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Quasigroups constructed from cycle systems

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Abstract

An m-cycle system of order n is a pair (S, C), where C is a collection of edge disjoint m-cycles which partitions the edge set of the complete undirected graph K_n with vertex set S. If the m-cycle system (S, C) has the additional property that every pair of vertices $a \neq b$ are joined by a path of length 2 (and therefore exactly one) in an m-cycle of C, then (S, C) is said to be 2-perfect. Now given an m-cycle system (S, C) we can define a binary operation "$\circ$" on S by $a \circ a = a$ and if $a \neq b$, $a \circ b = c$ and $b \circ a = d$ if and only if the cycle $(\dots, d, a, b, c, \dots) \in C$. This is called the Standard Construction and it is well known that the groupoid $(S, \circ)$ is a quasigroup (which can be considered to be the "multiplicative" part of a universal algebra quasigroup $(S, \circ, \backslash, /)$) if and only if (S, C) is 2-perfect. The class of 2-perfect m-cycle systems is said to be *equationally defined* if and only if there exists a *variety* of universal algebra quasigroups V such that the finite members of V are *precisely* all universal algebra quasigroups whose multiplicative parts can be constructed from 2-perfect m-cycle systems using the Standard Construction. This paper gives a survey of results showing that 2-perfect m-cycle systems can be equationally defined for $m = 3, 5$, and 7 only. Similar results are obtained for m-perfect $(2m+1)$-cycle systems using the Opposite Vertex Construction (too detailed to go into here). We conclude with a summary of similar results (without details) for 2-perfect and m-perfect directed cycle systems.

1. Introduction

Sometimes people in combinatorics, algebra, and universal algebra see things differently. The following three definitions are a good illustration of this principle.

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Definition 1.1. An $n \times n$ *latin square* (or a *latin square of order* n) is an $n \times n$ array such that each of the integers $1, 2, 3, \ldots, n$ occurs exactly once in each row and column.

Example 1.2. Latin square of order 5.

1	3	2	5	4
5	2	4	1	3
4	5	3	2	1
3	1	5	4	2
2	4	1	3	5

Definition 1.3. A *quasigroup* is a pair $(Q, \circ)$, where "$\circ$" is a binary operation on Q such that for all not necessarily distinct $a, b \in Q$, the equations

$$\begin{cases} a \circ x = b, \\ y \circ a = b \end{cases}$$

have unique solutions.

The fact that the solutions are unique guarantees that no element occurs twice in any row or column of the table for "$\circ$". If Q is finite, each element occurs exactly once in each row and column, and hence the table for a *finite* quasigroup of order n is nothing more than a latin square of order n with a headline and sideline.

Example 1.4. Quasigroup of order 5.

$\circ$	1	2	3	4	5
1	1	3	2	5	4
2	5	2	4	1	3
3	4	5	3	2	1
4	3	1	5	4	2
5	2	4	1	3	5

Definition 1.5. A *universal algebra quasigroup* of order n is an ordered 4-tuple $(Q, \circ, \backslash, /)$, where "$\circ$", "$\backslash$", and "$/$" are binary operations on the set Q called "multiplication", "left division", and "right division" respectively, satisfying the four identities

$$\begin{cases} x \circ (x \backslash y) = y, \\ x \backslash (x \circ y) = y, \\ (x/y) \circ y = x, \\ (x \circ y)/y = x. \end{cases}$$

This definition is a good bit more complicated than the first two, necessitating a more detailed explanation.

To begin with each of $(Q, \circ)$, $(Q, \backslash)$, and $(Q, /)$ is a quasigroup. For example to see that $(Q, \backslash)$ is a quasigroup, let $a, b \in Q$. Then $a \backslash (a \circ b) = b$ guarantees that the equation $a \backslash x = b$ has a solution. Further, if $a \backslash x_1 = a \backslash x_2$, then $x_1 = a \circ (a \backslash x_1) = a \circ (a \backslash x_2) = x_2$ guarantees that the solution is unique. Similarly the equation $y \backslash a = b$ has a unique solution. An analogous argument shows that $(Q, \circ)$ and $(Q, /)$ are quasigroups as well. Furthermore, the binary operations "$\circ$", "$\backslash$", and "$/$" have the symbiotic relationships

$$\begin{cases} a \circ b = c & \text{if and only if} \quad a \backslash c = b, \\ a \circ b = c & \text{if and only if} \quad c/b = a. \end{cases}$$

The first of these follows from the identities $x \backslash (x \circ y) = y$ and $x \circ (x \backslash y) = y$, while the second follows from the identities $(x \circ y)/y = x$ and $(x/y) \circ y = x$. Because of this symbiotic relationship only one of "$\circ$", "$\backslash$", and "$/$ is necessary to define all three. In everything that follows we will always use $(Q, \circ)$ to define $(Q, \circ, \backslash, /)$.

Example 1.6. Universal algebra quasigroup of order 5.

$\circ$	1	2	3	4	5
1	1	3	2	5	4
2	5	2	4	1	3
3	4	5	3	2	1
4	3	1	5	4	2
5	2	4	1	3	5

$\backslash$	1	2	3	4	5
1	1	3	2	5	4
2	4	2	5	3	1
3	5	4	3	1	2
4	2	5	1	4	3
5	3	1	4	2	5

$/$	1	2	3	4	5
1	1	4	5	2	3
2	5	2	1	3	4
3	4	1	3	5	2
4	3	5	2	4	1
5	2	3	4	1	5

On the other hand, *any* quasigroup $(Q, \circ)$ can be considered to be the multiplication part of a universal algebra quasigroup as follows: define "$\backslash$" and "$/$" in terms of "$\circ$" by

$$\begin{cases} a \backslash b = c & \text{if and only if} \quad a \circ c = b, \\ a/b = c & \text{if and only if} \quad c \circ b = a. \end{cases}$$

It is not difficult to see that $(Q, \circ, \backslash, /)$ satisfies the four identities $x \circ (x \backslash y) = y$, $x \backslash (x \circ y) = y$, $(x/y) \circ y = x$, and $(x \circ y)/y = x$. For example to see that the identity $(x \circ y)/y = x$ is satisfied, let $a \circ b = c$. Then $c/b = a$ and so $(a \circ b)/b = c/b = a$. The proofs that the other identities are satisfied by $(Q, \circ, \backslash, /)$ are just as easy.

Hence we can think of *any* quasigroup $(Q, \circ)$ as being the "multiplicative" part of a universal algebra quasigroup.

From now on we will use juxtaposition to indicate "multiplicative" in quasigroup identities. So, for example, the defining identities for a quasigroup become $x(x\backslash y) = y$, $x\backslash(xy) = y$, $(x/y)y = x$, and $(xy)/y = x$.

Now all of this might seem unnecessary at first, but for what we are going to do in this paper it is necessary! Here's the reason why. We are going to talk about *varieties* of quasigroups; i.e., classes of quasigroups defined by sets of quasigroup identities. Hence we need the universal algebra definition of a quasigroup.

2. A small amount of universal algrebra

Since any quasigroup can be considered to be the "multiplicative" part of a universal algebra quasigroup we will frequently drop the quantification "universal algebra" in front of quasigroup. The context will make clear what we are talking about.

A *variety* of quasigroups is a class of *universal algebra quasigroups* which is closed under the taking of subquasigroups, direct products, and homomorphic images. A very famous theorem due to G. Birkhoff [6] says that a variety V of quasigroups can be *equationally defined.* That is to say, if V is a variety of quasigroups, there exists a collection of quasigroup identities I such that V is *precisely* the set of *all* quasigroups which satisfy these identities. The identities I are called a *defining set* of identities for the variety V. (Actually Birkhoff proved a much more general result than this, but we are interested in quasigroups only, and so have edited the statement of Birkhoff's Theorem to quasigroups.) There is, of course, nothing unique about a defining set of identities. The converse is trivial; i.e., if I is a collection of quasigroup identities, the class of all quasigroups satisfying these identities is closed under the taking of subquasigroups, direct products, and homomorphic images, and so is a variety. Hence, to prove that a class of quasigroups $\mathcal{C}$ is NOT a variety, it suffices to produce a quasigroup in $\mathcal{C}$ having a homomorphic image which does NOT belong to $\mathcal{C}$. The following Folk Theorem and Folk Corollary are exactly what is needed to do this.

Theorem 2.1 (Folk Theorem). *The mapping α is a homomorphism of the universal algebra quasigroup $(Q_1, \circ_1, \backslash_1, /_1)$ onto the universal algebra quasigroup $(Q_2, \circ_2, \backslash_2, /_2)$ if and only if α is a homomorphism of $(Q_1, \circ_1)$ onto $(Q_2, \circ_2)$.*

Proof. One way is trivial. So let α be a homomorphism of the quasigroup $(Q_1, \circ_1)$ onto the quasigroup $(Q_2, \circ_2)$. Let $a\backslash_1 b = c$. Then $a \circ_1 c = b$, $(a \circ_1 c)\alpha = b\alpha$, $a\alpha \circ_2 c\alpha = b\alpha$, and $a\alpha\backslash_2 b\alpha = c\alpha$.

Similarly $a/_1 b = c$ gives $c \circ_1 b = a$, $(c \circ_1 b)\alpha = a\alpha$, $c\alpha \circ_2 b\alpha = a\alpha$, and $a\alpha/_2 b\alpha = c\alpha$. □

Corollary 2.2 (Folk Corollary). *A class of universal algebra quasigroups is closed under the taking of homomorphic images if and only if its class of multiplicative parts is closed under the taking of homomorphic images.* □

Hence, in order to show that a class of universal algebra quasigroups $\mathcal{C}$ is NOT a variety it suffices to construct a universal algebra quasigroup belonging to $\mathcal{C}$ whose multiplicative part has a homomorphic image onto a quasigroup which cannot be the multiplicative part of a universal algebra quasigroup belonging to $\mathcal{C}$.

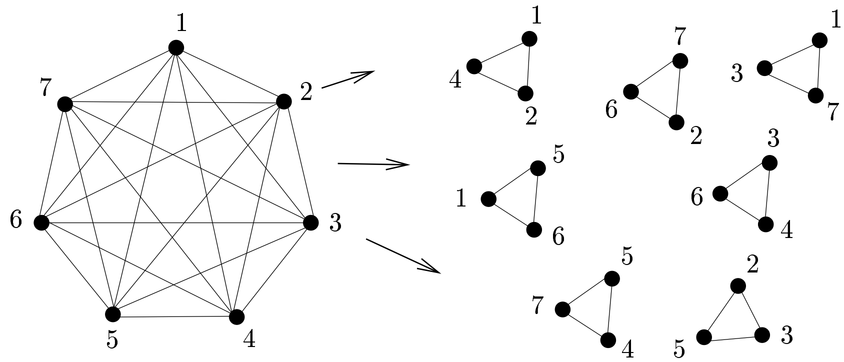
The object of this survey is an account of the struggle to achieve the solution to the problem of determining whether or not certain classes of quasigroups obtained from decomposing the edge set of the complete undirected graph into cycles form the *finite* members of a variety of quasigroups. We will now be a good deal more specific than this! And what better place to start than with Steiner triple systems!

3. Steiner triple systems

A *Steiner triple system* (or triple system) of order n is a pair (S, T), where T is a collection of edge disjoint triangles which partition the edge set of K_n (= the complete undirected graph on n vertices) with vertex set S.

It is well-known [12] that the *spectrum* (= the set of all n such that a triple system of order n exists) for triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$.

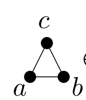
Example 3.1. Steiner triple system of order 7.



$\mathbf{K_7}$

Now given a triple system (S, T) we can define a groupoid $(S, \circ)$ as follows (the Standard Construction):

(1) $a \circ a = a$, for all $a \in S$, and

(2) if $a \neq b$, $a \circ b = b \circ a = c$, where  $\in T$.

Example 3.2. Groupoid constructed from Example 3.1.

$\circ$	1	2	3	4	5	6	7
1	1	4	7	2	6	5	3
2	4	2	5	1	3	7	6
3	7	5	3	6	2	4	1
4	2	1	6	4	7	3	5
5	6	3	2	7	5	1	4
6	5	7	4	3	1	6	2
7	3	6	1	5	4	2	7

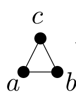
Inspection shows that the groupoid $(S, \circ)$ constructed above is actually a quasigroup which, as previously noted, is the multiplicative part of the universal algebra quasigroup $(S, \circ, \backslash, /)$.

Not only is $(S, \circ)$ a quasigroup, but it satisfies each of the *equivalent* sets of identities:

$$I_1 = \begin{cases} x^2 = x \\ (yx)x = y \\ xy = yx \end{cases} \qquad \text{and} \qquad I_2 = \begin{cases} x\backslash x = x \\ (yx)\backslash y = x \\ (yx)/y = x \end{cases}$$

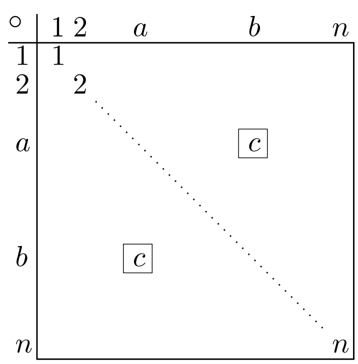
(A quasigroup satisfies the identities I_1 if and only if it satisfies the identities I_2.) In what follows we will *always* use the identities I_1.

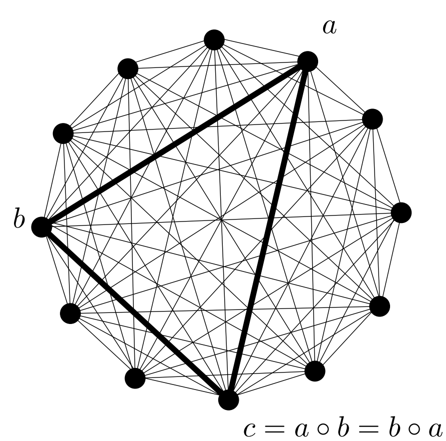
It turns out that the groupoid constructed from *any* triple system using the Standard Construction is always a quasigroup and always satisfies the identities I_1.

Denote the triangle 

by any cyclic shift of (a,b,c) or (b,a,c) and let (S,T) be a triple system and $(S,\circ)$ the groupoid constructed from T using the Standard Construction. Suppose $a \circ x = a \circ y$. If $a = x$, then $a = a \circ x = a \circ y$ implies $y = a$, since otherwise $(a,y,d) \in T$ and $a \circ y = d \neq a$. If $a \neq x$, then $a \neq y$, and (a,x,c), $(a,y,c) \in T$ implies $x = y$. Hence $(S,\circ)$ is row latin (= each element occurs exactly once in each row). Trivially $a \circ b = b \circ a$ ($(S,\circ)$ is commutative) and so $(S,\circ)$ is column latin as well. Hence $(S,\circ)$ is a quasigroup. As noted above $(S,\circ)$ satisfies $x^2 = x$ and $xy = yx$. To see that $(S,\circ)$ satisfies $(yx)x = y$ as well is easy. To begin with $(a \circ a) \circ a = a \circ a = a$. If $a \neq b$ and $(a,b,c) \in T$, then $(a \circ b) \circ b = c \circ b = a$.

What is of extreme importance to us in this discussion is that the converse is also true. That is to say, any finite quasigroup satisfying the three identities $x^2 = x$, $(yx)x = y$, and $xy = yx$ can be constructed from a triple system using the Standard Construction. So, let $(S,\circ)$ be a quasigroup of order n satisfying the three identities above, and define a collection C of triangles as follows: for each $a \neq b \in S$ place the triangle $(a, b, a \circ b = b \circ a = c)$ in C.





K_n

In order to show that the triangles in C are an edge disjoint collection of triangles which partition the edge set of K_n we must show that (i) every edge is in a triangle of C and (ii) the triangle $(a, b, a \circ b = b \circ a = c)$ constructed from the edge $\{a,b\}$ is the same triangle as the triangle constructed from each of the edges $\{b,c\}$ and $\{c,a\}$. Trivially each edge is in a triangle of C and so we can proceed to (ii). This is where the identities come into play. The triangle constructed from $\{b,c\}$ is $(b,c,b\circ c = c\circ b = (a\circ b)\circ b = a) = (b,c,a) = (a,b,c)$ and the triangle constructed from $\{c,a\}$ is $(c,a,c\circ a = a\circ c = a\circ(a\circ b) = b) = (c,a,b) = (a,b,c)$. Hence (S,C) is a triple system.

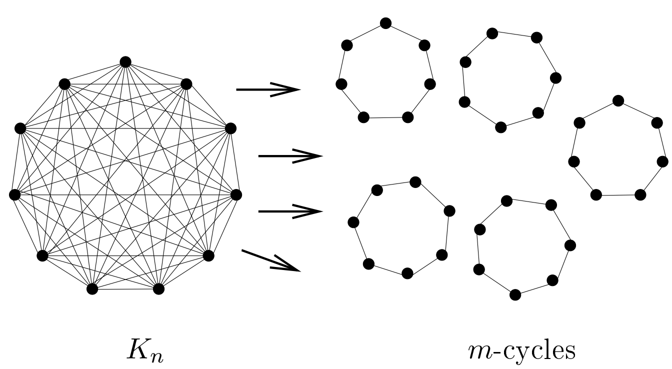
Not only is (S,C) a triple system but the triangles (a,b,c) in C all have the property that $a\circ b = b\circ a = c$, $b\circ c = c\circ b = a$, and $a\circ c = c\circ a = b$. It follows that if we apply the Standard Construction to (S,C) we get the quasigroup $(S,\circ)$ that we started with. We have the following theorem.

Theorem 3.3. *Let V be the variety of quasigroups defined by the identities $x^2 = x$, $xy = yx$, and $(yx)x = y$. A finite quasigroup belongs to V if and only if its multiplicative part can be constructed from a Steiner triple system using the Standard Construction.* □

Remark. Among other things Theorem 3.3 says that the spectrum for the finite quasigroups in the variety defined by the identities $x^2 = x$, $xy = yx$, and $(yx)x = y$ is precisely the set of all $n \equiv 1$ or $3 \pmod 6$, since this is the spectrum for Steiner triple systems. There is no particular reason for the defining identities to all be "multiplicative". However, the identities $I_1 = \{x^2 = x,\ xy = yx,\ (yx)x = y\}$ have been used "forever" to define *Steiner quasigroups* and there's no sense in changing now!

4. m-cycle systems

An *m-cycle system* is a pair (S,C), where C is a collection of edge disjoint m-cycles which partition the edge set of the complete undirected graph K_n with vertex set S. The number n is called the order of the m-cycle system (S,C).

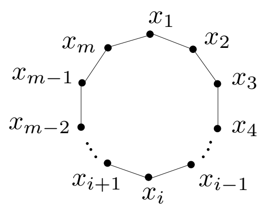


So, for example, a Steiner triple system is a 3-cycle system.

Fairly recently, the necessary and sufficient conditions for the existence of an m-cycle system of order n have been determined to be [1, 30];

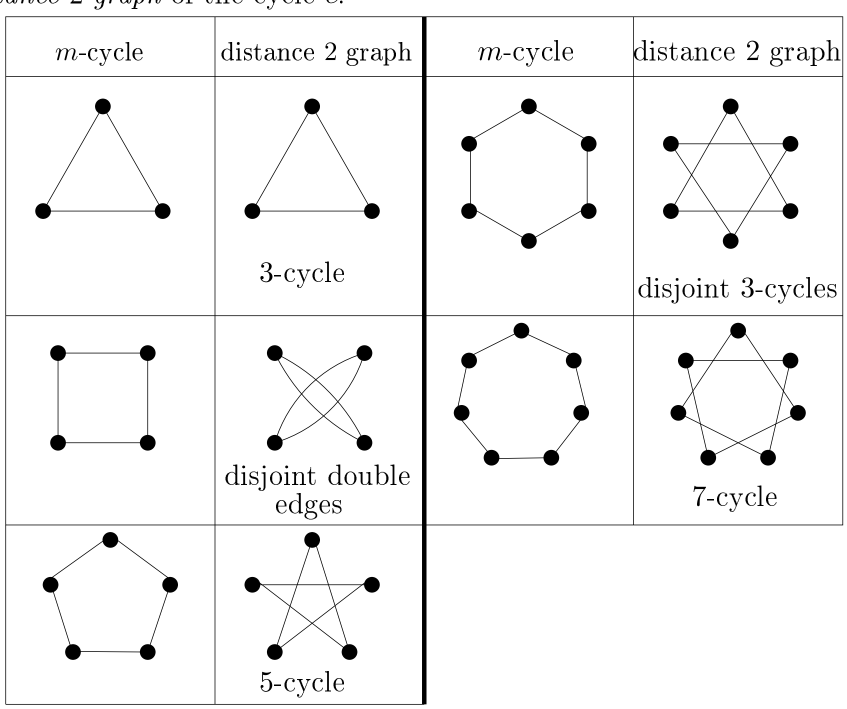
$$\begin{cases} (1) \quad n \geqslant m, \text{ if } n > 1, \\ (2) \quad n \text{ is odd, and} \\ (3) \quad n(n-1)/2m \text{ is an integer.} \end{cases}$$

In what follows we will denote the m-cycle



by any cyclic shift of $(x_1, x_2, x_3, \ldots, x_n)$ or $(x_1, x_m, x_{m-1}, x_{m-2}, \ldots, x_2)$.

If $c = (x_1, x_2, x_3, \ldots, x_m)$ is an m-cycle we will denote by $c(2)$ the collection of edges $\{x_i, x_{i+2}\}, i = 1, 2, \ldots, m$. The graph $c(2)$ is called the *distance* 2 *graph* of the cycle c.



Distance 2 graphs of m-cycles for $m = 3, 4, 5, 6,$ and 7

An m-cycle system (S, C) of *order* n is said to be 2-*perfect* provided the collection of graphs $C(2) = \{c(2) \mid c \in C\}$ covers the edge set of K_n. This is *equivalent* to saying that for every pair of vertices $a \neq b$, there is *exactly* one cycle of the form $(\cdots, a, x, b, \cdots) \in C$; i.e., *exactly* one cycle in C in which a and b are joined by a path of length 2.

Example 4.1. (two 6-cycle systems of order 13, one 2-perfect and one not 2-perfect.)

(i) 2-*perfect*: $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, and
$C_1 = \{(5, 9, 11, 8, 13, 12), (6, 10, 12, 9, 1, 13), (7, 11, 13, 10, 2, 1),$
$(8, 12, 1, 11, 3, 2), (9, 13, 2, 12, 4, 3), (10, 1, 3, 13, 5, 4), (11, 2, 4, 1, 6, 5),$
$(12, 3, 5, 2, 7, 6), (13, 4, 6, 3, 8, 7), (1, 5, 7, 4, 9, 8), (2, 6, 8, 5, 10, 9),$
$(3, 7, 9, 6, 11, 10), (4, 8, 10, 7, 12, 11)\}$.

(ii) *not* 2-*perfect*: $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, and
$C_2 = \{(1, 2, 13, 3, 12, 7), (2, 3, 1, 4, 13, 8), (3, 4, 2, 5, 1, 9), (4, 5, 3, 6, 2, 10),$
$(5, 6, 4, 7, 3, 11), (6, 7, 5, 8, 4, 12), (7, 8, 6, 9, 5, 13), (8, 9, 7, 10, 6, 1),$
$(9, 10, 8, 11, 7, 2), (10, 11, 9, 12, 8, 3), (11, 12, 10, 13, 9, 4),$
$(12, 13, 11, 1, 10, 5), (13, 1, 12, 2, 11, 6)\}$.

None of the edges $\{1, 4\}, \{2, 5\}, \{3, 6\}, \{4, 7\}, \{5, 7\}, \{6, 9\}, \{7, 10\}, \{8, 11\},$
$\{9, 12\}, \{10, 13\}, \{1, 11\}, \{2, 12\}, \{3, 13\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\},$
$\{6, 12\}, \{7, 12\}, \{1, 8\}, \{2, 9\}, \{3, 10\}, \{4, 11\}, \{5, 12\}, \{6, 13\}$
are covered by the graphs in $C_2(2)$. *Check it out*!

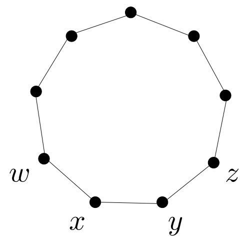
Although the spectrum for m-cycle systems (= set of all n such that an m-cycle system of order n exists) has recently been settled, the spectrum for m-cycle systems with the very strong additional property of being 2-perfect is far from settled. The spectrum for 2-perfect m-cycle systems has been determined for $m = 3, 5, 6,$ and 7 as well as for few other values of m [28]. However, knowing the spectrum for 2-perfect m-cycle systems is not necessary in what follows.

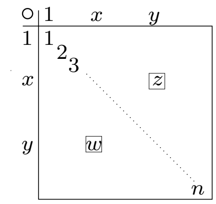
Given an m-cycle system (S, C) we can define a binary operation "$\circ$" on S called the *Standard Construction* (an *extrapolation* of the Standard Construction for Steiner triple systems) as follows:

THE STANDARD CONSTRUCTION

(1) $x \circ x = x$, for all $x \in S$, and

(2) if $x \neq y$, $x \circ y = z$ and $y \circ x = w$ if and only if $(\cdots, w, x, y, z, \cdots) \in C$.





Example 4.2. (The Standard Construction applied to the 6-cycle systems in Example 4.1.)

$\circ_1$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	10	13	2	7	5	11	9	12	4	3	8	6
2	7	2	11	1	3	8	6	12	10	13	4	5	9
3	10	8	3	12	2	4	9	7	13	11	1	6	5
4	6	11	9	4	13	3	5	10	8	1	12	2	7
5	8	7	12	10	5	1	4	6	11	9	2	13	3
6	4	9	8	13	11	6	2	5	7	12	10	3	1
7	2	5	10	9	1	12	7	3	6	8	13	11	4
8	5	3	6	11	10	2	13	8	4	7	9	1	12
9	13	6	4	7	12	11	3	1	9	5	8	10	2
10	3	1	7	5	8	13	12	4	2	10	6	9	11
11	12	4	2	8	6	9	1	13	5	3	11	7	10
12	11	13	5	3	9	7	10	2	1	6	4	12	8
13	9	12	1	6	4	10	8	11	3	2	7	5	13

$\circ_2$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	13	2	13	2	10	12	9	3	5	13	2	6
2	7	2	1	3	1	3	11	13	10	4	6	1	3
3	4	8	3	2	4	2	4	12	1	11	5	7	2
4	3	5	9	4	3	5	3	5	13	2	12	6	8
5	9	4	6	10	5	4	6	4	6	1	3	13	7
6	8	10	5	7	11	6	5	7	5	7	2	4	1
7	2	9	11	6	8	12	7	6	8	6	8	3	5
8	6	3	10	12	7	9	13	8	7	9	7	9	4
9	5	7	4	11	13	8	10	1	9	8	10	8	10
10	11	6	8	5	12	1	9	11	2	10	9	11	9
11	10	12	7	9	6	13	2	10	12	3	11	10	12
12	13	11	13	8	10	7	1	3	11	13	4	12	11
13	12	1	12	1	9	11	8	2	4	12	1	5	13

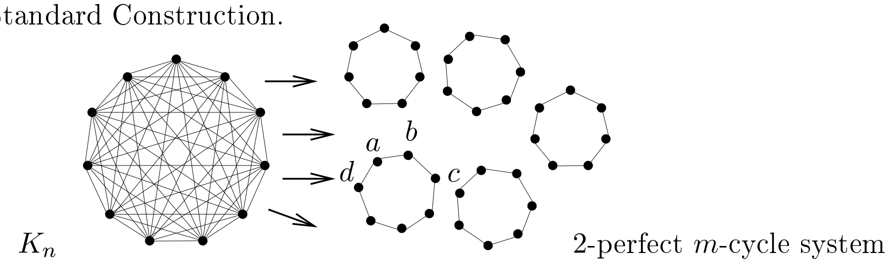
Now a cursory glance at the above example shows that the groupoid obtained from the 2-perfect 6-cycle system in Example 4.1 using the Standard Construction is a quasigroup, while the groupoid obtained from the 6-cycle system which is not 2-perfect using the Standard Construction is nowhere close to being a quasigroup. (For example, $1 \circ_2 2 = 1 \circ_2 4 = 13$.) The natural question to ask at this point is: does being 2-perfect have anything to do with the groupoid constructed from an m-cycle system being a quasigroup? The following Folk Theorem shows that the answer to this question is "**you bet it does**!"

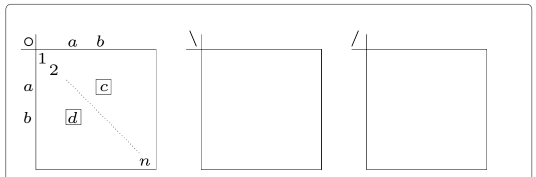
Theorem 4.3 (Folk Theorem). *Let (S, C) be an m-cycle system. The groupoid constructed from (S, C) using the Standard Construction is a quasigroup if and only if (S, C) is 2-perfect.*

Proof. Suppose (S, C) is 2-perfect. We need to show that for all a, $b \in S$ the equations $a \circ x = b$ and $y \circ a = b$ have *unique* solutions. Now $a \circ a = a$ by definition for all $a \in S$; and we cannot have $a \circ b = a$ or $c \circ a = a$ for $b \neq a$ or $c \neq a$, since we cannot have cycles in C that look like $(\cdots, a, b, a \cdots)$ or $(\cdots c, a, a, \cdots)$. Hence $a \circ x = a$ and $y \circ a = a$ have unique solutions. So let $a \neq b \in S$ and $(\cdots, y, a, b, \cdots) \in C$. Then $y \circ a = b$. This is unique since the edge $\{a, b\}$ belongs to exactly one m-cycle of C. Since (S, C) is 2-perfect there is *exactly* one cycle $(\cdots, a, x, b, \cdots) \in C$ and so $a \circ x = b$ is unique. Hence $(S, \circ)$ is a quasigroup.

Now assume $(S, \circ)$ is a quasigroup and let $a \neq b \in S$. Then the equation $a \circ x = b$ has a *unique* solution and so there is *exactly* one m-cycle of the form $(\cdots, a, x, b, \cdots)$ in C. Hence (S, C) is 2-perfect. $\square$

In what follows we will say that the class of 2-perfect m-cycle systems is *equationally defined* if and only if there exists a variety of quasigroups V with the property that the finite quasigroups in V are precisely the quasigroups whose multiplicative parts can be constructed from 2-perfect m-cycle systems using the Standard Construction. In other words, $(Q, \circ, \backslash, /) \in V$ if and only if $(Q, \circ)$ can be constructed from a 2-perfect m-cycle system using the Standard Construction.



$V =$ 

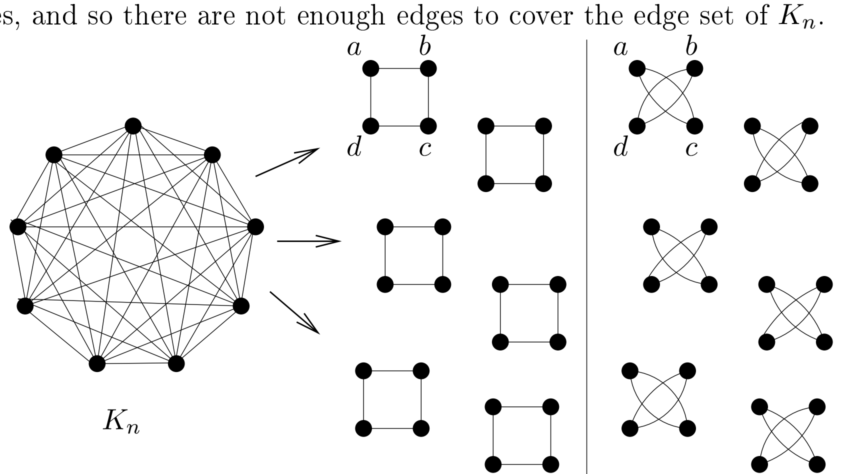
A finite quasigroup belongs to the variety of quasigroups V iff its multiplicative part can be constructed from a 2 perfect m cycle system using the Standard Construction.

Question. *For which $m \geqslant 3$ is the class of 2-perfect m-cycle systems equationally defined?*

Since a triangle is a 3-cycle, Theorem 3.3 shows that the class of Steiner triple systems is equationally defined. We will now show that the class of 2-perfect m-cycle systems is equationally defined for $m = 3, 5$, and 7 ONLY. In each case we will give a set of defining identities for the variety.

5. 4-cycle systems

It is well-known (see A. Kotzig [13]) that the spectrum for 4-cycle systems is the set of all $n \equiv 1 \pmod 8$. Unfortunately, 4-cycle systems are *never* 2-perfect. This is easy to see. Let (S, C) be a 4-cycle system of order n. Then $|C| = \binom{n}{2}/4$. The distance 2 graph of each 4-cycle (a, b, c, d) in C consists of the four edges $\{a, b\}, \{a, b\}, \{c, d\}, \{c, d\}$. Hence a distinct listing of the edges belonging to the distance 2 graphs contains at most $2|C| = \binom{n}{2}/2$ edges, and so there are not enough edges to cover the edge set of K_n.



Distance 2 graphs of a 4-cycle system

So much for 4-cycle systems!

6. 5-cycle systems

Unlike 3-cycle systems which are always 2-perfect and 4-cycle systems which are never 2-perfect, 5-cycle systems, just like the 6-cycle systems in Example 4.1, are sometimes 2-perfect and sometimes not 2-perfect.

Example 6.1. 2-perfect 5-cycle system of order 5 and two 5-cycle systems of order 11; one 2-perfect and the other not 2-perfect.

(1) 2-*perfect*: $S = \{1, 2, 3, 4, 5\}$, and $C = \{(1, 2, 3, 4, 5), (1, 3, 5, 2, 4)\}$.

(2) 2-*perfect*: $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and
$C_1 = \{(1, 3, 9, 5, 4), (2, 4, 10, 6, 5)$, $(3, 5, 11, 7, 6)$, $(4, 6, 1, 8, 7)$, $(5, 7, 2, 9, 8)$, $(6, 8, 3, 10, 9)$, $(7, 9, 4, 11, 10)$, $(8, 10, 5, 1, 11)$, $(9, 11, 6, 2, 1)$, $(10, 1, 7, 3, 2)$, $(11, 2, 8, 4, 3)\}$.

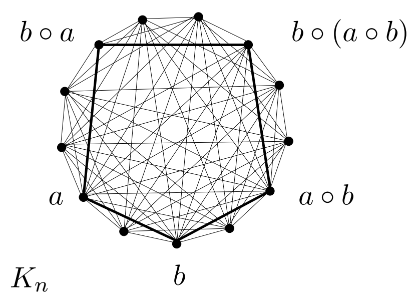
(3) *Not* 2-*perfect*: $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and
$C_2 = \{(1, 3, 10, 5, 4)$, $(2, 4, 11, 6, 5)$, $(3, 5, 1, 7, 6)$, $(4, 6, 2, 8, 7)$, $(5, 7, 3, 9, 8)$, $(6, 8, 4, 10, 9)$, $(7, 9, 5, 11, 10)$, $(18, 10, 6, 1, 11)$, $(9, 11, 7, 2, 1,)$, $(10, 1, 8, 3, 2)$, $(11, 2, 9, 4, 3)\}$.

In 1966 Alex Rosa [29] proved that the spectrum for 5-cycle systems is precisely the set of all $n \equiv 1$ or 5 (mod 10). Except for the unique 5-cycle system of order 5, none of the 5-cycle systems constructed by Rosa are 2-perfect. Almost 20 years later the 2-perfect spectrum for 5-cycle systems was determined by C. C. Lindner and D. R. Stinson [25] who showed that the 2-perfect spectrum is the same as the spectrum for 5-cycle systems, except that there does not exist a 2-perfect 5-cycle system of order 15. This is all quite interesting, but just as for 3-cycle systems, it plays no part in the determination of whether or not the class of 2-perfect 5-cycle systems is equationally defined.

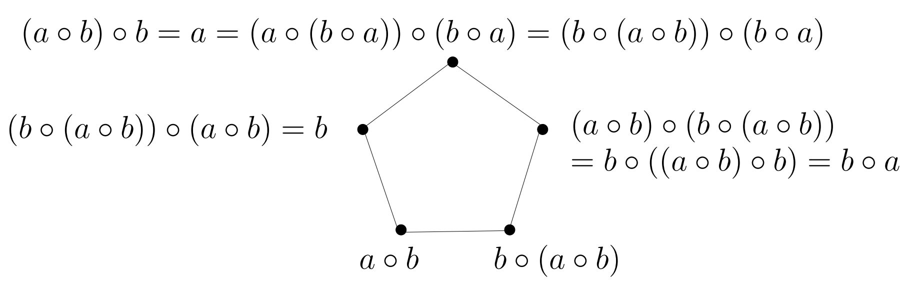
Theorem 6.2. *The class of* 2-*perfect* 5-*cycle systems can be equationally defined. The set of identities* $x^2 = x$, $(yx)x = y$, *and* $x(yx) = y(xy)$ *is a defining set of identities.*

Proof. Let V be the variety of quasigroups defined by the identities $x^2 = x$, $(yx)x = y$, and $x(yx) = y(xy)$. Let (S, C) be a 2-perfect 5-cycle system and $(S, \circ)$ the quasigroup constructed from (S, C) using the Standard Construction. To begin with, $(S, \circ)$ satisfies $x^2 = x$ by definition and so the other two identities are satisfied for $a = b$. Now suppose $a \neq b$ and $(a, b, c, d, e) \in C$. Then $(a \circ b) \circ b = c \circ b = a$ and $a \circ (b \circ a) = a \circ e = d = b \circ c = b \circ (a \circ b)$ and so the identities $(yx)x = y$ and $x(yx) = y(xy)$ are satisfied as well.

Hence $(S, \circ)$ belongs to the variety V. To finish the proof we need to show that every finite quasigroup belonging to V (= satisfying the defining set of identities) can be constructed from a 2-perfect 5-cycle system using the Standard Construction. The proof is more or less the same as for 3-cycle systems, except a bit more tedious. So, let $(S, \circ)$ be a quasigroup of order n satisfying the defining identities and define a collection of 5-cycles C as follows: for each $a \neq b \in S$, $(a, b, a \circ b, b \circ (a \circ b), b \circ a) \in C$.



We need to show that (i) $a, b, a\circ b, b\circ(a\circ b)$, and $b\circ a$ are distinct (so that $(a, b, a \circ b, b \circ (a \circ b), b \circ a)$ is indeed a 5-cycle), (ii) every edge of K_n belongs to a 5-cycle of C, (iii) each edge belonging to $(a, b, a \circ b, b \circ (a \circ b), b \circ a)$ determines exactly the same 5-cycle, and (iv) the Standard Construction applied to (S, C) gives the quasigroup $(S, \circ)$ that we started with. Parts (i) and (ii) are straightforward so we will go straight to (iii). We will show that the edge $\{a \circ b, b \circ (a \circ b)\}$ determines the same 5-cycle as the edge $\{a, b\}$. The best way to do this is with a picture.



The other cases are similar. This shows that (S, C) is a 2-perfect 5-cycle system (Theorem 4.3). Inherent in the proof of (iii) is $(x, a, b, y, z) \in C$ if and only if $a \circ b = y$ and $b \circ a = x$. Hence the Standard Construction applied to (S, C) gives the quasigroup $(S, \circ)$ that we started with (proving (iv)). $\square$

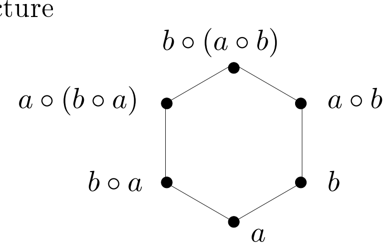
Remark . As with 3-cycle systems, there is no particular reason that the defining identities are all "multiplicative", other than the fact that the author happens to like them. Other collections involving all three operations are possible. For example, $x^2 = x$, $yx = y/x$, and $y/x = x\backslash(y(xy))$. The only requirement is that a quasigroup satisfies the defining identities if and only if its multiplicative part can be constructed from a 2-perfect 5-cycle system using the Standard Construction.

7. 2-perfect 6-cycle systems

The spectrum for 6-cycle systems is precisely the set of all $n \equiv 1$ or 9 (mod 12). This was determined by Alex Rosa and Charlotte Huang [11]. The 2-perfect spectrum is another matter and was determined in 1991 by C. C. Lindner, K. T. Phelps, C. Rodger, and E. J. Billington [4, 20] to be the same as for 6-cycle systems with the exception of $n = 9$, for which no 2-perfect 6-cycle system exists. As with the previous cases, knowing the 2-perfect spectrum has nothing to do with the problem of whether or not the class of 2-perfect 6-cycle systems can be equationally defined. In what follows a quasigroup $(S, \circ)$ is said to be *antisymmetric* provided $a \circ b \neq b \circ a$ for all $a \neq b \in S$. Denote by $\mathcal{C}$ the class of *all* finite antisymmetric quasigroups satisfying the three identities $x^2 = x, (yx)x = y$, and $(xy)(y(xy)) = x(yx)$.

Theorem 7.1. *$\mathcal{C}$ consists precisely of all quasigroups which can be constructed from 2-perfect 6-cycle systems using the Standard Construction.*

Proof. It is straightforward to see that the quasigroup constructed from a 2-perfect 6-cycle system (S, C) using the Standard Construction satisfies the three identities $x^2 = x$, $(yx)x = y$, and $(xy)(x(xy)) = x(yx)$. Antisymmetry comes from the fact that $(\cdots, d, a, b, c, \cdots) \in C$ gives $a \circ b = c \neq d = b \circ a$. Now let $(S, \circ)$ be an antisymmetric quasigroup satisfying the three identities above. Define a collection of 6-cycles C as follows: for each $a \neq b \in S$ place the 6-cycle $(a, b, a \circ b, b \circ (a \circ b), a \circ (b \circ a), b \circ a)$ in C. The proof that (S, C) is a 2-perfect 6-cycle system from which $(S, \circ)$ can be constructed using the Standard Construction follows the proof in Theorem 6.2 using the picture



□

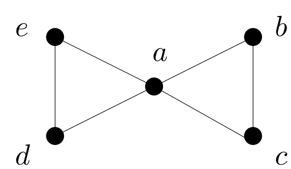
Corollary 7.2. *If I is a defining set of identities for 2-perfect 6-cycle systems and $V(I)$ is the variety of quasigroups defined by I, then the finite members of $V(I)$ are $\mathcal{C}$.* □

Here's where the trouble begins. Varieties are defined by identities *not* properties, and being antisymmetric is a property. Hence the three identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy)) = x(yx)$ PLUS antisymmetry does NOT define a variety of quasigroups. So the problem of determining whether or not the class of 2-perfect 6-cycle systems is equationally defined is equivalent to proving or disproving the existence of a collection of quasigroup identities I so that the variety $V(I)$ defined by I has the property that the finite members of $V(I)$ are $\mathcal{C}$. This remained an open problem for years until D. E. Bryant proved in 1992 that no such variety exists [7, 8]. The proof that Bryant gave was to construct a 2-perfect 6-cycle system (S, C) such that the quasigroup $(S, \circ) \in \mathcal{C}$ constructed from (S, C) using the Standard Construction has a homomorphic image onto a quasigroup which cannot be constructed from a 2-perfect 6-cycle system using the Standard Construction and so does not belong to $\mathcal{C}$. It then follows that $\mathcal{C}$ cannot constitute the finite members of a variety $V(I)$, since any homomorphic image of any quasigroup in $\mathcal{C}$ would have to be in $V(I)$, and since it is finite would have to be in $\mathcal{C}$ as well (and therefore constructable from a 2-perfect 6-cycle system using the Standard Construction).

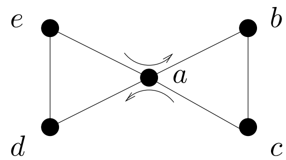
We give a sketch of Bryant's proof in the next section.

8. 2-perfect 6-cycle systems cannot be equationally defined

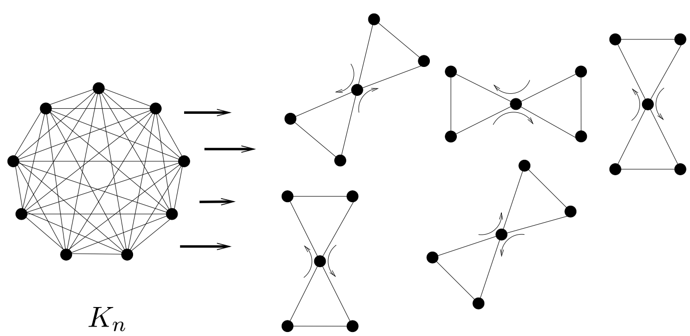
In order to prove that the class of 2-perfect 6-cycle systems cannot be equationally defined we will need to use a decomposition of K_n called a *bowtie system*. This calls for a definition. A *bowtie* is a closed 6-trail of the form (a, b, c, a, d, e), where a, b, c, d, and e are *distinct*. So that there is no confusion, the closed 6-trail (a, b, c, a, d, e) consists of the 6 edges $\{a, b\}$, $\{b, c\}$, $\{c, a\}$, $\{a, d\}$, $\{d, e\}$, and $\{e, a\}$. Now, the graph of these edges is



which is also the graph of the bowtie (a, b, c, a, e, d). To differentiate between these two bowties we will use the picture

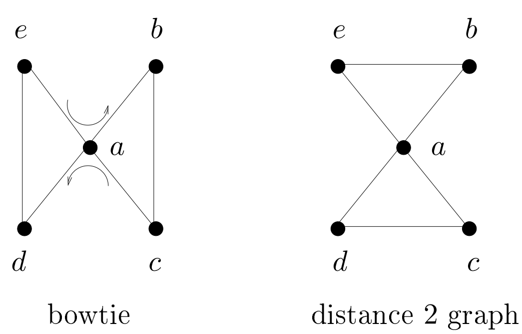


to represent the bowtie (a, b, c, a, d, e). A *bowtie system* of order n is a pair (S, B), where B is a collection of bowties which partition the edge set of K_n with vertex set S.



Bowtie system of order n

Just as with m-cycle systems, the bowtie system (S, B) is said to be 2-perfect provided the collection of distance 2 graphs of the bowties in B covers the edge set of K_n.



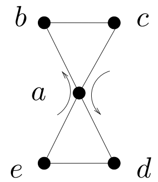
Theorem 8.1 (E. J. Billington and C. C. Lindner [5]). *The spectrum for 2-perfect bowtie systems is the set of all $n \equiv 1$ or $9 \pmod{12}$, $n \geqslant 21$.* $\square$

Now given a 2-perfect bowtie system (S, B) we can define a binary operation "$\circ$" on S called the Standard Construction as follows:

THE STANDARD CONSTRUCTION

(1) $x \circ x = x$, for all $x \in S$, and

(2) if $x \neq y$, $x \circ y = z$ and $y \circ x = w$ if and only if $(\cdots w, x, y, z, \cdots) \in B$.



$\circ$	a	b	c	d	e	n
a		$\boxed{c}$	$\boxed{b}$	$\boxed{e}$	$\boxed{d}$	
b	$\boxed{e}$		$\boxed{a}$			
c	$\boxed{d}$	$\boxed{a}$				
d	$\boxed{c}$				$\boxed{a}$	
e	$\boxed{b}$			$\boxed{a}$		
n						n

Just as with the Standard Construction for m-cycle systems the groupoid $(S, \circ)$ is a quasigroup if and only if (S, B) is 2-perfect. The proof follows the proof of Theorem 4.3. With all of the above information in hand we can now sketch the proof of the following theorem.

Theorem 8.2 (D. E. Bryant [7, 8]). *The class of 2-perfect 6-cycle systems CANNOT be equationally defined.*

Proof. We will construct a quasigroup of order 273 which can be constructed from a 2-perfect 6-cycle system using the Standard Construction having a homomorphic image of order 21 which cannot be constructed from a 2-perfect 6-cycle system using the Standard Construction.

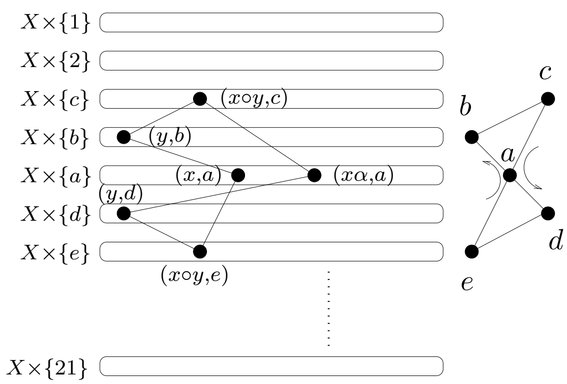
Let (X, C_1) be the 2-perfect 6-cycle system of order 13 in Example 4.1 and let (Y, B) be a 2-perfect bowtie system of order 21 (see Theorem 8.1). Let $S = X \times Y$ and define a collection of 6-cycles C_2 as follows:

(1) $((x, i), (y, i), (z, i), (u, i), (v, i), (w, i)) \in C_2$ for every $(x, y, z, u, v, w) \in C_1$ and every $i \in Y$, and

(2) let $(X, \circ)$ be any quasigroup of order 13, α a derangement on X, and for each $(a, b, c, a, d, e) \in B$ and each x, $y \in X$ (x, y not necessarily distinct) place the 6-cycle $((x, a), (y, b), (x \circ y, c), (x\alpha, a), (y, d), (x \circ y, e))$ in C_2.

It is straightforward to see that (S, C_2) is a 2-perfect 6-cycle system.

place a copy of (X, C_1) on each $X \times \{i\}$



Now let $(S, \circ_2)$ be the quasigroup constructed from (S, C_2) using the Standard Construction and $(Y, \circ_3)$ the quasigroup constructed from (Y, B) using the Standard Construction.

Define the mapping $\beta : S \overset{\text{onto}}{\longrightarrow} Y$ by $(x, i)\beta = i$. It is straightforward to see that β is a homomorphism of $(S, \circ_2)$ onto $(Y, \circ_3)$ (and therefore a homomorphism of $(S, \circ_2, \backslash_2, /_2)$ onto $(Y, \circ_3, \backslash_3, /_3)$). Now let (a, b, c, a, d, e) be any bowtie in (Y, B). Then $b \circ_3 c = a = c \circ_3 b, b \neq c$, and so $(Y, \circ_3)$ is definitely NOT antisymmetric. Since the multiplicative part of a quasigroup constructed from a 2-perfect 6-cycle system using the Standard Construction is *always* antisymmetric, $(Y, \circ_3)$ cannot be constructed from a 2-perfect 6-cycle system using the Standard Construction. It follows that the class of 2-perfect 6-cycle systems cannot be equationally defined. □

9. 7-cycle systems

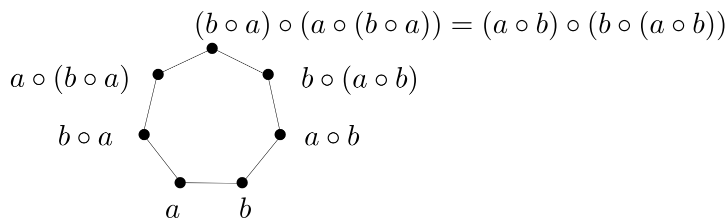
So far we have shown that the classes of 2-perfect 3-cycle and 5-cycle systems can be equationally defined, the class of 2-perfect 6-cycle systems cannot be equationally defined, and 4-cycle systems are not even worth discussing.

The spectrum for 7-cycle systems is the set of all $n \equiv 1$ or 7 (mod 14). (See A. Rosa [29].) The spectrum for 2-perfect 7-cycle systems is *exactly the same* as for 7-cycle systems and was determined in 1991 by Elisabetta Manduchi [26].

It turns out that the class of 2-perfect 7-cycle systems can be equationally defined.

Theorem 9.1. *The class of* 2*-perfect* 7*-cycle systems can be equationally defined. The set of identities* $x^2 = x$, $(yx)x = y$, *and* $(xy)(y(xy)) = (yx)(x(yx))$ *is a defining set of identities.*

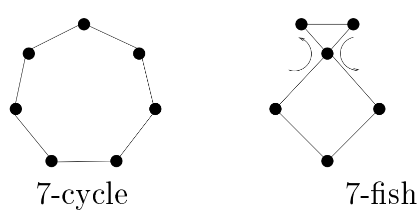
Proof. Let V be the variety of quasigroups defined by the identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy)) = (yx)(x(yx))$, and let (S, C) be a 2-perfect 7-cycle system and $(S, \circ)$ the quasigroup constructed from (S, C) using the Standard Construction. Since $a \circ a = a$, for all $a \in S$, by definition, all three identities are satisfied for $a = b$. Now suppose $a \neq b$ and let $(a, b, c, d, e, f, g) \in C$. Then $(b \circ a) \circ a = g \circ a = b$ and $(a \circ b) \circ (b \circ (a \circ b)) = c \circ (b \circ c) = c \circ d = e = g \circ f = (b \circ a) \circ (a \circ g) = (b \circ a) \circ (a \circ (b \circ a))$ and so the identities $(yx)x = y$ and $(xy(y(xy)) = (yx)(x(yx))$ are satisfied. It follows that every quasigroup constructed from a 2-perfect 7-cycle system using the Standard Construction belongs to the variety V defined by the three identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy)) = (yx)(x(yx))$. We must now show that every finite quasigroup belonging to the variety V can be constructed from a 2-perfect 7-cycle system using the Standard Construction. The proof is a bit tedious, but perfectly straightforward, and follows the proofs in Theorems 6.2 and 7.1 using the picture



□

Remark. Other collections of defining identities are possible of course, including collections involving all three operations.

Remark. The interested reader may feel a bit *uneasy* at this point for the following reason. Maybe the class of 2-perfect 7-cycle systems really cannot be equationally defined. Why not copy the argument in Theorem 8.2 to construct a quasigroup from a 2-perfect 7-cycle system having a homomorphism onto a quasigroup constructed from a 2-perfect "closed 7-trail system" so that the quasigroup constructed from this closed 7-trail system cannot be constructed from a 2-perfect 7-cycle system? The answer is simple: there are only two closed trails of length 7; here they are!



Closed trails of length 7

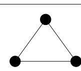
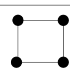
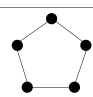
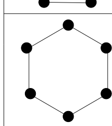
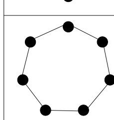
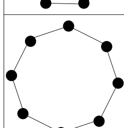
Since 4-cycle systems cannot be 2-perfect, "7-fish systems" cannot be 2-perfect. Hence the argument in Theorem 8.2 is not possible.

10. 2-perfect m-cycle systems cannot be equationally defined for m $\geqslant$ 8

Let's recap what we've done so far: the classes of 2-perfect 3, 5, and 7 cycle systems can be equationally defined; the class of 6-cycle systems *cannot* be equationally defined; and the Standard Construction applied to 4-cycle systems never gives a quasigroup. In [9] it is shown that the class of 2-perfect m-cycle systems cannot be equationally defined for $m \geqslant slant8$. The construction to show this is an extrapolation of the construction used to show that 2-perfect 6-cycle systems cannot be equationally defined. Although the construction is similar the details are extremely tedious and since this is a survey with the intent of popularizing connections between universal algebra and graph theory, the author has decided to omit these details and refer the interested reader to [9, 19].

Theorem 10.1. *If $m \geqslant slant8$, 2-perfect m-cycle systems cannot be equationally defined.* □

11. Summary of results for 2-perfect m-cycle systems

K_n	*2 perfect spectrum*	*Equationally defined*
	$n \equiv 1$ *or* $3(mod 6)$ *T.P.Kirkman* [12]	YES $\begin{cases} x^2 = x \\ (yx)x = y \\ xy = yx \end{cases}$
		
	$n \equiv 1$ *or* $5(mod\ 10)$ $n \neq 15$ *C.C.Lindner* *D.R.Stinson* [25]	YES $\begin{cases} x^2 = x \\ (yx)x = y \\ x(yx) = y(xy) \end{cases}$
	$n \equiv 1$ *or* $9(mod\ 12)$ $n \neq 9$ *E.J.Billington* *C.C.Lindner* *K.T.Phelps* *C.A.Rodger* [4, 20]	NO *D.E.Bryant* [7, 8]
	$n \equiv 1(mod\ 14)$ *or* $n \equiv 7(mod\ 14)$ *E.Manduchi* [26]	YES $\begin{cases} x^2 = x \\ (yx)x = y \\ (xy)(y(xy)) = \\ (yx)(x(yx)) \end{cases}$
 m*−cycle* $m \geqslant 8$	*See* [28]	NO *D.E.Bryant* *C.C.Lindner* [9]

12. The Opposite Vertex Construction

Now it doesn't take the wisdom of a saint to see that there are lots of binary operations that can be defined on an m-cycle system other than the Standard Construction. One such binary operation is the *Opposite Vertex Construction.*

Let (S, C) be an m-cycle system of order n and denote by $C(k)$ the collection of distance k graphs of cycles in C. If the graphs in $C(k)$ partition K_n with vertex set S, then (S, C) is said to be *k-perfect.* This is *equivalent* to saying that for every pair of vertices $a \neq b$, there is *exactly* one cycle belonging to C in which a and b are joined by a path of length k. Up to now we have considered only 2-perfect m-cycle systems.

Now given a $(2m+1)$-cycle system we can define an idempotent $(x^2 = x)$ commutative $(xy = yx)$ groupoid as follows.

The opposite vertex construction

Let (S, C) be a $(2m+1)$ cycle system and define a binary operation $\circ$ by:
(1) $x \circ x = x$, all $x \in S$, and
(2) if $x \neq y, x \circ y = y \circ x =$ the vertex opposite the edge $\{x, y\}$ in the cycle containing $\{x, y\}$.

It is immediate that $(S, \circ)$ is a quasigroup if and only if (S, C) is *m-perfect.*

Example 12.1. 2-perfect 5-cycle system of order 11 and quasigroup constructed using the Opposite Vertex Construction.

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $C = \{(1, 9, 4, 3, 5), (2, 10, 5, 4, 6), (3, 11, 6, 5, 7), (4, 1, 7, 6, 8), (5, 2, 8, 7, 9), (6, 3, 9, 8, 10), (7, 4, 10, 9, 11), (8, 5, 11, 10, 1), (9, 6, 1, 11, 2), (10, 7, 2, 1, 3), (11, 8, 3, 2, 4)\}$.

$\circ$	1	2	3	4	5	6	7	8	9	10	11
1	1	10	7	6	4	2	8	11	3	5	9
2	10	2	11	8	7	5	3	9	1	4	6
3	7	11	3	1	9	8	6	4	10	2	5
4	6	8	1	4	2	10	9	7	5	11	3
5	4	7	9	2	5	3	11	10	8	6	1
6	2	5	8	10	3	6	4	1	11	9	7
7	8	3	6	9	11	4	7	5	2	1	10
8	11	9	4	7	10	1	5	8	6	3	2
9	3	1	10	5	8	11	2	6	9	7	4
10	5	4	2	11	6	9	1	3	7	10	8
11	9	6	5	3	1	7	10	2	4	8	11

Inspection reveals that this quasigroup satisfies the 3 quasigroup identities $I(5) = \{x^2 = x,\ xy = yx,\ ((xy)\backslash x)y = (xy)\backslash y\}$.

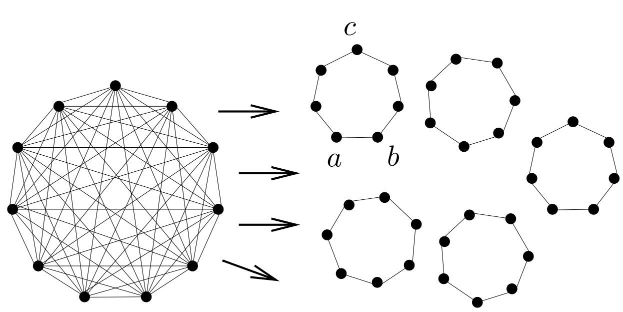
Example 12.2. 3-perfect 7-cycle system of order 7 and quasigroup constructed using the Opposite Vertex Construction.
$S = \{1, 2, 3, 4, 5, 6, 7\}$,
$C = \{(1, 2, 3, 4, 5, 6, 7),\ (1, 3, 5, 7, 2, 4, 6),\ (1, 4, 7, 3, 6, 2, 5)\}$.

$\circ$	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

Inspection reveals that this quasigroup satisfies the 3 quasigroup identities $I(7) = \{x^2 = x,\ xy = yx,\ x\backslash((xy)\backslash x) = ((xy)\backslash x)(y\backslash((xy)\backslash y))\}$.

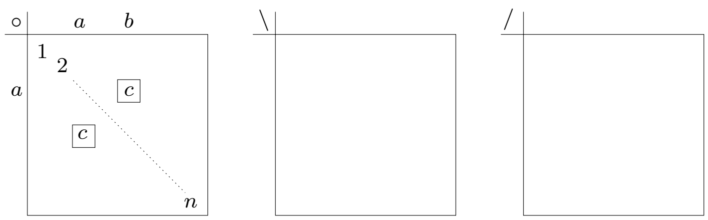
Examples 12.1 and 12.2 *illustrate* the fact (which is easy to prove) that the quasigroups constructed from 2-perfect 5-cycle systems using the Opposite Vertex Construction *always* satisfy the identities in $I(5)$ and the quasigroups constructed from 3-perfect 7-cycle systems using the Opposite Vertex Construction *always* satisfy the identities in $I(7)$.

In what follows to say that m-perfect $(2m + 1)$-cycle systems are *equationally defined* means that there exists a variety of quasigroups V such that a *finite* quasigroup belongs to the variety V if and only if its multiplicative part can be constructed from an m-perfect $(2m+1)$-cycle system using the Opposite Vertex Construction.



K_n m-perfect $(2m + 1)$-cycle system

$V =$



A *finite* quasigroup belongs to the variety of quasigroups V iff its multiplicative part can be constructed from an m-perfect $2m+1$-cycle system using the Opposite Vertex Construction.

We will show that m-perfect $(2m+1)$-cycle systems can be equationally defined for $m = 1, 2$, and 3 only. We already know that 1-perfect 3-cycle systems (= Steiner triple systems) can be equationally defined, since the Opposite Vertex Construction and the Standard Construction are the same for 3-cycles.

The following two lemmas establish a fundamental relationship between 2-perfect $(2m+1)$-cycle systems and m-perfect $(2m+1)$-cycle systems.

Lemma 12.3. *If* (Q, C) *is a* 2*-perfect* $(2m+1)$*-cycle system, then* $C(2)$ *is an* m*-perfect* $(2m+1)$*-cycle system and* $C(2)(m) = C$*. Furthermore, if* $(Q, \circ_1, \backslash_1, /_1)$ *is the quasigroup constructed from* (Q, C) *using the Standard Construction and* $(Q, \circ_2, \backslash_2, /_2)$ *is the quasigroup constructed from* $C(2)$ *using the Opposite Vertex Construction, then* $\circ_1 = \backslash_2, \backslash_1 = \circ_2$*, and* $/_1$ *and* $/_2$ *are transposes.*

Proof. Let $(x_1, x_2, x_3, \ldots, x_{2m+1}) \in C$. Then

$$(x_1, x_3, x_5, \ldots, x_{2m+1}, x_2, x_4, \ldots, x_{2m}) \in C(2).$$

Since (Q, C) is 2-perfect, $(Q, C(2))$ is a $(2m+1)$-cycle system. It is immediate that the distance m graph of $(x_1, x_3, x_5, \ldots, x_{2m+1}, x_2, x_4, \ldots, x_{2m})$ is $(x_1, x_2, x_3, \ldots, x_{2m+1})$ and so $C(2) = C$ and $(Q, C(2))$ is m-perfect. Now let $(Q, \circ_1, \backslash_1, /_1)$ be the quasigroup constructed from (Q, C) using the Standard Construction and $(Q, \circ_2, \backslash_2, /_2)$ the quasigroup constructed from $(Q, C(2))$ using the Opposite Vertex Construction. Then

$$\begin{cases} x_1 \circ_1 x_2 = x_3, & x_1 \circ_2 x_3 = x_2 \\ x_1 \backslash_1 x_3 = x_2, & x_1 \backslash_2 x_2 = x_3 \\ x_3 /_1 x_2 = x_1, & x_2 /_2 x_3 = x_1. \end{cases}$$

It follows that $\circ_1 = \backslash_2, \backslash_1 = \circ_2$, and $/_1$ and $/_2$ are transposes. $\square$

Lemma 12.4. *If (Q, C) is an m-perfect $(2m{+}1)$-cycle system, then $C(m)$ is a 2-perfect $(2m{+}1)$-cycle system and $C(m)(2) = C$. Further, if $(Q, \circ_1, \backslash_1, /_1)$ is the quasigroup constructed from (Q, C) using the Opposite Vertex Construction and $(Q, \circ_2, \backslash_2, /_2)$ the quasigroup constructed from $C(m)$ using the Standard Construction, then $\circ_1 = \backslash_2$, $\backslash_1 = \circ_2$, and $/_1$ and $/_2$ are transposes.*

Proof. Similar to Lemma 12.3. $\square$

Now let $w(x, y)$ be any quasigroup word in the free quasigroup on the two generators x and y. Denote by $sw(x, y)$ the word obtained from $w(x, y)$ by replacing "$\circ$" with "$\backslash$", "$\backslash$" with "$\circ$", and any subword of the form "$a(x, y)/b(x, y)$" with "$b(x, y)/a(x, y)$". If I is any set of quasigroup identities set $S(I) = \{sw(x, y) = sv(x, y) \mid w(x, y) = v(x, y) \in I\}$. (Note that $S(S(I)) = I$.) We have the following lemma.

Lemma 12.5. *If $(Q, \circ_1, \backslash_1, /_1)$ and $(Q, \circ_2, \backslash_2, /_2)$ are quasigroups where $\circ_1 = \backslash_2$, $\backslash_1 = \circ_2$, and $/_1$ and $/_2$ are transposes, then one of these quasigroups satisfies the set of identities I if and only if the other quasigroup satisfies the identities $S(I)$.* $\square$

Example 12.6. Let $(Q, \circ_1, \backslash_1, /_1)$ and $(Q, \circ_2, \backslash_2, /_2)$ be given by the accompanying quasigroups. Then $\circ_1 = \backslash_2, \backslash_1 = \circ_2$, and $/_1$ and $/_2$ are transposes. It is straightforward to see that $(Q, \circ_1, \backslash_1, /_1)$ satisfies the identities

$$I = \{x^2 = x, \quad y(x/(xy)) = (xy)\backslash y, \quad ((xy)\backslash y)x = x/(xy)\}$$

and $(Q, \circ_2, \backslash_2, /_2)$ satisfies the identities

$$S(I) = \{x\backslash x = x, \quad y\backslash((x\backslash y)/x) = (x\backslash y)y, \quad ((x\backslash y)y)\backslash x = (x\backslash y)/x\}.$$

$\circ_1$	1	2	3	4	5
1	1	4	2	5	3
2	4	2	5	3	1
3	2	5	3	1	4
4	5	3	1	4	2
1	3	1	4	2	5

$\circ_2$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Lemma 12.7. *m-perfect $(2m+1)$-cycle systems can be equationally defined if and only if 2-perfect $(2m+1)$-cycle systems can be equationally defined.*

Proof. Suppose 2-perfect $(2m+1)$-cycle systems are equationally defined and let I be a defining set of identities. *Claim*: $S(I)$ is a defining set of identities for m-perfect $(2m+1)$-cycle systems. If $(Q, \circ_1, \backslash_1, /_1)$ satisfies $S(I)$, then the quasigroup $(Q, \circ_2, \backslash_2, /_2)$ satisfies I, where $\circ_2 = \backslash_1, \backslash_2 = \circ_1$, and $/_2$ and $/_1$ are transposes (Lemma 12.5). Let (Q, C) be the 2-perfect $(2m+1)$-cycle system from which $(Q, \circ_2, \backslash_2, /_2)$ is constructed using the Standard Construction. Then $(Q, C(2))$ is m-perfect and if $(Q, \circ_3, \backslash_3, /_3)$ is the quasigroup constructed from $(Q, C(2))$ using the Opposite Vertex Construction, then $\circ_3 = \backslash_2 = \circ_1$, $\backslash_3 = \circ_2 = \backslash_1$, and $/_3$ and $/_2$ are transposes (Lemma 12.3). Since $/_2$ and $/_1$ are transposes $/_3 = /_1$. Hence $(Q, \circ_3, \backslash_3, /_3) = (Q, \circ_1, \backslash_1, /_1)$ and so $(Q, \circ_1, \backslash_1, /_1)$ can be constructed from an m-perfect $(2m+1)$-cycle system using the Opposite Vertex Construction.

Now let (Q, C) be m-perfect. Then $(Q, C(m))$ is 2-perfect and so the quasigroup $(Q, \circ_2, \backslash_2, /_2)$ constructed from $(Q, C(m))$ using the Standard Construction satisfies the identities I. If $(Q, \circ_1, \backslash_1, /_1)$ is the quasigroup constructed from (Q, C) using the Opposite Vertex Construction, then (Lemma 12.4) $\circ_1 = \backslash_2$, $\backslash_1 = \circ_2$, and $/_1$ and $/_2$ are transposes. Hence by Lemma 12.5 the quasigroup $(Q, \circ_1, \backslash_1, /_1)$ satisfies the identities $S(I)$. Combining all of the above shows that if 2-perfect $(2m+1)$-cycle systems are equationally defined then so are m-perfect $(2m+1)$-cycle systems.

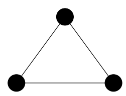
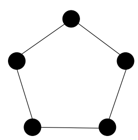
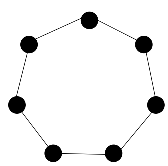
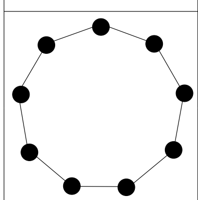
The proof of the converse is identical. □

Theorem 12.8 (C. C. Lindner and C. A. Rodger [21]). *m-perfect $(2m+1)$-cycle systems can be equationally defined for $m = 1, 2$ and 3 only.*

Proof. 2-perfect $(2m+1)$-cycle systems can be equationally defined for $m = 1, 2$, and 3 only. □

13. Summary of results for m-perfect $(2m+1)$-cycle systems

The accompaning table is a summary of the results on equationally defining m-perfect $(2m+1)$-cycle systems (using the Opposite Vertex Construction). The defining identities in each case are not necessarily “translations” of the corresponding identities used to define 2-perfect $(2m+1)$-cycle systems. It is nevertheless straightforward to see that, in fact, they are defining identities. The reason for their inclusion here is that they are appealing to the author.

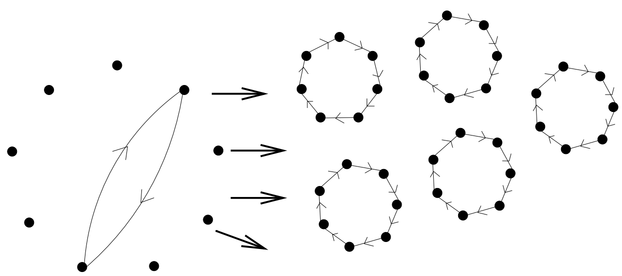
K_n	m-perfect spectrum	Equationally defined (using the Opposite Vertex Construction)
	$n \equiv 1$ or $3 \pmod 6$ Steiner triple system [12]	YES $\left\{\begin{array}{l} x^2 = x \\ xy = yx \\ (yx)x = y \end{array}\right.$
	$n \equiv 1$ or $5 \pmod{10}$ $n \neq 15$ C.C.Lindner D.R.Stinson [25]	YES $\left\{\begin{array}{l} x^2 = x \\ xy = yx \\ ((xy) \setminus x)y = (xy) \setminus y \end{array}\right.$
	$n \equiv 1$ or $7 \pmod{14}$ E.Manduchi [26]	YES $\left\{\begin{array}{l} x^2 = x \\ xy = yx \\ x \setminus ((xy) \setminus x) = \\ ((xy) \setminus x)(y \setminus ((xy) \setminus y)) \end{array}\right.$
 $(2m+1)$-cycle $2m+1 \geqslant 9$	See [28]	NO C.C.Lindner C.A.Rodger [21]

14. Directed cycle systems

A natural question to ask at this point is: "are there analogues of the Standard Construction and Opposite Vertex Construction for *directed m-cycle systems* (as opposed to the results we have surveyed so far = undirected m-cycle systems)?" The answer is YES!

Since there is a limit to the length of this paper, we will give here the directed analogues of the Standard and Opposite Vertex Constructions *without details*. The interested reader can find plenty of details in [10, 21].

A *directed m-cycle system* of order n is a pair (S, C), where C is an edge disjoint collection of directed m-cycles which partitions the edge set of D_n (the complete directed graph on n vertices) with vertex set S.

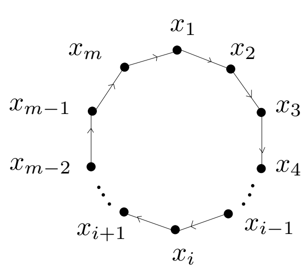


D_n directed m-cycles

Quite recently, the necessary and sufficient conditions for the existence of a directed m-cycle system of order n have been determined to be [2]:

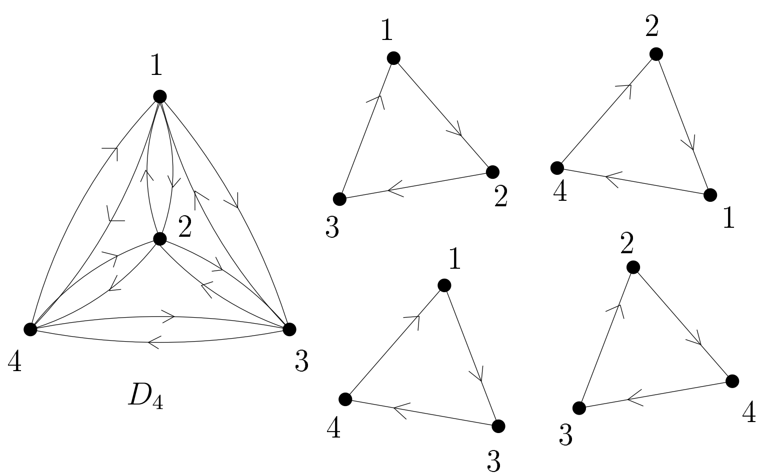
$$\begin{cases} (1) \ \ n \geqslant m, \text{ if } n > 1, \\ (2) \ \ n(n-1)/m \quad \text{is an integer, and} \\ (3) \ \ (n,m) \neq (4,4),(6,3), \text{ or } (6,6). \end{cases}$$

In what follows we will denote the directed m-cycle

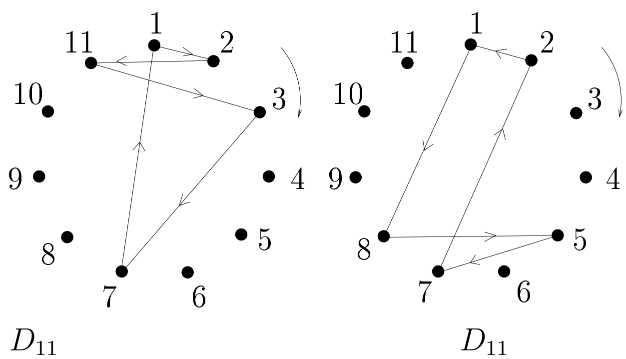


by any cyclic shift of $< x_1, x_2, x_3, \ldots, x_m >$ and the edge from a to b by $< a, b >$.

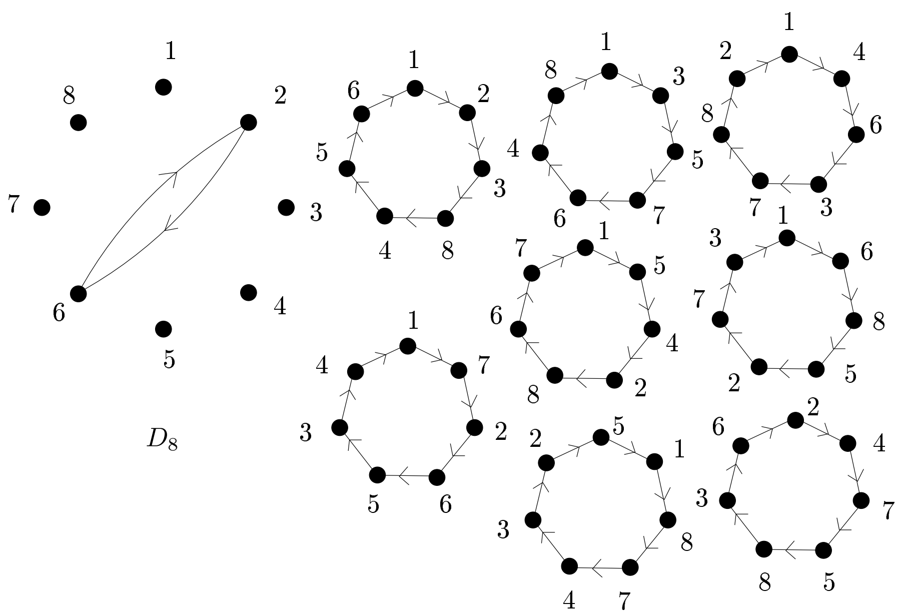
Example 14.1. Directed 3-cycle system of order 4.



Example 14.2. Directed 5-cycle system of order 11.



Example 14.3. Directed 7-cycle system of order 8.



Now given a directed cycle system (S, C) we can define two binary operations on S as follows:

The directed standard construction

Let (S, C) be a directed m-cycle system of order n and define a binary operation $\circ$ on S by:

(1) $x \circ x = x$, for all $x \in S$, and

(2) if $x \neq y$, $x \circ y = z$ if and only if $< \ldots, x, y, z, \cdots > \in S$.

THE DIRECTED OPPOSITE VERTEX CONSTRUCTION

Let (S, C) be a directed $(2m+1)$-cycle system of order n and define a binary operation $\circ$ on S by:

(1) $x \circ x = x$, for all $x \in S$, and

(2) if $x \neq y, x \circ y =$ the vertex opposite the edge $< x, y >$ in the directed cycle containing $< x, y >$.

CAUTION. Since the edges $< a, b >$ and $< b, a >$ belong to different cycles, the vertex opposite $< a, b >$ is not necessarily the same vertex as the vertex opposite $< b, a >$. For example, in Example 14.2 the vertex opposite the edge $< 1, 2 >$ is 3, while the vertex opposite the edge $< 2, 1 >$ is 5.

The directed m-cycle system (S, C) of order n is said to be *k-perfect* if and only if $(S, C(k))$ partitions D_n, where $C(k)$ is the collection of distance k graphs of the cycles in C. This is equivalent to saying that for each $a \neq b \in S$, a and b are connected by a path of length k from a to b in a cycle of C and a path of length k from b to a in a cycle of C.

It is straightforward to see that the directed 3-cycle system of order 4 in Example 14.1 is 2-perfect; the directed 5-cycle system in Example 14.2 is NOT 2-perfect; and the directed 7-cycle system in Example 14.3 is 3-perfect.

As with undirected cycles, the groupoid (S, C) constructed from a directed m-cycle system using the Directed Standard Construction is a quasigroup if and only if (S, C) is 2-perfect and the groupoid constructed from a directed $(2m+1)$-cycle system using the Directed Opposite Vertex Construction is a quasigroup if and only if (S, C) is m perfect. This is easy to prove (so we will omit the proof).

Example 14.4. Quasigroup constructed from Example 14.1 using the Directed Standard Construction.

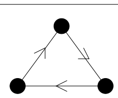
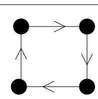
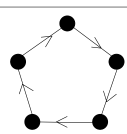
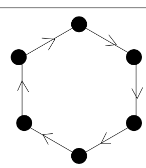
$\circ$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Example 14.5. Quasigroup constructed from Example 14.3 using the Directed Opposite Vertex Construction.

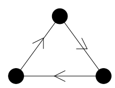
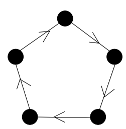
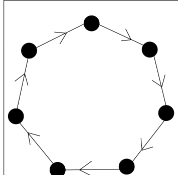
$\circ$	1	2	3	4	5	6	7	8
1	1	4	6	7	8	2	5	3
2	3	2	5	8	7	4	6	1
3	5	8	3	2	4	7	1	6
4	6	7	1	4	2	8	3	5
5	4	1	7	6	5	3	8	2
6	8	5	2	3	1	6	4	7
7	2	3	8	5	6	1	7	4
8	7	6	4	1	3	5	2	8

Just as was the case for undirected cycle systems, the class of 2-perfect (m-perfect) directed m-cycle ($(2m+1)$-cycle) systems is *equationally defined* if and only if there exists a variety of quasigroups V such that the finite quasigroups in V are *precisely* the quasigroups whose multiplicative parts can be constructed from 2-perfect (m-perfect) directed m-cycle ($(2m+1)$-cycle) systems using the Directed Standard Construction (Directed Opposite Vertex Construction).

Summary of results for 2-perfect directed m-cycle systems

D_n	2-perfect spectrum	Equationally defined
	$n \equiv 0$ or 1 (mod 3) $n \neq 6$ N.S.Mendelsohn [27]	$\begin{cases} x^2 = x \\ x(yx) = y \end{cases}$
	$n \equiv 0$ or 1 (mod 4) $n \neq 4$ or 8 [3]	$\begin{cases} x^2 = x \\ (xy)(y(xy)) = x \end{cases}$
	$n \equiv 0$ or 1 (mod 5) $n \neq 6$ or 10 and possibly $n = 15$ and 20 [3]	$\begin{cases} x^2 = x \\ (y(xy))((xy)(y(xy))) = x \end{cases}$
 m-cycle $m \geqslant 6$	?	NO D.E.Bryant C.C.Lindner [10]

Summary of results for m-perfect directed $(2m+1)$-cycle systems

D_n	m-perfect spectrum	Equationally defined
	$n \equiv 0$ or $1 \pmod 3$ $n \neq 6$ N.S.Mendelsohn [27]	$\begin{cases} x^2 = x \\ x(yx) = y \end{cases}$
	$n \equiv 0$ or $1 \pmod 5$ $n \neq 6$ or 10 and possibly $n = 15$ and 20 [3]	$\begin{cases} x^2 = x \\ y(x/(xy)) = (xy)\backslash y \\ ((xy)\backslash y)x = x/(xy) \end{cases}$
 $(2m+1)$-cycle $2m+1 \geqslant 7$	?	NO C.C.Lindner C.A.Rodger [21]

15. Concluding remarks

The initial part of this survey is a rewriting of a survey paper by the author for a talk in Adelaide at the AustMS meetings at Flinders University in 1996 [17]. The sections on the Opposite Vertex Construction and the directed analogues of the Standard Construction and the Opposite Vertex Construction have been added. Other “perfect” graph decompositions are possible, and the interested reader is referred to [15, 16, 17, 22, 23, 24] for additional reading on the subject.

Finally, as mentioned throughout this set of notes, knowing the spectrum for 2-perfect m-cycle systems and 2-perfect directed m-cycle systems is not necessary in determining whether or not a 2-perfect class is equationally defined. However, it is certainly comforting to know the spectrum for the 2-perfect classes that can be equationally defined. The author would like to point out that the determination of the 2-perfect spectrums for $m = 5, 6$ and 7 for undirected cycles and $m = 4$ and 5 for directed cycles is a difficult undertaking.

The general problem of determining the 2-perfect spectrum for both undirected and directed m-cycle systems is an open and extremely difficult problem.

Well, I could go on and on. However, this set of notes is long enough as it is, and so I will end with the immortal words of Porky Pig:

THAT'S ALL FOLKS!

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