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On medial-like identities

Mirko Polonijo

Abstract

The description of the quasigroups that satisfy the identities of the form $(a \cdot b) \cdot (c \cdot d) = (\pi(a) \cdot \pi(b)) \cdot (\pi(c) \cdot \pi(d))$, where π is a certain permutation on $\{a, b, c, d\}$, is given. Those quasigroups include internally medial ($ab \cdot cd = ac \cdot bd$), externally medial ($ab \cdot cd = db \cdot ca$) and palindromic ($ab \cdot cd = dc \cdot ba$) quasigroups. There are six identities that are the equivalents of commutativity, and fourteen identities are the equivalents of commutative mediality.

It is well-known that a groupoid $(Q, \cdot)$ is medial ([1]; entropic in [5]) if it satisfies

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d) \qquad (M)$$

i.e.

$$ab \cdot cd = ac \cdot bd \qquad (M)$$

for all $a, b, c, d \in Q$. In the identity (M) we interchange the internal pair of the variables and now we could look for the identity in which the external pair is interchanged

$$ab \cdot cd = db \cdot ca \qquad (M_e)$$

or the identity in which the both pairs are interchanged

$$ab \cdot cd = dc \cdot ba\,. \qquad (P)$$

Therefore, we could call (M) *internal mediality* and (M_e) *external mediality* (paramediality in [2], [3]). The identity (P) we shall call *palindromity.*

Proposition 1. *For any groupoid $(Q, \cdot)$, any two of the three identities (M), (M_e) and (P) imply the third one.*

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Proof. $((M)\&(M_e) \Rightarrow (P))$ $\quad ab \cdot cd = ac \cdot bd = dc \cdot ba.$
$((P)\&(M_e) \Rightarrow (M))$ $\quad ab \cdot cd = dc \cdot ba = ac \cdot bd.$

$((P)\&(M) \Rightarrow (M_e))$ $\quad ab \cdot cd = dc \cdot ba = db \cdot ca.$ □

Proposition 2. *Let $(Q, \cdot)$ be a commutative groupoid. Then $(Q, \cdot)$ is palindromic. Further, the constraints (M) and (M_e) are equivalent, i.e. a commutative groupoid $(Q, \cdot)$ is internally medial if and only if it is externally medial.*

Proof. The first statement is obvious, and the second one follows from the previous proposition. □

Proposition 3. *Let $(Q, \cdot)$ be an idempotent groupoid. If it is externally medial or palindromic, then it is commutative.*

Proof. Any externally medial groupoid satisfies $xx \cdot yy = yx \cdot yx$, and for palindromic quasigroup $xx \cdot yy = yy \cdot xx$ is valid. Therefore, if the groupoid is idempotent, i.e. $xx = x$ holds for all $x \in Q$, it is commutative. □

Remark 1. There are idempotent internally medial groupoids (moreover quasigroups) which are not commutative. For instance, take Z_3 and define multiplication by $x \cdot y = x + 2y$.

Proposition 4. *Let $(Q, \cdot)$ be an internally medial or externally medial or palindromic quasigroup. Its center is empty or Q.*

Proof. The center is the set of all $c \in Q$ which commutes with all elements of Q. Therefore, if the center is not empty and c is in the center, then any $a, b \in Q$ can be written in the form $a = cx = xc$, $b = cy = yc$ for some $x, y \in Q$. Then (M) implies $ab = cx \cdot yc = cy \cdot xc = ba$, (M_e) implies $ab = xc \cdot cy = yc \cdot cx = ba$, (P) implies $ab = cx \cdot yc = cy \cdot xc = ba$, i.e. in any case $(Q, \cdot)$ is commutative. □

Proposition 5. *A loop is internally medial or externally medial if and only if it is an abelian group.*

Proof. $(\Rightarrow)$ If $(Q, \cdot)$ is internally medial or externally medial loop it is commutative, since a loop has nonempty center. Now, put the unit element for b and associativity follows. Sufficiency $(\Leftarrow)$ is obvious. □

Proposition 6. *A loop is palindromic if and only if it is commutative.*

Proof. Notice that a loop has nonempty center. □

Corollary 1. *A group is internally medial or externally medial or palindromic if and only if it is an abelian group.*

Remark 2. As we know, every commutative quasigroup (groupoid) is palindromic, but the converse is not true. If we take an abelian group $(Q, +)$, then quasigroup $(Q, -)$ satisfies (P), but is not commutative. Notice that $(Q, -)$ is internally medial and externally medial.

Proposition 7. *A quasigroup $(Q, \cdot)$ is palindromic if and only if exists its automorphism α such that*

$$\alpha(x \cdot y) = y \cdot x$$

holds for all $x, y \in Q$.

Proof. $(\Rightarrow)$ For arbitrary $a, b \in Q$ put $ab = u, ba = v$ and take permutation $\alpha = L_v^{-1} R_u$ (where L_v is left translation for v and R_u is right translation for u i.e. $L_v(x) = v \cdot x$ and $R_u(x) = x \cdot u$, for any $x \in Q$).

Then we have $L_v \alpha(xy) = R_u(xy) = xy \cdot ab = ba \cdot yx = L_v(yx)$ and therefore $\alpha(xy) = yx$. Further, for any $x, y \in Q$ taking $x = x_1x_2, y = y_1y_2$ we have $\alpha(xy) = yx = y_1y_2 \cdot x_1x_2 = x_2x_1 \cdot y_2y_1 = \alpha(x_1x_2) \cdot \alpha(y_1y_2) = \alpha(x) \cdot \alpha(y)$ i.e. α is an automorphism.

$(\Leftarrow)$ If α is an automorphism such that $\alpha(x \cdot y) = y \cdot x$ then follows $ab \cdot cd = \alpha(cd \cdot ab) = \alpha(cd) \cdot \alpha(ab) = dc \cdot ba$ i.e. quasigroup is palindromic. □

Proposition 8. *If $(Q, \cdot)$ is internally or externally medial quasigroup, then it satisfies Thomsen's closure condition, i.e.*

$$x_1y_2 = x_2y_1 \ \ \textit{and} \ \ x_1y_3 = x_3y_1 \ \ \textit{imply} \ \ x_2y_3 = x_3y_2$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in Q$. Therefore, any internally and any externally medial quasigroup is an abelian group isotope.

Proof. Let us suppose $x_1y_2 = x_2y_1$ and $x_1y_3 = x_3y_1$ hold and take $z \in Q$. Now, for internally medial quasigroup we get

$$x_2y_3 \cdot y_1z = x_2y_1 \cdot y_3z = x_1y_2 \cdot y_3z = x_1y_3 \cdot y_2z = x_3y_1 \cdot y_2z = x_3y_2 \cdot y_1z$$

and for externally medial quasigroup we have

$$x_2y_3 \cdot zx_1 = x_1y_3 \cdot zx_2 = x_3y_1 \cdot zx_2 = x_2y_1 \cdot zx_3 = x_1y_2 \cdot zx_3 = x_3y_2 \cdot zx_1.$$

Hence, in both cases, $x_2y_3 = x_3y_2$.

Since Thomsen's closure condition is valid in $(Q, \cdot)$ it follows that $(Q, \cdot)$ is isotopic to an abelian group (cf. [1], [5]). □

Proposition 9. *For a quasigroup $(Q, \cdot)$ and $e, f \in Q$ let us define binary operation $+$ on Q by*

$$xe \ + \ fy = xy$$

for all $x, y \in Q$. If $(Q, \cdot)$ is internally or externally medial quasigroup, then $(Q, +)$ is an abelian group.

Proof. It is well-known (and easy to check) that $(Q, +)$ is a loop (with the unity $0 = fe$). If $(Q, \cdot)$ is internally or externally medial quasigroup then it is isotopic to an abelian group and therefore loop $(Q, +)$ is an abelian group isotope too. Because of Albert's theorem (cf. [1]), $(Q, +)$ is an abelian group. □

Proposition 10. ([6], [4]) *Let $(Q, \cdot)$ be internally or externally medial quasigroup. Then there is an abelian group $(Q, +)$, an element $q \in Q$ and group automorphisms α, β such that*

$$x \cdot y = \alpha(x) + \beta(y) + q$$

for all $x, y \in Q$. For internally medial quasigroup $\alpha\beta = \beta\alpha$ is fulfilled, and for externally medial quasigroup $\alpha\alpha = \beta\beta$.

Proof. Let $(Q, +)$ be the abelian group defined in the previous proposition and $\varphi(x) = R_e(x) = xe$, $\psi(x) = L_f(x) = fx$ for all $x \in Q$. For internally medial quasigroup and externally medial quasigroup we get respectively

$$\varphi(\varphi(a) + \psi(b)) + \psi(\varphi(c) + \psi(d)) = \varphi(\varphi(a) + \psi(c)) + \psi(\varphi(b) + \psi(d)),$$

$$\varphi(\varphi(a) + \psi(b)) + \psi(\varphi(c) + \psi(d)) = \varphi(\varphi(d) + \psi(b)) + \psi(\varphi(c) + \psi(a)).$$

The first equality implies

$$\varphi(a + b) + \psi(\varphi(0) + \psi(0)) = \varphi(a + \psi(0)) + \psi(\varphi\psi^{-1}(b) + \psi(0)),$$

$$\varphi(\varphi(0) + \psi(0)) + \psi(c + d) = \varphi(\varphi(0) + \psi\varphi^{-1}(c)) + \psi(\varphi(0) + d),$$

and the second one gives

$$\varphi(a+b)+\psi(\varphi(0)+\psi(0))=\varphi(\varphi(0)+b)+\psi(\varphi(0)+\psi\varphi^{-1}(a)),$$

$$\varphi(\varphi(0)+\psi(0))+\psi(c+d)=\varphi(\varphi\psi^{-1}(d)+\psi(0))+\psi(c+\psi(0)).$$

In both cases it follows that there are such permutations $\varphi_1,\varphi_2,\psi_1,\psi_2$ on Q for which

$$\varphi(a+b)=\varphi_1(a)+\varphi_2(b),\quad \psi(c+d)=\psi_1(c)+\psi_2(d).$$

Hence, φ and ψ are quasi-automorphisms of the abelian group $(Q,+)$. It implies that there are automorphisms α,β of $(Q,+)$ and $q_1,q_2\in Q$ such that

$$\varphi(x)=\alpha(x)+q_1,\quad \psi(x)=\beta(x)+q_2\,.$$

Therefore, putting $q=q_1+q_2$ we have

$$x\cdot y=\alpha(x)+\beta(y)+q\,.$$

Now, for internally medial quasigroup we get

$$\alpha(\alpha(a)+\beta(b))+\beta(\alpha(c)+\beta(d))=\alpha(\alpha(a)+\beta(c))+\beta(\alpha(b)+\beta(d))$$

and putting $a=c=d=0$ we obtain $\alpha\beta=\beta\alpha$.

For externally medial quasigroup we have

$$\alpha(\alpha(a)+\beta(c))+\beta(\alpha(b)+\beta(d))=\alpha(\alpha(d)+\beta(c))+\beta(\alpha(b)+\beta(a))$$

and putting $b=c=d=0$ it follows $\alpha\alpha=\beta\beta$. □

Remark 3. It is widely known that K. Toyoda (cf. [6]) proved the previously mentioned proposition for internally medial quasigroups, which is commonly named Toyoda's theorem (see also [1], [5]). The proposition was proved in [4] for externally medial quasigroups (see also [2], [3]). We gave the above proof to stress that it is the same for both types of quasigroups, as is expected.

Any of the identities (M), (M_e), (P) is of the form

$$ab\cdot cd=(\pi(a)\cdot\pi(b))\cdot(\pi(c)\cdot\pi(d))$$

where π is a certain permutation on $\{a,b,c,d\}$. Therefore we would like to look on such identities on the quasigroups for any permutation π. Beside the identities $(M), (M_e), (P)$ and trivial identity $ab\cdot cd = ab\cdot cd$ which is fulfilled in any groupoid, we have the following twenty "medial-like" identities more:

$$ab \cdot cd = ab \cdot dc \quad (C_1), \qquad ab \cdot cd = ba \cdot cd \quad (C_2),$$

$$ab \cdot cd = ba \cdot dc \quad (C_3), \qquad ab \cdot cd = cd \cdot ab \quad (C_4),$$

$$ab \cdot cd = cd \cdot ba \quad (C_5), \qquad ab \cdot cd = dc \cdot ab \quad (C_6),$$

$$ab \cdot cd = ac \cdot db \quad (CM_1), \qquad ab \cdot cd = ad \cdot bc \quad (CM_2),$$

$$ab \cdot cd = ad \cdot cb \quad (CM_3), \qquad ab \cdot cd = bc \cdot ad \quad (CM_4),$$

$$ab \cdot cd = bc \cdot da \quad (CM_5), \qquad ab \cdot cd = bd \cdot ac \quad (CM_6),$$

$$ab \cdot cd = bd \cdot ca \quad (CM_7), \qquad ab \cdot cd = ca \cdot bd \quad (CM_8),$$

$$ab \cdot cd = ca \cdot db \quad (CM_9), \qquad ab \cdot cd = cb \cdot ad \quad (CM_{10}),$$

$$ab \cdot cd = cb \cdot da \quad (CM_{11}), \qquad ab \cdot cd = da \cdot bc \quad (CM_{12}),$$

$$ab \cdot cd = da \cdot cb \quad (CM_{13}), \qquad ab \cdot cd = db \cdot ac \quad (CM_{14}).$$

Proposition 11. *For a quasigroup $(Q, \cdot)$ and $i \in \{1, 2, \ldots, 6\}$, (C_i) is valid if and only if the quasigroup is commutative.*

Proof. $(\Leftarrow)$ is obvious. $(\Rightarrow)$ is evident for $(C_1), (C_2), (C_4)$. For (C_3) put $c = d$; for (C_5) put $c = b,\ d = a$; for (C_6) put $c = a,\ d = b$. □

Proposition 12. *For a quasigroup $(Q, \cdot)$ and $i \in \{1, 2, \ldots, 14\}$, (CM_i) holds if and only if the quasigroup is both commutative and internally medial.*

Proof. $((CM_1), \Leftarrow)$ is obvious.

$((CM_1), \Rightarrow)$ Put $c = b$ and commutativity follows; hence (M).

$((CM_2), \Leftarrow)$ $ab \cdot cd = ba \cdot cd = bc \cdot ad = ad \cdot bc$.

$((CM_2), \Rightarrow)$ Put $d = b$ and commutativity follows; therefore $ab \cdot cd = ba \cdot cd = bd \cdot ac = ac \cdot bd$.

$((CM_3), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_3) follows.

$((CM_3), \Rightarrow)$ Put $c = a$ and commutativity follows; hence (CM_2) and therefore (M).

$((CM_4), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_4) follows.

$((CM_4), \Rightarrow)$ Put $c = a$ and commutativity follows; hence (CM_2) and therefore (M).

$((CM_5), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_5) follows.

$((CM_5), \Rightarrow)$ Because of $ab \cdot cd = bc \cdot da = cd \cdot ab$ commutativity follows; hence (CM_2) and therefore (M).

$((CM_6), \Leftarrow)$ is obvious.

$((CM_6), \Rightarrow)$ Put $c = b$ and commutativity follows; hence (M).

$((CM_7), \Leftarrow)$ is obvious.

$((CM_7), \Rightarrow)$ Put $d = a$ and commutativity follows; hence (M).

$((CM_8), \Leftarrow)$ is obvious.

$((CM_8), \Rightarrow)$ Put $c = b$ and commutativity follows; hence (M).

$((CM_9), \Leftarrow)$ is obvious.

$((CM_9), \Rightarrow)$ Put $d = a$ and commutativity follows; hence (M).

$((CM_{10}), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_{10}) follows.

$((CM_{10}), \Rightarrow)$ Put $d = b$ and commutativity follows; hence (CM_2) and therefore (M).

$((CM_{11}), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_{11}) follows.

$((CM_{11}), \Rightarrow)$ Put $c = a$ and commutativity follows; hence (CM_2) and therefore (M).

$((CM_{12}), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_{12}) follows.

$((CM_{12}), \Rightarrow)$ Because of $ab \cdot cd = da \cdot bc = cd \cdot ab$ commutativity follows; hence (CM_2) and therefore (M).

$((CM_{13}), \Leftarrow)$ Commutative internally medial quasigroup satisfies (CM_2), hence (CM_{13}) follows.

$((CM_{13}), \Rightarrow)$ Put $d = b$ and commutativity follows; hence (CM_2) and therefore (M).

$((CM_{14}), \Leftarrow)$ is obvious.

$((CM_{14}), \Rightarrow)$ Put $d = a$ and commutativity follows; hence (M). □

Corollary 2. *For a quasigroup* $(Q, \cdot)$ *and* $i \in \{1, 2, \ldots, 14\}$, (CM_i) *is valid if and only if the quasigroup is both commutative and externally medial.*

Corollary 3. ([3]) *If* (CM_i) *is fulfilled in a quasigroup* $(Q, \cdot)$ *for some* $i \in \{1, 2, \ldots, 14\}$, *i.e. if* $(Q, \cdot)$ *is internally or externally medial quasigroup which is commutative, then there is an abelian group* $(Q, +)$, *an element* $q \in Q$ *and group automorphisms* α *such that*

$$x \cdot y = \alpha(x + y) + q$$

is valid for all $x, y \in Q$. □

References

[1] **V. D. Belousov**: *Osnovy teorii kvazigrupp i lup*, (Russian), Nauka, Moskva, 1967.

[2] **J. R. Cho, J. Ježek, T. Kepka**: *Paramedial groupoids*, Czechoslovak. Math. J. **49** (1999), 277 – 290.

[3] **W. Förg, A. Krapež**: *Equations which preserve the height of variables*, Aequationes Math, to appear.

[4] **P. Nemec, T. Kepka**: *T-quasigroups, Part I*, Acta Univ. Carol. Math. Phys. **12** (1971), 39 – 49.

[5] **H. O. Pflugfelder**: *Quasigroups and Loops: Introduction*, Sigma Ser. in Pure Math, vol. 7, Heldermann Verlag, Berlin 1990.

[6] **K. Toyoda**: *On axioms of linear functions*, Proc. Imp. Acad. Jap. **17** (1941), 221 – 227.

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University of Zagreb
Department of Mathematics
Bijenička c. 30
10000 Zagreb
Croatia