:

$*$	1	2	3	4	5	6	7
1	1	2	3	7	6	4	5
2	2	3	1	6	7	5	4
3	3	1	2	5	4	7	6
4	7	5	6	4	1	2	3
5	6	4	7	1	5	3	2
6	4	7	5	3	2	6	1
7	5	6	4	2	3	1	7

which is isotopic to the initial *D*-loop $Q(\circ)$ but it is not a *D*-loop. This means that in Theorem 2.6 the assumption on a can not be ignored. $\square$

Proposition 2.9. *An element $a \in Q$ used in Theorem 2.6 has the same inverse element in $Q(\cdot)$ and $Q(\circ)$ defined by (9) if and only if*

$$x \cdot a = x \circ a \quad \textit{and} \quad a' \cdot x = a' \circ x \tag{10}$$

for all $x \in Q$.

Proof. Let $a' \in Q$ be inverse to a in $Q(\cdot)$ and $Q(\circ)$. Then $a = (a')'$ and

$$z = (z \circ a') \circ a = (R_{a'}(z) \cdot L_a(a')) \circ a = R_{a'}(z) \circ a,$$

which for $z = x \cdot a$ gives $x \cdot a = x \circ a$.

Similarly,

$$z = a' \circ (a \circ z) = a' \circ (R_{a'}(a) \cdot L_a(z)) = a' \circ L_a(z)$$

for $z = a' \cdot x$ implies $a' \cdot x = a' \circ x$.

Conversely, if $a' \in Q$ is inverse to a in $Q(\cdot)$ and (10) are satisfied, then

$$(x \circ a) \circ a' = (x \cdot a) \circ a' = R_{a'}(x \cdot a) \cdot L_a(a') = ((x \cdot a) \cdot a') \cdot (a \cdot a') = x.$$

Analogously $a' \circ (a \circ x) = a' \cdot (a \circ x) = x$. Hence a' is inverse to a in $Q(\circ)$. $\square$

Corollary 2.10. *An element $a \in Q$ used in Theorem 2.6 has the same inverse element in $Q(\cdot)$ and $Q(\circ)$ defined by (9) if and only if the multiplication tables of these two loops have the same a–columns and the same a'–rows.*

Proposition 2.11. *An element $a \in Q$ used in Theorem 2.6 has the same inverse element in $Q(\cdot)$ and $Q(\circ)$ defined by (9) if and only if*

$$L_a L_a = L_{a^2} \quad \textit{and} \quad R_a R_a = R_{a^2} \tag{11}$$

where L_a and R_a are translations in $Q(\cdot)$.

Proof. Let $a \in Q$ has the same inverse a' in $Q(\cdot)$ and $Q(\circ)$. Then for every $x \in Q$ we have

$$\begin{aligned} x = a' \circ (a \circ x) &= R_{a'}(a') \cdot L_a(R_{a'}(a) \cdot L_a(x)) \\ &= R_{a'}(a')L_aL_a(x) = L_{a' \cdot a'}L_aL_a(x) = L_{(a^2)'}L_aL_a(x), \end{aligned}$$

whence, applying (1), we get $L_aL_a = L^{-1}_{(a^2)'} = L_{a^2}$.

Similarly, for every $z \in Q$ we have

$$z \circ a = R_{a'}(z) \cdot L_a(a) = R_{a^2}R_{a'}(z),$$

which for $z = R_a(x)$, by (1), gives

$$R_a(x) \circ a = R_{a^2}R_{a'}R_a(x) = R_{a^2}(x).$$

Hence $R_aR_a = R_{a^2}$. This proves (11).

The converse statement is obvious. □

Below we present a simple method of construction of new loops from given loops. This method is based on *exchange of tracks*. Next, this method will be applied to the construction of D-loops.

Let $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ be tracks of a D-loop $Q(\cdot)$ with the identity 1. We say that for $i \neq j \neq 1$ tracks φ_i, φ_j are *decomposable* if there exist two nonempty subsets X, Y of Q such that $Q = X \cup Y$, $X \cap Y = \emptyset$, $1 \in X$ and

$$\begin{cases} \varphi_i = \bar{\varphi}_i\hat{\varphi}_i \\ \varphi_j = \bar{\varphi}_j\hat{\varphi}_j \end{cases} \tag{12}$$

where $\bar{\varphi}_i$, $\bar{\varphi}_j$ are permutations of X, $\hat{\varphi}_i, \hat{\varphi}_j$ are permutations of Y.

Putting

$$\begin{cases} \psi_i = \bar{\varphi}_i\hat{\varphi}_j \\ \psi_j = \bar{\varphi}_j\hat{\varphi}_i \end{cases} \tag{13}$$

and $\psi_k = \varphi_k$ for $k \notin \{i, j\}$ we obtain the new system of tracks which defines on Q the new loop $Q(\circ)$ with the same identity as in $Q(\cdot)$.

Example 2.12. The loop $Q(\cdot)$ defined by

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	8	7	6	1	4	3	2
6	6	5	8	7	2	1	4	3
7	7	6	5	8	3	2	1	4
8	8	7	6	5	4	3	2	1

is a group (so, it is a D-loop) with the following tracks:

$\varphi_1 = (1.24.3.4.5.6.7.8.)$ $\varphi_2 = (12.34.5876.)$ $\varphi_3 = (13.2.4.57.68.)$
$\varphi_4 = (14.23.5678.)$ $\varphi_5 = (15.37.2846.)$ $\varphi_6 = (16.38.2547.)$
$\varphi_7 = (17.35.2648.)$ $\varphi_8 = (18.36.2745.)$

For $(i, j) \in \{(2,3),(2,4),(3,4),(5,7),(6,8)\}$ tracks φ_i, φ_j are decomposable. In the case $(i, j) = (6, 8)$ we have

$$\begin{cases} \varphi_6 = \bar{\varphi}_6\hat{\varphi}_6, & \text{where } \bar{\varphi}_6 = (16.38.),\ \hat{\varphi}_6 = (2547.) \\ \varphi_8 = \bar{\varphi}_8\hat{\varphi}_8, & \text{where } \bar{\varphi}_8 = (18.36.),\ \hat{\varphi}_8 = (2745.) \end{cases}$$

whence, according to (13), we obtain

$$\begin{cases} \psi_6 = \bar{\varphi}_6\hat{\varphi}_8 = (16.38.2745.) \\ \psi_8 = \bar{\varphi}_8\hat{\varphi}_6 = (18.36.2547.) \end{cases}$$

and $\psi_k = \varphi_k$ for $k = 1, 2, 3, 4, 5, 7$.

This new system of tracks $\{\psi_1, \psi_2, \dots, \psi_8\}$ defines the loop $Q(\circ)$:

$\circ$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	$\boxed{8}$	7	$\boxed{6}$	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	$\boxed{6}$	5	$\boxed{8}$	7
5	5	$\boxed{6}$	7	$\boxed{8}$	1	4	3	2
6	6	5	8	7	2	1	4	3
7	7	$\boxed{8}$	5	$\boxed{6}$	3	2	1	4
8	8	7	6	5	4	3	2	1

where items changed by tracks ψ_6 and ψ_8 are entered in the box. □

This new loop $Q(\circ)$ can be used for the construction of another loop since it has the same pair of decomposable tracks as $Q(\cdot)$. So, for the construction of new loops we can use not only one but also two or more pairs of decomposable tracks. Using different pairs of decomposable tracks we obtain different loops which may not be isotopic. Obtained loops may not be isotopic to the initial loop $Q(\cdot)$, too.

Example 2.13. Direct computations show that this loop

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	5	7	3	8	4	6
3	3	8	6	1	4	2	5	7
4	4	6	1	5	7	3	8	2
5	5	7	4	2	8	1	6	3
6	6	4	8	3	1	7	2	5
7	7	5	2	8	6	4	3	1
8	8	3	7	6	2	5	1	4

is a D-loop. It hasn't got any decomposable pair of tracks. □

Theorem 2.14. *Let $Q(\cdot)$ be a D-loop with the identity* 1*. If φ_i, φ_j, where $i \cdot j = 1$ and $i \neq j$, are decomposable tracks of $Q(\cdot)$, then a loop $Q(\circ)$ obtained from $Q(\cdot)$ by exchange of tracks is a D-loop.*

Proof. Since $Q(\circ)$ is a loop it is sufficient to show that $\psi_1\psi_k\psi_1 = \psi_{k^{-1}}^{-1}$ for every $k \in Q$ (Theorem 2.2). For $k \notin \{i,j\}$ we have $\psi_k = \varphi_k$, so for $k \notin \{i,j\}$ this condition is satisfied by the assumption. For $k = i$ we have

$$\psi_1\psi_i\psi_1 = \varphi_1\bar{\varphi}_i\hat{\varphi}_j\varphi_1 = (\varphi_1\bar{\varphi}_i\varphi_1)(\varphi_1\hat{\varphi}_j\varphi_1) = \bar{\varphi}_j{}^{-1}\hat{\varphi}_i{}^{-1} = \psi_j^{-1} = \psi_{i^{-1}}^{-1}$$

because $\varphi_1^2 = \varepsilon$, $\bar{\varphi}_i\hat{\varphi}_j = \hat{\varphi}_j\bar{\varphi}_i$ and $i \cdot j = 1$.

For $k = j$ the proof is analogous. So, $Q(\circ)$ is a D-loop. □

The assumption $i \cdot j = 1$ is essential. Indeed, in Example 2.12 tracks φ_3, φ_4 are decomposable, $4 \cdot 3 \neq 1$, $4^{-1} = 2$ and $\psi_1\psi_4\psi_1 = \psi_4 \neq \psi_2^{-1}$. So, a loop determined by tracks $\psi_1, \dots, \psi_8$ is not a D-loop.

The D-loop $Q(\circ)$ constructed in Example 2.12 is not isotopic to the initial group $Q(\cdot)$ since $(7 \circ 7) \circ 2 \neq 7 \circ (7 \circ 2)$. In this loop we also have $7 \circ (7 \circ 2) \neq 2$, so it is not an IP-loop, too.

Example 2.15. The loop $Q(\cdot)$ defined by

·	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	5	7	3	8	4	6
3	3	8	4	1	7	2	6	5
4	4	6	1	3	8	7	5	2
5	5	7	8	2	6	1	3	4
6	6	4	7	8	1	5	2	3
7	7	5	2	6	4	3	8	1
8	8	3	6	5	2	4	1	7

is a D-loop with the following tracks:

$\varphi_1 = (1.2.34.56.78.)$ $\varphi_2 = (12.367.485.)$ $\varphi_3 = (13.4.25768.)$
$\varphi_4 = (14.3.27586.)$ $\varphi_5 = (15.6.23847.)$ $\varphi_6 = (16.5.28374.)$
$\varphi_7 = (17.8.24635.)$ $\varphi_8 = (18.7.26453.)$

For $(i,j) \in \{(3,4),(5,6),(7,8)\}$ tracks φ_i, φ_j are decomposable. Each pair of such tracks gives a D-loop. Obtained loops are isotopic but they are not isotopic to $Q(\cdot)$ since they and $Q(\cdot)$ have different indicators Φ^*. (Isotopic loops have the same indicators – see [7].)

If for the construction of a new loop we use two pairs of decomposable tracks: φ_3, φ_4 and φ_5, φ_6, or φ_3, φ_4 and φ_7, φ_8, or φ_5, φ_6 and φ_7, φ_8, then we obtain three isotopic D-loops. These loops are not isotopic either to $Q(\cdot)$ or to the previous because have different indicators Φ^*.

Also in the case when we use three pairs of decomposable tracks obtained D-loop. It is not isotopic to any of the previous.

So, from this D-loop we obtain three nonisotopic D-loops which also are not isotopic to the initial D-loop $Q(\cdot)$. $\square$

As it is well known with each quasigroup $Q(\cdot)$ we can conjugate five new quasigroups (called *parastrophes* of $Q(\cdot)$) by permuting the variables in the defining equation. Namely, if $Q_0 = Q(\cdot)$ is a fixed quasigroup, then its parastrophes have the form

$$\begin{array}{lccc} Q(\backslash) & x\backslash z = y & \Longleftrightarrow & x\cdot y = z, \\ Q(/) & z/y = x & \Longleftrightarrow & x\cdot y = z, \\ Q(*) & y * x = z & \Longleftrightarrow & x\cdot y = z, \\ Q(\bullet) & y\bullet z = x & \Longleftrightarrow & x\cdot y = z, \\ Q(\triangleleft) & z\triangleleft x = y & \Longleftrightarrow & x\cdot y = z. \end{array}$$

Theorem 2.16. *Parastrophes of a D-loop $Q(\cdot)$ are isomorphic to one of the following quasigroups: $Q(\cdot)$, $Q(\backslash)$, $Q(/)$.*

Proof. Indeed, if $Q(\cdot)$ is a D-loop, φ_1 – its track determined by the identity of $Q(\cdot)$, then, according to the definition of D-loops, we have

$$\varphi_1(x\cdot y) = \varphi_1(y)\cdot\varphi_1(x).$$

Hence

$$\varphi_1(y * x) = \varphi_1(z) \Longleftrightarrow \varphi_1(x\cdot y) = \varphi_1(z) \Longleftrightarrow \varphi_1(y)\cdot\varphi_1(x) = \varphi_1(z).$$

So, $\varphi_1(y * x) = \varphi_1(y)\cdot\varphi_1(x)$, i.e., $Q(*)$ and $Q(\cdot)$ are isomorphic.

Further,

$$\begin{aligned} \varphi_1(y\bullet z) = \varphi_1(x) \quad &\Longleftrightarrow \varphi_1(x\cdot y) = \varphi_1(z) \Longleftrightarrow \varphi_1(y)\cdot\varphi_1(x) = \varphi_1(z) \\ &\Longleftrightarrow \varphi_1(y)\backslash\varphi_1(z) = \varphi_1(x). \end{aligned}$$

Thus, $\varphi_1(y\bullet z) = \varphi_1(y\backslash z)$. Consequently, $Q(\bullet) \cong Q(\backslash)$.

Analogously,

$$\begin{aligned} \varphi_1(z\triangleleft x) = \varphi_1(y) \quad &\Longleftrightarrow \varphi_1(x\cdot y) = \varphi_1(z) \Longleftrightarrow \varphi_1(y)\cdot\varphi_1(x) = \varphi_1(z) \\ &\Longleftrightarrow \varphi_1(z)/\varphi_1(x) = \varphi_1(y), \end{aligned}$$

whence $\varphi_1(z\triangleleft x) = \varphi_1(z/x)$. So, $Q(\triangleleft) \cong Q(/)$. $\square$

3. Loops isotopic to D-loops

As was mentioned earlier, loops isotopic to D-loops are not D-loops in general, but in some cases principal isotopes of D-loops are D-loops. Below we find conditions under which D-loops are isotopic to groups and conditions under which a principal isotope of a D-loop is a D-loop.

Definition 3.1. Let $Q(\cdot)$, where $Q = \{1, 2, \dots, n\}$, be a quasigroup. By a *spin* of a quasigroup $Q(\cdot)$ we mean a permutation

$$\varphi_{ij} = \varphi_i \varphi_j^{-1} = \varphi_i \lambda_j \,,$$

where φ_i and λ_j are right and left tracks of $Q(\cdot)$ respectively.

Obviously $\varphi_{ii} = \varepsilon$ for $i \in Q$ and $\varphi_{ij} \neq \varphi_{ik}$ for $j \neq k$, but the situation where $\varphi_{ij} = \varphi_{kl}$ for some $i, j, k, l \in Q$ also is possible (cf. [6]). Hence the collection Φ of all spins of a given quasigroup $Q(\cdot)$ can be divided into disjoint subsets $\Phi_i = \{\varphi_{ij} : j \in Q\}$ (called *spin-basis*) in which all elements are different. Generally, Φ_i are not closed under the composition of permutations but in some cases Φ_i are groups.

In [6] the following result is proved.

Theorem 3.2. *A quasigroup $Q(\cdot)$ is isotopic to some group if and only if its spin-basis Φ_1 is a group.* □

In this case $\Phi_1 = \Phi_i$ for all $i \in Q$.

Theorem 3.3. *In D-loops we have $\Phi = \langle \Phi_1 \rangle = \{\varphi_{1i}\varphi_{1j} : i, j \in Q\}$.*

Proof. Indeed, by Corollary 2.3

$$\varphi_{1i}\varphi_{1j} = \varphi_1\varphi_i^{-1}\varphi_1\varphi_j^{-1} = (\varphi_1\varphi_i^{-1}\varphi_1)\varphi_j^{-1} = \varphi_{i^{-1}}\varphi_j^{-1} = \varphi_{i^{-1}j} \in \Phi$$

and conversely, each $\varphi_{ij} \in \Phi$ can be written in the form $\varphi_{ij} = \varphi_{1i^{-1}}\varphi_{1j}$. □

Corollary 3.4. *A D-loop is isotopic to a group if and only if $\langle \Phi_1 \rangle = \Phi_1$.*

Proof. If a D-loop $Q(\cdot)$ is isotopic to a group, then, by Theorem 3.2, Φ_1 is a group. Hence $\langle \Phi_1 \rangle = \Phi_1$.

Conversely, if $\langle \Phi_1 \rangle = \Phi_1$, then $\varphi_{1i}\varphi_{i1} = \varphi_{11}$ implies $\varphi_{1i}^{-1} = \varphi_{i1} \in \Phi = \langle \Phi_1 \rangle = \Phi_1$ which means that Φ_1 is a group. Thus $Q(\circ)$ is isotopic to some group. □

Corollary 3.5. *A D-loop is isotopic to a group if and only if Φ_1 is closed under a composition of permutations.*

Proof. If a D-loop $Q(\cdot)$ is isotopic to a group, then, by Theorem 3.2, Φ_1 is a group. Hence Φ_1 is closed under a composition of permutations.

Conversely, if Φ_1 is closed under a composition of permutations, then, in view of Theorem 3.3, from $\varphi_{1i}\varphi_{i1} = \varphi_{11}$ it follows $\varphi_{1i}^{-1} = \varphi_{i1} \in \Phi = \langle \Phi_1 \rangle = \Phi_1$, which means that Φ_1 is a group. Thus $Q(\cdot)$ is isotopic to some group. □

Corollary 3.6. *A D-loop $Q(\cdot)$ is isotopic to a group if and only if for all $i, j \in Q$ there exists $k \in Q$ such that $\varphi_i\varphi_1\varphi_j = \varphi_k$.*

Proof. Indeed, $\varphi_{1j}\varphi_{1i} = \varphi_{1k}$ means that $\varphi_1\varphi_j^{-1}\varphi_1\varphi_i^{-1} = \varphi_1\varphi_k^{-1}$. Thus $\varphi_j^{-1}\varphi_1\varphi_i^{-1} = \varphi_k^{-1}$. Hence $\varphi_k = \varphi_i\varphi_1\varphi_j$. □

Theorem 3.7. *If a quasigroup $Q(\cdot)$ is isotopic to a D-loop $Q(\circ)$, then there exists a permutation σ of Q and an element $p\in Q$ and such that*

$$\varphi_p\varphi_i^{-1}\varphi_p=\varphi_{\sigma(i)} \tag{14}$$

for all tracks φ_i of $Q(\cdot)$.

Proof. Let a quasigroup $Q(\cdot)$ be isotopic to a D-loop $Q(\circ)$. Then

$$\gamma(x\cdot y)=\alpha(x)\circ\beta(y) \tag{15}$$

for some permutations α,β,γ of Q. Thus for all $i,x\in Q$ we have

$$\gamma(i)=\gamma(x\cdot\varphi_i(x))=\alpha(x)\circ\beta\varphi_i(x),$$

where φ_i is a right track of $Q(\cdot)$. Hence

$$\gamma(i)=z\circ\beta\varphi_i\alpha^{-1}(z)$$

for all $i\in Q$ and $z=\alpha(x)$. This together with $\gamma(i)=z\circ\psi_{\gamma(i)}(z)$ gives

$$\beta\varphi_i\alpha^{-1}=\psi_{\gamma(i)},$$

i.e.,

$$\varphi_i=\beta^{-1}\psi_{\gamma(i)}\alpha,\qquad \varphi_i^{-1}=\alpha^{-1}\psi_{\gamma(i)}^{-1}\beta. \tag{16}$$

Thus for $p=\gamma^{-1}(1)$, where 1 is the identity of $Q(\circ)$, we obtain

$$\varphi_p\varphi_i^{-1}\varphi_p=(\beta^{-1}\psi_1\alpha)(\alpha^{-1}\psi_{\gamma(i)}^{-1}\beta)(\beta^{-1}\psi_1\alpha)=\beta^{-1}(\psi_1\psi_{\gamma(i)}^{-1}\psi_1)\alpha.$$

Since $Q(\circ)$ is a D-loop, for $k=\gamma^{-1}\psi_1\gamma(i)$, by Corollary 2.3, we have

$$\beta^{-1}(\psi_1\psi_{\gamma(i)}^{-1}\psi_1)\alpha=\beta^{-1}\psi_{\gamma(i)^{-1}}\alpha=\beta^{-1}\psi_{\psi_1\gamma(i)}\alpha=\beta^{-1}\psi_{\gamma(k)}\alpha=\varphi_k.$$

So, $\varphi_p\varphi_i^{-1}\varphi_p=\varphi_k$, which means that (14) is valid for $\sigma=\gamma^{-1}\psi_1\gamma$. □

The converse statement is more complicated.

Theorem 3.8. *Let a quasigroup $Q(\cdot)$ and a loop $Q(\circ)$ with the identity* 1 *be isotopic, i.e., let* (15) *holds. If φ_i are tracks of $Q(\cdot)$, ψ_i – tracks of $Q(\circ)$ and* (14) *is satisfied for $p=\gamma^{-1}(1)$, $\sigma=\gamma^{-1}\psi_1\gamma$ and all $i\in Q$, then $Q(\circ)$ is a D-loop.*

Proof. Indeed, (15) holds, then for $p=\gamma^{-1}(1)$ and any $i\in Q$, in view of (16), we have

$$\begin{aligned}\psi_1\psi_i^{-1}\psi_1&=(\beta\varphi_{\gamma^{-1}(1)}\alpha^{-1})(\alpha\varphi_{\gamma^{-1}(i)}^{-1}\beta^{-1})(\beta\varphi_{\gamma^{-1}(1)}\alpha^{-1})=\beta(\varphi_p\varphi_{\gamma^{-1}(i)}^{-1}\varphi_p)\alpha^{-1}\\&=\beta\varphi_{\sigma(\gamma(i))}\alpha^{-1}=\beta\varphi_{\gamma^{-1}\varphi_1(i)}\alpha^{-1}=\psi_{\psi_1(i)}=\psi_{i^{-1}},\end{aligned}$$

where i^{-1} is calculated in $Q(\circ)$.

Thus $\psi_1\psi_i^{-1}\psi_1=\psi_{i^{-1}}$, which means that $Q(\circ)$ is a D-loop. □

Lemma 3.9. *A loop $Q(\circ)$ is a principal isotope of a quasigroup $Q(\cdot)$ if and only if*

$$x \circ y = R_b^{-1}(x) \cdot L_a^{-1}(y),$$

for some $a, b \in Q$ such that $a \cdot b = 1$, where 1 is the identity of $Q(\circ)$ and L_a, R_b are translations of $Q(\cdot)$.

Proof. Indeed, if $Q(\circ)$ is a loop with the identity 1 and $x \circ y = \alpha(x) \cdot \beta(y)$ for some permutations α, β of Q, then for $a = \alpha(1)$, $b = \beta(1)$ we have

$$1 = 1 \circ 1 = \alpha(1) \cdot \beta(1) = a \cdot b,$$
$$x = x \circ 1 = \alpha(x) \cdot \beta(1) = \alpha(x) \cdot b,$$
$$y = 1 \circ y = \alpha(1) \cdot \beta(y) = a \cdot \beta(y).$$

Thus

$$\alpha(x) = R_b^{-1}(x), \qquad \beta(y) = L_a^{-1}(y).$$

Hence $x \circ y = R_b^{-1}(x) \cdot L_a^{-1}(y)$.

The converse statement is obvious. □

Corollary 3.10. *A quasigroup $Q(\cdot)$ is a principal isotope of a loop $Q(\circ)$ with the identity 1 if and only if*

$$x \cdot y = R_b(x) \circ L_a(y),$$

for some translations L_a, R_b of $Q(\cdot)$ such that $a \cdot b = 1$. □

Proposition 3.11. *In any principal isotope $Q(\cdot)$ of a D-loop $Q(\circ)$ with the identity 1 we have*

$$\varphi_1 \varphi_i^{-1} \varphi_1 = \varphi_{i^{-1}},$$

where i^{-1} is calculated in $Q(\circ)$.

Proof. It is a consequence of (15) and (16). □

Corollary 3.12. *A principal isotope $Q(\cdot)$ of a D-loop $Q(\circ)$ is a D-loop if and only if $Q(\cdot)$ and $Q(\circ)$ have the same inverse elements.* □

Corollary 3.13. *A principal isotope $Q(\cdot)$ of a D-loop $Q(\circ)$ is a D-loop if and only if $Q(\cdot)$ and $Q(\circ)$ have the same tracks induced by the identity of $Q(\circ)$, i.e., if and only if φ_1 and ψ_1, where 1 he identity of $Q(\circ)$.* □

4. Proper D-loops

A *D*-loop is *proper* if it is not an IP-loop. The smallest *D*-loop has six elements. Below we present a full list of all nonisotopic proper *D*-loops of order 6. They represent (respectively) the classes 8.1.1, 9.1.1, 10.1.1 and 11.1.1 mentioned in the book [4].

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	3	4
3	3	6	1	2	4	5
4	4	5	2	1	6	3
5	5	3	4	6	1	2
6	6	4	5	3	2	1

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	2	5	6	4
4	4	6	5	1	2	3
5	5	4	6	2	3	1
6	6	5	4	3	1	2

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	6	5	4	3
3	3	5	1	2	6	4
4	4	6	2	1	3	5
5	5	3	4	6	2	1
6	6	4	5	3	1	2

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	5	6	3
3	3	4	2	6	1	5
4	4	5	6	2	3	1
5	5	6	1	3	2	4
6	6	3	5	1	4	2

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