Quasigroups and Related Systems **20** (2012), 97 − 112

Right k-weakly regular hemirings

Muhammad Shabir and Rukhshanda Anjum

Abstract. In this paper we define right k-weakly regular hemirings, which are generalization of k-regular hemirings. We characterize these hemirings by the properties of their right k-ideals and also by the properties of their fuzzy right k-ideals.

1. Introduction

There are many concepts of universal algebra generalizing an associative ring $(R, +, \cdot)$. Some of them, nearrings and several kinds of semirings, have been proven very useful. The notion of semiring was introduced by H. S. Vandiver in 1934 [12]. Semirings provide a common generalization of rings and distributive lattices, appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics. Hemirings, semirings with commutative addition and zero element, have also proved to be an important algebraic tool in theoretical computer science. The concept of a fuzzy set, introduced by Zadeh [14], was applied by many researchers to generalize some of the basic concepts of algebra. The notions of automata and formal languages have been generalized and extensively studied in a fuzzy frame work.

Ideals of semirings play a central role in the structure theory and are useful for many purposes. However in general, they do not coincide with usual ring ideals. For this, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Henriksen defined in [6] a more restricted class of ideals in semirings, which is called the class of k-ideals. These ideals have the property that if the semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal.

Investigations of fuzzy semirings were initiated in [2]. Fuzzy k-ideals are studied in [3, 5, 7, 11]. In this paper we characterize hemirings in which each right k-ideal is idempotent and those hemirings for which each fuzzy right k-ideal is idempotent. We also study right pure and purely prime k-ideals and fuzzy right pure and fuzzy purely prime k-ideals in hemirings.

2010 Mathematics Subject Classification: 20N05
Keywords: hemiring, right weakly regular hemiring, k-ideal, right pure k-ideal, purely prime k-ideal, right pure fuzzy k-ideal, purely prime fuzzy k-ideal.

2. Preliminaries

For the definitions of semiring, hemiring, left (right) ideal we refer to [4].

A left (right) ideal A of a hemiring R is called a *left* (*right*) *k-ideal* of R if for any $a, b \in A$ and $x \in R$ from $x + a = b$ it follows $x \in A$.

The *k-closure* of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a = b \text{ for some } a, b \in A\}.$$

It is clear that if A is a left (right) ideal of R, then $\overline{A}$ is the smallest left (right) k-ideal of R containing A. Also, $\overline{A} = A$ for all left (right) k-ideals of R. Obviously $\overline{\overline{A}} = \overline{A}$ for each non-empty $A \subseteq R$. Also $\overline{A} \subseteq \overline{B}$ for all $A \subseteq B \subseteq R$. A right k-ideal A with the property $\overline{A^2} = A$ is called *k-idempotent*.

Lemma 2.1. *$\overline{AB} = \overline{\overline{A}\,\overline{B}}$ for any subsets A, B of a hemiring R.* □

Lemma 2.2. [10] *If A and B are right and left k-ideals of a hemiring R respectively, then $\overline{AB} \subseteq A \cap B$.* □

An element a of a hemiring R is called *regular* if there exists $x \in R$ such that $a = axa$. A hemiring R is called *regular* if each element of R is regular. Generalizing the concept of regularity, in [1, 9] k-regular hemirings are defined as a hemiring in which for each $a \in R$, there exist $x, y \in R$ such that $a + axa = aya$.

Obviously, every regular hemiring is a k-regular but the converse is not true. If R is a ring, then the regular and k-regular coincide.

Theorem 2.3. [9] *A hemiring R is k-regular if and only if for any fuzzy right k-ideal A and any fuzzy left k-ideal B, we have $\overline{AB} = A \cap B$.* □

For any fuzzy subsets λ and μ of X we define

$$\lambda \leqslant \mu \iff \lambda(x) \leqslant \mu(x),$$
$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\},$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\}$$

for all $x \in X$.

More generally, if $\{\lambda_i : i \in I\}$ is a collection of fuzzy subsets of X, then by the *intersection* and the *union* of this collection we mean the fuzzy subsets

$$\Big(\bigwedge_{i\in I} \lambda_i\Big)(x) = \bigwedge_{i\in I} \lambda_i(x) = \inf_{i\in I}\{\lambda_i(x)\},$$
$$\Big(\bigvee_{i\in I} \lambda_i\Big)(x) = \bigvee_{i\in I} \lambda_i(x) = \sup_{i\in I}\{\lambda_i(x)\},$$

respectively.

A fuzzy subset λ of a hemiring R is called a *fuzzy left* (*right*) *ideal* of R if for all $a, b \in R$ we have

(1) $\lambda(a+b) \geqslant \lambda(a) \wedge \lambda(b)$,

(2) $\lambda(ab) \geqslant \lambda(b)$, $(\lambda(ab) \geqslant \lambda(a))$.

Note that $\lambda(0) \geqslant \lambda(x)$ for all $x \in R$.

A fuzzy left (right) ideal λ of a hemiring R is called a *fuzzy left (right) k-ideal* if $x+y=z \Longrightarrow \lambda(x) \geqslant \lambda(y) \wedge \lambda(z)$ holds for all $x, y, z \in R$.

A fuzzy right k-ideal is defined analogously. The basic properties of fuzzy k-ideals in semirings are described in [3].

Let λ be a fuzzy subset of a universe X and $t \in [0,1]$. Then the subset $U(\lambda; t) = \{x \in X \ : \ \lambda(x) \geqslant t\}$ is called *level subset* of λ.

The following Proposition is a consequence of transfer principle [8].

Proposition 2.4. *Let A be a non-empty subset of a hemiring R. Then a fuzzy set λ_A defined by*

$$\lambda_A(x) = \begin{cases} t & if\ x \in A \\ s & otherwise \end{cases}$$

where $0 \leqslant s < t \leqslant 1$, is a fuzzy left (right) k-ideal of R if and only if A is a left (right) k-ideal of R. □

Corollary 2.5. *Let A be a non-empty subset of a hemiring R. Then the characteristic function χ_A of A is a fuzzy right k-ideal of R if and only if A is a right k-ideal of R.* □

Proposition 2.6. *If A, B are subsets of a hemiring R such that $Im\lambda_A = Im\lambda_B$ then*

(1) $A \subseteq B \Longleftrightarrow \lambda_A \leqslant \lambda_B$,

(2) $\lambda_A \wedge \lambda_B = \lambda_{A\cap B}$. □

Definition 2.7. [11] The *k-product* of two fuzzy subsets μ and ν on R is defined by

$$(\mu \odot_k \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \nu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \nu(b'_j)\right] \right]$$

and $(\mu \odot_k \nu)(x) = 0$ if x cannot be expressed as $x + \sum\limits_{i=1}^{m} a_i b_i = \sum\limits_{j=1}^{n} a'_j b'_j$.

A fuzzy subset λ such that $\lambda \odot_k \lambda = \lambda$ is called *k-idempotent*.

Proposition 2.8. *Let μ, ν, ω, λ be fuzzy subsets on R. Then*

(1) $\mu \leqslant \omega$ *and* $\nu \leqslant \lambda \Longrightarrow \mu \odot_k \nu \leqslant \omega \odot_k \lambda$.

(2) $\chi_A \odot_k \chi_B = \chi_{\overline{AB}}$ *for characteristic functions of $A, B \subset R$.* □

Lemma 2.9. *If μ, ν are fuzzy left (right) k-ideals of a hemiring R, then $\mu \wedge \nu$ is also a fuzzy left (right) k-ideal of R.* □

Theorem 2.10. [11]
(*i*) *If λ and μ are fuzzy k-ideals of R, then so is $\lambda \odot_k \mu$. Moreover, $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$.*
(*ii*) *If λ is fuzzy right k-ideal of R and μ a fuzzy left k-ideals of R, then $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$.* □

Theorem 2.11. [11] *A hemiring R is k-regular if and only if for any fuzzy right k-ideal μ and any fuzzy left k-ideal ν of R we have $\mu \odot_k \nu = \mu \wedge \nu$.* □

3. Right k-weakly regular hemirings

Definition 3.1. A hemiring R is called *right* (*left*) *k-weakly regular* if for each $x \in R$, $x \in \overline{(xR)^2}$ $\left(\text{res. } x \in \overline{(Rx)^2}\right)$.

That is for each $x \in R$ we have $r_i, s_i, t_j, p_j \in R$ such that $x + \sum_{i=1}^{n} x r_i x s_i = \sum_{j=1}^{m} x t_j x p_j$ $\left(x + \sum_{i=1}^{n} r_i x s_i x = \sum_{j=1}^{m} t_j x p_j x\right)$. Thus each k-regular hemiring with identity is right k-weakly regular but the converse is not true. However for a commutative hemiring both the concept coincide.

Proposition 3.2. *The following statements are equivalent for a hemiring R with identity:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *$\overline{BA} = B \cap A$ for all right k-ideals B and two-sided k-ideals A of R.*

Proof. (1) $\Longrightarrow$ (2) Let R be a right k-weakly regular hemiring and B be a right k-ideal of R. Clearly $\overline{B^2} \subseteq B$.

Let $x \in B$. Since R is right k-weakly regular, so $x \in \overline{(xR)^2}$ where xR is the right ideal of R generated by x and so $\overline{xR}$ is the right k-ideal of R generated by x. Thus $xR \subseteq B$, this implies $x \in \overline{(xR)(xR)} \subseteq \overline{BB} = \overline{B^2}$. Thus $B \subseteq \overline{B^2}$. So, $\overline{B^2} = B$.

(2) $\Longrightarrow$ (3) Let B be a right k-ideal of R and A a two-sided k-ideal of R, then by Lemma 2.2, $\overline{BA} \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$ and xR and RxR are right ideal and two-sided ideal of R generated by x, respectively. Thus $xR \subseteq B$ and $RxR \subseteq A$.

$$x \in xR \subseteq \overline{xR} = \overline{\overline{xR}\,\overline{xR}} = \overline{xRxR} = \overline{(xR)(xR)} = \overline{x(RxR)} \subseteq \overline{xA} \subseteq \overline{BA}$$

Hence $B \cap A \subseteq \overline{BA}$ and so $B \cap A = \overline{BA}$.

(3) $\Longrightarrow$ (1) Let $x \in R$ and RxR and xR be the two-sided ideal and right ideal of R generated by x, respectively. Then

$$x \in xR \cap RxR \subseteq \overline{xR} \cap \overline{RxR} = \overline{\overline{xR}\ \overline{RxR}} = \overline{xRRxR} = \overline{xR^2xR} = \overline{(xR)^2}.$$

Hence R is right k-weakly regular hemiring. □

Theorem 3.3. *For a hemiring R with identity, the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all fuzzy right k-ideals of R are k-idempotent,*
3. *$\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.*

Proof. (1) $\Longrightarrow$ (2) Let λ be a fuzzy right k-ideal of R, then $\lambda \odot_k \lambda \leqslant \lambda$.

For the reverse inclusion, let $x \in R$. Since R is right k-weakly regular so there exist $s_i, t_i, s'_j, t'_j \in R$ such that $x + \sum_{i=1}^{m} xs_ixt_i = \sum_{j=1}^{n} xs'_jxt'_j$. Hence

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leqslant \bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)).$$

Also

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leqslant \bigwedge_{j=1}^{n} \left(\lambda(xs'_j) \wedge \lambda(xt'_j)\right).$$

Therefore

$$\begin{aligned} \lambda(x) &\leqslant \bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)) \wedge \bigwedge_{j=1}^{n} \left(\lambda(xs'_j) \wedge \lambda(xt'_j)\right) \\ &\leqslant \bigvee_{x+\sum\limits_{i=1}^{m} xs_ixt_i = \sum\limits_{j=1}^{n} xs'_jxt'_j} \left[\bigwedge_{i=1}^{m} (\lambda(xs_i) \wedge \lambda(xt_i)) \wedge \bigwedge_{j=1}^{n} \left(\lambda(xs'_j) \wedge \lambda(xt'_j)\right)\right] \\ &= (\lambda \odot_k \lambda)(x). \end{aligned}$$

Hence $\lambda \leqslant \lambda \odot_k \lambda$, which proves $\lambda \odot_k \lambda = \lambda$.

(2) $\Longrightarrow$ (3) Let λ and μ be fuzzy right and two sided k-ideal of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right k-ideal of R. By Theorem 2.10, $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$. By hypothesis,$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leqslant \lambda \odot_k \mu$. Hence $\lambda \odot_k \mu = \lambda \wedge \mu$.

(3) $\Longrightarrow$ (1) Let B be a right k-ideal of R and A be a two-sided k-ideal of R, then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided k-ideal of R, respectively. Hence by the hypothesis and Propositions 2.6 and 2.8, we have $\chi_B \odot_k \chi_A = \chi_B \wedge \chi_A$, i.e., $\chi_{\overline{BA}} = \chi_{B\cap A}$, which implies $\overline{BA} = B \cap A$. Thus, by Proposition 3.2, R is right k-weakly regular hemiring. □

Theorem 3.4. *For a hemiring R with identity, the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *$\overline{BA} = B \cap A$ for all right k-ideals B and two-sided k-ideals A of R,*
4. *all fuzzy right k-ideals of R are k-idempotent,*
5. *$\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R.*

If R is commutative, then the above statements are equivalent to

6. *R is k-regular.*

Proof. $1, 2, 3$ are equivalent by Proposition 3.2. $1, 4, 5$ are equivalent by Theorem 3.3. Finally, if R is commutative, then by Theorem 2.3, also 1 and 6 are equivalent. □

Definition 3.5. [11] The *k-sum* $\lambda +_k \mu$ of fuzzy subsets λ and μ of R is defined by

$$(\lambda +_k \mu)(x) = \bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)],$$

where $x, a_1, b_1, a_2, b_2 \in R$.

Theorem 3.6. [11] *The k-sum of fuzzy k-ideals of R is also a fuzzy k-ideal of R.* □

Theorem 3.7. *The collection of all k-ideals of a right k-weakly regular hemiring R forms a complete distributive lattice.*

Proof. The collection $\mathcal{L}_R$ of all k-ideals of a right k-weakly regular hemiring R is a partially ordered set under the inclusion of sets and is a complete lattice under the operations $\sqcup$, $\sqcap$ defined as $A \sqcup B = \overline{A + B}$ and $A \sqcap B = A \cap B$.

Let $A, B, C \in \mathcal{L}_R$, then obviously $\overline{(A \cap B) + (A \cap C)} \subseteq A \cap (\overline{B + C})$. For the reverse inclusion, let $x \in A \cap (\overline{B + C}) = \overline{A(\overline{B + C})}$. Then $x + a = b$ for some $a, b \in A(\overline{B + C})$. Hence $a = a_1 y_1$ and $b = a_2 y_2$ for some $a_1, a_2 \in A$ and $y_1, y_2 \in (\overline{B + C})$. Then $y_1 + b_1 + c_1 = b_2 + c_2$ and $y_2 + b_3 + c_3 = b_4 + c_4$ for some $b_1, b_2, b_3, b_4 \in B$ and $c_1, c_2, c_3, c_4 \in C$. Thus $a_1 y_1 + a_1 b_1 + a_1 c_1 = a_1 b_2 + a_1 c_2$ yields $a + a_1 b_1 + a_1 c_1 = a_1 b_2 + a_1 c_2$ which implies $a \in \overline{AB + AC}$. Similarly $b \in \overline{AB + AC}$ and thus $x \in \overline{AB + AC}$. Hence $A \cap (\overline{B + C}) = \overline{A(\overline{B + C})} \subseteq \overline{AB + AC} \subseteq \overline{\overline{AB} + \overline{AC}} = \overline{(A \cap B) + (A \cap C)}$. Thus $\overline{(A \cap B) + (A \cap C)} = A \cap (\overline{B + C})$. □

The following example shows that if the collection of all k-ideals of a hemiring R is a complete distributive lattice then R is not necessarily a right k-weakly regular hemiring.

Example 3.8. Consider the hemiring $R = \{0, a, b\}$ with $+$ and $\cdot$ defined by $x + y = \max\{x, y\}$, where $0 < a < b$ and $x \cdot y = b$ for $x = y = b$ and $x \cdot y = 0$ otherwise.

The k-ideals of R are $\{0\}, \{0, a\}$ and R. Since $\{0\} \subseteq \{0, a\} \subseteq R$. So the collection of k-ideals is a complete distributive lattice but R is not right k-weakly regular hemiring. □

Theorem 3.9. *If R is a right k-weakly regular hemiring, then the set $\mathcal{L}_R$ of all fuzzy k-ideals of R (ordered by $\leqslant$) is a distributive lattice.*

Proof. The set $\mathcal{L}_R$ of all fuzzy k-ideals of R (ordered by $\leqslant$) is clearly a lattice under the k-sum and intersection of fuzzy k-ideals. Now we show that $\mathcal{L}_R$ is a distributive lattice, that is for any fuzzy k-ideals λ, μ, δ of R we have $(\lambda \wedge \delta) + \mu = (\lambda + \mu) \wedge (\delta + \mu)$.

For any $x \in R$

$$\begin{aligned}
[(\lambda \wedge \delta) + \mu](x) &= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} (\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \\ (\mu)(b_1) \wedge (\mu)(b_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \\ \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \wedge \\ [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \end{bmatrix} \\
&= \Big(\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \Big) \\
&\quad \wedge \Big(\bigvee_{x+(a_1+b_1)=(a_2+b_2)} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \Big) \\
&= (\lambda + \mu)(x) \wedge (\delta + \mu)(x) = [(\lambda + \mu) \wedge (\delta + \mu)](x).
\end{aligned}$$
□

4. Prime and Fuzzy prime right k-ideals

Definition 4.1. A right k-ideal P of a hemiring R is called *k-prime* (*k-semiprime*) if for any right k-ideals A, B of R,

$$AB \subseteq P \Longrightarrow A \subseteq P \text{ or } B \subseteq P \quad \big(A^2 \subseteq P \Longrightarrow A \subseteq P\big).$$

P is *k-irreducible* (*k-strongly irreducible*) if for any right k-ideals A, B of R

$$A \cap B = P \Longrightarrow A = P \text{ or } B = P \quad (A \cap B \subseteq P \Longrightarrow A \subseteq P \text{ or } B \subseteq P).$$

A fuzzy right k-ideal μ of a hemiring R is called a *fuzzy k-prime* (*k-semiprime*) *right k-ideal* of R if for any fuzzy k-right ideals λ, δ of R,

$$\lambda \odot_k \delta \leqslant \mu \Longrightarrow \lambda \leqslant \mu \;\text{ or }\; \delta \leqslant \mu \;\; (\lambda \odot_k \lambda \leqslant \mu \Longrightarrow \lambda \leqslant \mu).$$

μ is called a *fuzzy k-irreducible* (*k-strongly irreducible*) if for any fuzzy right k-ideals λ, δ of R,

$$\lambda \wedge \delta = \mu \Longrightarrow \lambda = \mu \text{ or } \delta = \mu \ (\lambda \wedge \delta \leqslant \mu \Longrightarrow \lambda \leqslant \mu \text{ or } \delta \leqslant \mu).$$

Lemma 4.2. *In any hemiring R*

(a) *a (fuzzy) k-prime right k-ideal is a (fuzzy) k-semiprime right k-ideal,*

(b) *an intersection of (fuzzy) k-prime right k-ideals is a (fuzzy) k-semi prime right k-ideal.* □

Theorem 4.3. *Each proper right k-ideal of a right k-weakly regular hemiring R is the intersection of right k-irreducible k-ideals which contain it.*

Proof. Let I be a proper right k-ideal of R and let $\{I_\alpha : \alpha \in \Lambda\}$ be a family of right k-irreducible k-ideals of R which contain I. Clearly $I \subseteq \cap_{\alpha\in\Lambda} I_\alpha$. Suppose $a \notin I$. Then by Zorn's Lemma there exists a right k-ideal I_β such that I_β is maximal with respect to the property $I \subseteq I_\beta$ and $a \notin I_\beta$. We will show that I_β is k-irreducible. Let A, B be right k-ideals of R such that $I_\beta = B \cap A$. Suppose $I_\beta \subset B$ and $I_\beta \subset A$. Then by the maximality of I_β, we have $a \in B$ and $a \in A$. But this implies $a \in B \cap A = I_\beta$, which is a contradiction. Hence either $I_\beta = B$ or $I_\beta = A$. So there exists a k-irreducible k-ideal I_β such that $a \notin I_\beta$ and $I \subseteq I_\beta$. Hence $\cap I_\alpha \subseteq I$. Thus $I = \cap I_\alpha$. □

Proposition 4.4. *Let R be a right k-weakly regular hemiring. If λ is a fuzzy right k-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in (0,1]$, then there exists a fuzzy k-irreducible right k-ideal δ of R such that $\lambda \leqslant \delta$ and $\delta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu$ is a fuzzy right k-ideal of R, $\mu(a) = \alpha$ and $\lambda \leqslant \mu\}$. Then $X \neq \emptyset$, since $\lambda \in X$. Let F be a totally ordered subset of X, say $F = \{\lambda_i : i \in I\}$. We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy right k-ideal of R. For any $x, r \in R$, we have

$$\Big(\bigvee_i \lambda_{i\in I}\Big)(x) = \bigvee_{i\in I} (\lambda_i(x)) \leqslant \bigvee_{i\in I} (\lambda_i(xr)) = \Big(\bigvee_{i\in I} \lambda_i\Big)(xr).$$

Let $x, y \in R$, consider

$$\begin{aligned}
\Big(\bigvee_{i\in I}\lambda_i\Big)(x)\wedge\Big(\bigvee_{i\in I}\lambda_i\Big)(y) &= \Big(\bigvee_{i\in I} \lambda_i(x)\Big) \wedge \Big(\bigvee_{j\in I} \lambda_j(y))\Big) \\
&= \bigvee_{j\in I}\Big(\bigvee_{i\in I}(\lambda_i(x) \wedge \lambda_j(y))\Big) \\
&\leqslant \bigvee_{j\in I}\Big(\bigvee_{i\in I}(\max\{\lambda_i(x), \lambda_j(x)\} \wedge \max\{\lambda_i(y), \lambda_j(y)\})\Big) \\
&\leqslant \bigvee_{j\in I}\Big(\bigvee_{i\in I}\max\{\lambda_i(x+y), \lambda_j(x+y)\}\Big) \\
&\leqslant \bigvee_{i\in I}\max\{\lambda_i(x+y), \lambda_j(x+y)\} = \Big(\bigvee_{i\in I}\lambda_i\Big)(x+y).
\end{aligned}$$

Now, let $x+a=b$, where $a,b\in R$. Then

$$\begin{aligned}\Big(\bigvee_{i\in I}\lambda_i\Big)(a)\wedge\Big(\bigvee_{i\in I}\lambda_i\Big)(b) &= \Big(\bigvee_{i\in I}\lambda_i(a)\Big)\wedge\Big(\bigvee_{j\in I}\lambda_j(b)\Big)\\ &= \bigvee_{j\in I}\Big(\bigvee_{i\in I}\lambda_i(a)\wedge\lambda_j(b)\Big)\\ &\leqslant \bigvee_{j\in I}\Big(\bigvee_{i\in I}\max\{\lambda_i(a),\lambda_j(a)\}\wedge\max\{\lambda_i(b),\lambda_j(b)\}\Big)\\ &= \bigvee_{i,j\in I}\max\{\lambda_i(x),\lambda_j(x)\}\leqslant\bigvee_{i\in I}\lambda_i(x).\end{aligned}$$

Thus $\bigvee_{i\in I}\lambda_i$ is a fuzzy right k-ideal of R. Clearly $\lambda\leqslant\bigvee_i\lambda_i$ and $\bigvee_i\lambda_i\,(a)=\alpha$. Thus $\bigvee_i\lambda_i$ is the l.u.b of F. Hence by Zorn's lemma there exists a fuzzy right k-ideal δ of R which is maximal with respect to the property that $\lambda\leqslant\delta$ and $\delta\,(a)=\alpha$.

We will show that δ is fuzzy k-irreducible right k-ideal of R. Let $\delta=\delta_1\wedge\delta_2$, where δ_1,δ_2 are fuzzy right k-ideals of R. Thus $\delta\leqslant\delta_1$ and $\delta\leqslant\delta_2$. We claim that either $\delta=\delta_1$ or $\delta=\delta_2$. Suppose $\delta\neq\delta_1$ and $\delta\neq\delta_2$. Since δ is maximal with respect to the property that $\delta\,(a)=\alpha$ and since $\delta\lneqq\delta_1$ and $\delta\lneqq\delta_2$, so $\delta_1\,(a)\neq\alpha$ and $\delta_2\,(a)\neq\alpha$. Hence $\alpha=\delta\,(a)=(\delta_1\wedge\delta_2)\,(a)=(\delta_1)\,(a)\wedge(\delta_2)\,(a)\neq\alpha$, which is impossible. Hence $\delta=\delta_1$ or $\delta=\delta_2$. Thus δ is a fuzzy k-irreducible right k-ideal of R. □

Theorem 4.5. *Every fuzzy right k-ideal of a hemiring R is the intersection of all fuzzy k-irreducible right k-ideals of R which contain it.*

Proof. Let λ be the fuzzy right k-ideal of R and let $\{\lambda_\alpha:\alpha\in\Lambda\}$ be the family of all fuzzy k-irreducible right k-ideals of R which contain λ. Obviously $\lambda\leqslant\bigwedge_{\alpha\in\Lambda}\lambda_\alpha$. We show that $\bigwedge_{\alpha\in\Lambda}\lambda_\alpha\leqslant\lambda$. Let a be any element of R, then by Proposition 4.4, there exists a fuzzy k-irreducible right k-ideal λ_β such that $\lambda\leqslant\lambda_\beta$ and $\lambda\,(a)=\lambda_\beta\,(a)$. Hence $\lambda_\beta\in\{\lambda_\alpha:\alpha\in\Lambda\}$. Hence $\bigwedge_{\alpha\in\Lambda}\lambda_\alpha\leqslant\lambda_\beta$, so $\bigwedge_{\alpha\in\Lambda}\lambda_\alpha\,(a)\leqslant\lambda_\beta\,(a)=\lambda\,(a)$, i.e., $\bigwedge_{\alpha\in\Lambda}\lambda_\alpha\leqslant\lambda$. Hence $\bigwedge_{\alpha\in\Lambda}\lambda_\alpha=\lambda$. □

Theorem 4.6. *A hemiring with identity is right k-weakly regular if and only if each its right k-ideal is k-semiprime.*

Proof. Suppose every right k-ideal is idempotent. Let I,J be right k-ideals of R, such that $J^2\subseteq I$. Thus $\overline{J^2}\subseteq\overline{I}$. By Theorem 3.4, $J=\overline{J^2}$, so $J\subseteq I$. Hence I is a k-semiprime right k-ideal of R.

Conversely, if each each right k-ideal I of R is k-semiprime, then $\overline{I^2}$ is also a right k-ideal of R and $I^2\subseteq\overline{I^2}$. Hence by hypothesis $I\subseteq\overline{I^2}$. But $\overline{I^2}\subseteq I$ always. Hence $I=\overline{I^2}$. Thus by Theorem 3.4, R is right k-weakly regular. □

Theorem 4.7. *For a hemiring R with identity the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all fuzzy right k-ideals of R are k-idempotent,*
3. *$\lambda \odot_k \mu = \lambda \wedge \mu$ for all fuzzy right k-ideals λ and all fuzzy two-sided k-ideals μ of R,*
4. *each fuzzy right k-ideal of R is also fuzzy k-semiprime.*

Proof. $1, 2, 3$ are equivalent by Theorem 3.3.

If δ is a fuzzy right k-ideal of R, then $\lambda \odot_k \lambda \leqslant \delta$, where λ is a fuzzy right k-ideal of R. By (2) $\lambda \odot_k \lambda = \lambda$, so $\lambda \leqslant \delta$. Thus δ is a fuzzy k-semiprime right k-ideal of R.

Conversely, if δ is a fuzzy right k-ideal of R, then also $\delta \odot_k \delta$ is a fuzzy right k-ideal of R and so by (4) $\delta \odot_k \delta$ is a fuzzy k-semiprime right k-ideal of R. As $\delta \odot_k \delta \leqslant \delta \odot_k \delta$ we have $\delta \leqslant \delta \odot_k \delta$. But $\delta \odot_k \delta \leqslant \delta$ always. So $\delta \odot_k \delta = \delta$. □

Theorem 4.8. *If every right k-ideal of a hemiring R is k-prime , then R is a right k-weakly regular hemiring and the set of k-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each right k-ideal is prime right k-ideal. Let A be a right k-ideal of R then $\overline{A^2}$ is a right k-ideal of R. As $A^2 \subseteq \overline{A^2} \Longrightarrow A \subseteq \overline{A^2}$. But $\overline{A^2} \subseteq A$ always. Hence $A = \overline{A^2}$. Thus R is right k-weakly regular.

Let A, B be any k-ideals of R then $AB \subseteq A \cap B$. As $A \cap B$ is a k-ideal of R, so a k-prime right k-ideal. Thus either $A \subseteq A \cap B$ or $B \subseteq A \cap B$. That is either $A \subseteq B$ or $B \subseteq A$. □

Theorem 4.9. *If R is a right k-weakly regular hemiring and the set of all right k-ideals of R is totally ordered, then every right k-ideal of R is k-prime.*

Proof. Let A, B, C be right k-ideals of R such that $AB \subseteq C$. Since the set of all right k-ideals of R is totally ordered, so we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then $A = \overline{AA} \subseteq \overline{AB} \subseteq C$. If $B \subseteq A$ then $B = \overline{BB} \subseteq \overline{AB} \subseteq C$. Thus C is a k-prime right k-ideal. □

Theorem 4.10. *If every fuzzy right k-ideal of a hemiring R is a fuzzy k-prime right k-ideal, then R is a right k-weakly regular hemiring and the set of fuzzy k-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each fuzzy right k-ideal is fuzzy prime. Let λ be a fuzzy right k-ideal of R. Then $\lambda \odot_k \lambda$ is also a fuzzy right k-ideal of R. As $\lambda \odot_k \lambda \leqslant \lambda \odot_k \lambda \Longrightarrow \lambda \leqslant \lambda \odot_k \lambda$. But $\lambda \odot_k \lambda \leqslant \lambda$ always. Hence $\lambda = \lambda \odot_k \lambda$. Thus R is a right k-weakly regular hemiring.

Let λ, μ be any fuzzy k-ideals of R. Then $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$. As $\lambda \wedge \mu$ is a fuzzy k-ideal of R so it is fuzzy k-prime. Thus either $\lambda \leqslant \lambda \wedge \mu$ or $\mu \leqslant \lambda \wedge \mu$. That is either $\lambda \leqslant \mu$ or $\mu \leqslant \lambda$. □

Theorem 4.11. *If the set of all fuzzy right k-ideals of a right k-weakly regular hemiring R is totally ordered, then every fuzzy right k-ideal of R is a fuzzy k-prime right k-ideal of R.*

Proof. Let λ, μ, ν be fuzzy right k-ideals of R such that $\lambda \odot_k \mu \leqslant \nu$. Since the set of all fuzzy right k-ideals of R is totally ordered, so we have $\lambda \leqslant \mu$ or $\mu \leqslant \lambda$. If $\lambda \leqslant \mu$ then $\lambda = \lambda \odot_k \lambda \leqslant \lambda \odot_k \mu \leqslant \nu$. If $\mu \leqslant \lambda$ then $\mu = \mu \odot_k \mu \leqslant \lambda \odot_k \mu \leqslant \nu$. Thus ν is a fuzzy k-prime right k-ideal. □

Example 4.12. Consider the set $R = \{0, x, 1\}$ in which $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ are defined by the chains $0 < 1 < x$ and $0 < x < 1$. Then $(R, +, \cdot)$ is a hemiring.

The right k-ideals of R are $\{0\}, \{0, x\}, \{0, x, 1\}$. The k-ideals $\{0\}$ $\{0, x, 1\}$ are idempotent.

In order to examine the right fuzzy k-ideals of R, we observe the following facts.

Fact 1. A fuzzy subset λ of R is a fuzzy right ideal if and only if $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$.

Indeed, since $0 = x \cdot 0 = 1 \cdot 0$ so $\lambda(0) \geqslant \lambda(x)$ and $\lambda(0) \geqslant \lambda(1)$. Also $\lambda(x) = \lambda(1 \cdot x) \geqslant \lambda(1)$. Thus $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$.

Conversely, If λ is a fuzzy subset of R such that $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$, then by the definition of $+$ in R, we have $m + m' = m$ or m' for every $m, m' \in R$, and certainly $\lambda(m) \wedge \lambda(m') \leqslant \lambda(m)$ and $\lambda(m) \wedge \lambda(m') \leqslant \lambda(m')$. Thus $\lambda(m + m') \geqslant \lambda(m) \wedge \lambda(m')$. By the definition of $\cdot$ defined on R, it is easy to verify that $\lambda(ma) \geqslant \lambda(m)$ for all m, a in R. Hence λ is a fuzzy right ideal of R.

Fact 2. λ is a fuzzy right k-ideal of R if and only if $\lambda(0) \geqslant \lambda(x) = \lambda(1)$.

Indeed, by the Fact 1 we have $\lambda(0) \geqslant \lambda(x) \geqslant \lambda(1)$. Since $1 + x = x$, so $\lambda(1) \geqslant \lambda(x) \wedge \lambda(x) = \lambda(x)$. Thus $\lambda(0) \geqslant \lambda(x) = \lambda(1)$. Conversely, if $\lambda(0) \geqslant \lambda(x) = \lambda(1)$, then by the Fact 1, λ is a fuzzy right ideal of R.

If $x + a = b$ for $a, b, x \in R$ then $\lambda(x) \geqslant \lambda(a) \wedge \lambda(b)$. So λ is a fuzzy right k-ideal of R.

Obviously R is a right k-weakly regular hemiring. But each fuzzy right k-ideal of R is not k-prime. Because λ, μ, ν defined by $\lambda(0) = 0.8, \lambda(x) = \lambda(1) = 0.6$, $\mu(0) = 0.9$, $\mu(x) = \mu(1) = 0.5$ and $\nu(0) = 0.85$, $\nu(x) = \nu(1) = 0.55$ are fuzzy k-ideals of R such that $\lambda \odot_k \mu \leqslant \nu$ but neither $\lambda \leqslant \nu$ nor $\mu \leqslant \nu$. □

5. Right pure k-ideals

In this section we define right pure k-ideals of a hemiring R and also right pure fuzzy k-ideals of R. We prove that a two-sided k-ideal I of a hemiring R is right pure if and only if for every right k-ideal A of R, we have $A \cap I = \overline{AI}$.

Definition 5.1. A k-ideal I of a hemiring R is called *right pure* if for each $x \in I$, $x \in \overline{xI}$, i.e., if for each $x \in I$ there exist $y, z \in I$ such that $x + xy = xz$.

Lemma 5.2. *A k-ideal I of a hemiring R is right pure if and only if $A \cap I = \overline{AI}$ for every right k-ideal A of R.*

Proof. Suppose that I is a right pure k-ideal of R and A is a right k-ideal of R. Then $\overline{AI} \subseteq A \cap I$. Clearly, $a \in A \cap I$ implies $a \in A$ and $a \in I$. Since I is right pure, so $a \in \overline{aI} \subseteq \overline{AI}$. Thus $A \cap I \subseteq \overline{AI}$. Hence $A \cap I = \overline{AI}$.

Conversely, assume that $A \cap I = \overline{AI}$ for every right k-ideal A of R. Let $x \in I$. Take A, the principal right k-ideal generated by x, that is, $A = \overline{xR + \mathbb{N}_\circ x}$, where $\mathbb{N}_\circ = \{0, 1, 2, ...\}$. By hypothesis $A \cap I = \overline{AI} = \overline{\overline{(xR + \mathbb{N}_\circ x)}I} = \overline{(xR + \mathbb{N}_\circ x)I} \subseteq \overline{xI}$. So $x \in \overline{xI}$. Hence I is a right pure k-ideal of R. □

Definition 5.3. A fuzzy k-ideal λ of a hemiring R is called *right pure* if and only if $\mu \wedge \lambda = \mu \odot_k \lambda$ for every fuzzy right k-ideal μ of R.

Proposition 5.4. *The characteristic function of a non-empty subset A of a hemiring R is its right pure fuzzy k-ideal if and only if A is a right pure k-ideal of R.*

Proof. Let A be a right pure k-ideal of R. Then χ_A is a fuzzy k-ideal of R. To prove that χ_A is right pure we have to show that for any fuzzy right k-ideal μ of R, $\mu \wedge \chi_A = \mu \odot_h \chi_A$. Now if $x \notin A$, then

$$(\mu \wedge \chi_A)(x) = \mu(x) \wedge \chi_A(x) = 0 \leqslant (\mu \odot_h \chi_A)(x).$$

For the case $x \in A$, as A is a right pure k-ideal of R, so there exist $a, b \in A$, such that $x + xa = xb$. As $x, a, b \in A$, this implies $\chi_A(x) = \chi_A(a) = \chi_A(b) = 1$. Now,

$$\begin{aligned}(\mu \odot_k \chi_A)(x) &= \bigvee_{x + \sum_{i=1}^{m} a_i b_i = \sum_{j=1}^{n} a'_j b'_j} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \chi_A(b_i)] \wedge \bigwedge_{j=1}^{n} [\mu(a'_j) \wedge \chi_A(b'_j)] \right] \\ &\geqslant \min[\mu(x) \wedge \chi_A(a) \wedge \mu(x) \wedge \chi_A(b)] \\ &\geqslant \min[\mu(x) \wedge \chi_A(x) \wedge \mu(x) \wedge \chi_A(x)] \\ &\geqslant \mu(x) \wedge \chi_A(x) = (\mu \wedge \chi_A)(x).\end{aligned}$$

So, in both the cases $\mu \odot_k \chi_A \geqslant \mu \wedge \chi_A$. But $\mu \odot_k \chi_A \leqslant \mu \wedge \chi_A$ is always true. Thus, $\mu \wedge \chi_A = \mu \odot_k \chi_A$. So, χ_A is right pure fuzzy k-ideal of R.

Conversely, let χ_A be a right pure fuzzy k-ideal of R. Then A is a k-ideal of R. Let B be a right k-ideal of R, then χ_B is a fuzzy right k-ideal of R. Hence by hypothesis $\chi_B \odot_k \chi_A = \chi_B \wedge \chi_A = \chi_{B \cap A}$. By Proposition 2.8, $\chi_B \odot_k \chi_A = \chi_{\overline{BA}}$. This implies that $B \cap A = \overline{BA}$. Therefore A is a right pure k-ideal of R. □

Proposition 5.5. *Intersection of right pure k-ideals of R is a right pure k-ideal of R.*

Proof. Let A, B be right pure k-ideals of R and I be any right k-ideal of R. Then $I \cap (A \cap B) = (I \cap A) \cap B = (\overline{IA}) \cap B = \overline{(\overline{IA})B} = \overline{(IA)B} = \overline{I(AB)} = \overline{I(A \cap B)}$ because $(\overline{IA})$ is a right k-ideal. Hence $A \cap B$ is a right pure k-ideal of R. □

Proposition 5.6. *Let λ_1, λ_2 are right pure fuzzy k-ideals of R, then so is $\lambda_1 \wedge \lambda_2$.*

Proof. Indeed, $\lambda_1 \wedge \lambda_2$ is a fuzzy k-ideal of R. We have to show that, for any fuzzy right k-ideal μ of R, $\mu \odot_k (\lambda_1 \wedge \lambda_2) = \mu \wedge (\lambda_1 \wedge \lambda_2)$.

Since λ_2 is right pure fuzzy k-ideal of R so it follows that $\lambda_1 \odot_k \lambda_2 = \lambda_1 \wedge \lambda_2$. Hence $\mu \odot_k (\lambda_1 \odot_k \lambda_2) = \mu \odot_k (\lambda_1 \wedge \lambda_2)$.

Also, $\mu \wedge (\lambda_1 \wedge \lambda_2) = (\mu \wedge \lambda_1) \wedge \lambda_2 = (\mu \odot_k \lambda_1) \wedge \lambda_2 = (\mu \odot_k \lambda_1) \odot_k \lambda_2 = \mu \odot_k (\lambda_1 \odot_k \lambda_2)$ since $\mu \odot_k \lambda_1$ is a fuzzy right k-ideal of R.

Thus $\mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \odot_k (\lambda_1 \wedge \lambda_2)$. □

Proposition 5.7. *For a hemiring R with identity the following statements are equivalent:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *every k-ideal of R is right pure.*

Proof. 1 and 2 are equivalent by Proposition 3.2.

(1) $\Longrightarrow$ (3) Let I and A be k-ideal and right k-ideal of R, respectively. Then $A \cap I = \overline{AI}$. Thus by Lemma 5.2, A is right pure.

(3) $\Longrightarrow$ (1) Let I be a k-ideal of R and A a right k-idealof R, then by hypothesis, I is right pure and so $A \cap I = \overline{AI}$. Thus, by Proposition 3.2, R is right k-weakly regular. □

Proposition 5.8. *The following statements are equivalent for a hemiring R with identity:*

1. *R is right k-weakly regular hemiring,*
2. *all right k-ideals of R are k-idempotent,*
3. *every k-ideal of R is right pure,*
4. *all fuzzy right k-ideals of R are k-idempotent,*
5. *every fuzzy k-ideal of R is right pure.*

If R is commutative, then the above statements are equivalent to

6. *R is k-regular.*

Proof. $1, 2, 3$ are equivalent by Proposition 5.7, $1, 4$ by Theorem 3.3.

(4) $\Longrightarrow$ (5) Let λ and μ be fuzzy right and two sided k-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right k-ideal of R. By Theorem 2.10, $\lambda \odot_k \mu \leqslant \lambda \wedge \mu$. By hypothesis,$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_k (\lambda \wedge \mu) \leqslant \lambda \odot_k \mu$. Hence $\lambda \odot_k \mu = \lambda \wedge \mu$. Thus μ is right pure.

(5) $\Longrightarrow$ (1) Let B be a right k-ideal of R and A be a two-sided k-ideal of R then the characteristic functions χ_B and χ_A are fuzzy right and fuzzy two-sided k-ideals of R, respectively. Hence $\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A$ implies $\chi_{\overline{BA}} = \chi_{B\cap A}$, i.e., $\overline{BA} = B \cap A$. Thus by Proposition 3.2, R is right k-weakly regular.

Finally, for a commutative hemiring, by Theorem 2.11, 1 and 6 are equivalent. □

6. Purely prime k-ideals

Definition 6.1. A proper right pure k-ideal I of a hemiring R is called *purely prime* if for any right pure k-ideals A, B of R, $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$, or equivalently, if $\overline{AB} \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Definition 6.2. A proper right pure k-ideal μ of a hemiring R is called *purely prime* if for any right pure fuzzy k-ideals λ, δ of R, $\lambda \wedge \delta \leqslant \mu$ implies $\lambda \leqslant \mu$ or $\delta \leqslant \mu$, or equivalently, if $\lambda \odot_k \delta \leqslant \mu$ implies $\lambda \leqslant \mu$ or $\delta \leqslant \mu$.

Proposition 6.3. *For a k-ideal I of a right k-weakly regular hemiring R with identity the following statements are equivalent:*

1. $A \cap B = I \Longrightarrow A = I$ *or* $B = I$,
2. $A \cap B \subseteq I \Longrightarrow A \subseteq I$ *or* $B \subseteq I$,

where A, B are k-ideals of R.

Proof. (1) $\Longrightarrow$ (2) Suppose A, B are k-ideals of R such that $A \cap B \subseteq I$. Then by Theorem 3.4, $I = \overline{(A \cap B) + I} = \overline{(A + I)} \cap \overline{(B + I)}$. Hence by the hypothesis $I = \overline{(A + I)}$ or $I = \overline{(B + I)}$, i.e., $A \subseteq I$ or $B \subseteq I$.

(2) $\Longrightarrow$ (1) Suppose A, B are k-ideals of R such that $A \cap B = I$. Then $I \subseteq A$ and $I \subseteq B$. On the other hand by hypothesis $A \subseteq I$ or $B \subseteq I$. Thus $A = I$ or $B = I$. □

Proposition 6.4. *Let R be a right k-weakly regular hemiring. Then any proper right pure k-ideal of R is contained in a purely prime k-ideal of R.*

Proof. Let I be a proper right pure k-ideal of a weakly regular hemiring R and $a \in R$ such that $a \notin I$. Consider the set X of all proper right pure k-ideals J of R containing I and such that $a \notin J$. Then X is non-empty because $I \in X$. By Zorn's Lemma this family contains a maximal element, say M. This maximal element is purely prime. Indeed, let $A \cap B = M$ for some some right pure k-ideals A, B of R. If A, B both properly contains M, then by the maximality of M, $a \in A$ and $a \in B$. Thus $a \in A \cap B = M$, which is a contradiction. Hence either $A = M$ or $B = M$. □

Proposition 6.5. *Let R be a right k-weakly regular hemiring. Then each proper right pure k-ideal is the intersection of all purely prime k-ideals of R which contain it.*

Proof. The proof is similar to the proof of Theorem 4.3. □

Proposition 6.6. *Let R be a right k-weakly regular hemiring. If λ is a right pure fuzzy k-ideal of R with $\lambda(a) = t$ where $a \in R$ and $t \in [0,1]$, then there exists a purely prime fuzzy k-ideal μ of R such that $\lambda \leqslant \mu$ and $\mu(a) = t$.*

Proof. The proof is similar to the proof of Proposition 4.4. □

Proposition 6.7. *Let R be a right k-weakly regular hemiring. Then each proper fuzzy right pure k-ideal is the intersection of all purely prime fuzzy k-ideals of R which contain it.*

Proof. The proof is similar to the proof of Theorem 4.5. □

References

[1] **M. R. Adhikari, M. K. Sen, H. J. Weinert**, *On k-regular semirings*, Bull. Cal. Math. Soc. **88** (1996), 141 – 144.

[2] **J. Ahsan, K. Saifullah, M. Farid Khan**, *Fuzzy semirings*, Fuzzy Sets Syst. **60** (1993), 309 – 320.

[3] **S. I. Baik, H. S. Kim**, *On Fuzzy k-ideals in semirings*, Kangweon-Kyungki Math. J. **8** (2000), 147 – 154.

[4] **W.A. Dudek, M. Shabir, R. Anjum**, *Characterizations of hemirings by their h-ideals*, Computer Math. Appl. **59** (2010), 3167 – 3179.

[5] **S. Ghosh**, *Fuzzy k-ideals of semirings*, Fuzzy Sets Syst. **95** (1998), 103 – 108.

[6] **M. Henriksen**, *Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices **6** (1958), 321.

[7] **C. B. Kim, M. Park**, *k-fuzzy ideals in semirings*, Fuzzy Sets Syst. **81** (1996), 281 – 286.

[8] **M. Kondo, W. A. Dudek**, *On the transfer principle in fuzzy theory*, Mathware Soft Comput. **12** (2005), 41 – 55.

[9] **P. Mukhopadhyay**, *Characterizations of regular semirings*, Mat. Vesnik **48** (1996), 83 – 85.

[10] **M. K. Sen, P. Mukhopadhyay**, *Von Neumannian regularity in semirings*, Kyungpook Math. J. **35** (1995), 249 – 258.

[11] **M. Shabir, R. Anjum**, *Characterization of hemirings by their k-ideals*, (submitted)

[12] **H. S. Vandiver**, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934), 914 – 920.

[13] **Y. Yin, H. Li**, *The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings*, Inform. Sci. **178** (2008), 3451 – 3464.

[14] **L.A. Zadeh**, *Fuzzy Sets*, Inf. Contr. **8** (1965), 338 – 353.

[15] **J.Zhan, W. A. Dudek**, *Fuzzy h-ideals of hemirings*, Inform. Sci. **177** (2007), $676 - 886$.

Received July 23, 2010
Revised March 25, 2011

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mails: mshabirbhatti@yahoo.co.uk (M.Shabir), rukh-28@hotmail.com (R.Anjum)