Capitolul 6. Serii numerice.

1. **Să se stabilească natura seriilor următoare, calculând limita şirului sumelor parţiale:**

1.1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{n-1}.$

1.2. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{n-1}.$

1.3. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}.$

1.4. $\displaystyle\sum_{n=1}^{\infty}\frac{2n}{5^n}.$

1.5. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$

1.6. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)}.$

1.7. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2+n-2}.$

1.8. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+5n+6}.$

1.9. $\displaystyle\sum_{n=1}^{\infty}\frac{12}{36n^2+12n-35}.$

1.10. $\displaystyle\sum_{n=1}^{\infty}\frac{5}{25n^2-5n-6}.$

1.11. $\displaystyle\sum_{n=1}^{\infty}\frac{7}{49n^2+7n-12}.$

1.12. $\displaystyle\sum_{n=1}^{\infty}\frac{6}{36n^2-24n-5}.$

1.13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}.$

1.14. $\displaystyle\sum_{n=2}^{\infty}\frac{3n-5}{n(n^2-1)}.$

1.15. $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(n-2)(n+2)}.$

1.16. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)(2n+5)}.$

1.17. $\displaystyle\sum_{n=2}^{\infty}\frac{5n+4}{(n-1)n(n+2)}.$

1.18. $\displaystyle\sum_{n=1}^{\infty}\frac{n-1}{n(n+1)(n+2)}.$

1.19. $\displaystyle\sum_{n=1}^{\infty}\frac{3-n}{n(n+1)(n+3)}.$

1.20. $\displaystyle\sum_{n=1}^{\infty}\frac{2-n}{n(n+1)(n+2)}.$

1.21. $\displaystyle\sum_{n=2}^{\infty} \frac{n-\sqrt{n^2-1}}{\sqrt{n(n-1)}}$.

1.22. $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$.

1.23. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{(n+2)!}$.

1.24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+m)}$, $m \in \mathbb{N}$.

2. Folosind criteriul general de convergenţă al lui Cauchy să se stabilească natura seriilor:

2.1. $\displaystyle\sum_{n=1}^{\infty} q^n \sin(2n)$, $|q| < 1$.

2.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

2.3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

2.4. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+4}$.

2.5. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{2^n}$, $x \in \mathbb{R}$.

2.6. $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$.

2.7. $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{10^n}$, $|a_n| < 10$.

2.8. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 2^n}{n^2}$.

2.9. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n(n+1)}$, $\alpha \in \mathbb{R}$.

2.10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$.

3. Utilizând condiţia de convergenţă, să se demonstreze divergenţa seriilor:

3.1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$.

3.2. $\displaystyle\sum_{n=1}^{\infty} \operatorname{arctg}(n-1)$.

3.3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n+2}$.

3.4. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n^2+4}{3n^2+2}\right)^{n^2}$.

3.5. $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{2n+3}{3n+5}}$.

3.6. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n+1}}{\ln^2(n+2)}$.

3.7. $\displaystyle\sum_{n=1}^{\infty} n \operatorname{arctg} \frac{1}{n+1}.$

3.8. $\displaystyle\sum_{n=1}^{\infty} (n^2+1) \ln \frac{n^2+1}{n^2}.$

3.9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-1}{n+2} \arcsin \frac{1}{n^2+1}.$

3.10. $\displaystyle\sum_{n=2}^{\infty} \sqrt[n]{0,05}.$

4. Utilizând criteriile de comparaţie, să se studieze natura seriilor:

4.1. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n\sqrt[3]{n}}{n\sqrt[3]{n}}.$

4.2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+1)}{-\sqrt[5]{n^9}}.$

4.3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(\pi n)}{n(n+1)(n+2)}.$

4.4. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n-\ln n}.$

4.5. $\displaystyle\sum_{n=2}^{\infty} \frac{\arcsin \frac{(-1)^n n}{n+1}}{n^2+2}.$

4.6. $\displaystyle\sum_{n=1}^{\infty} \frac{3+(-1)^n}{2^{n+2}}.$

4.7. $\displaystyle\sum_{n=2}^{\infty} \frac{\operatorname{arctg}[2+(-1)^n]}{\ln n}.$

4.8. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^3}.$

4.9. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\ln n}}.$

4.10. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sqrt{\ln\left(1+\frac{1}{n}\right)} \right).$

4.11. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+n^3}{4^n+\ln^2(n+1)}.$

4.12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n+1}{\sqrt{n^6+n^3+1}}.$

4.13. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n-1}.$

4.14. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+1}{5^n+2}.$

4.15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{(3n+1)(3n+2)}}.$

4.16. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n+2}-\sqrt{n-2}}{\sqrt{n+1}}.$

4.17. $\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}}-1\right)\sin\frac{1}{\sqrt{n+2}}.$

4.18. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\operatorname{arctg}\frac{1}{\sqrt{n}}.$

4.19. $\sum_{n=1}^{\infty}\ln\frac{n^2+3}{n^2+2}.$

4.20. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n}}\arcsin\frac{1}{\sqrt[3]{n^2}}.$

5. Utilizând criteriul D'Alembert, să se stabilească natura următoarelor serii:

5.1. $\sum_{n=1}^{\infty}\frac{n+2}{3^n n!}.$

5.2. $\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}.$

5.3. $\sum_{n=1}^{\infty}\frac{(n+1)!}{n^n}.$

5.4. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}.$

5.5. $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}.$

5.6. $\sum_{n=1}^{\infty}\frac{(3n)!}{(n!)^3 4^{3n}}.$

5.7. $\sum_{n=1}^{\infty}\frac{(n+1)!(2n+3)!}{(3n+3)!}.$

5.8. $\sum_{n=1}^{\infty}\frac{(2n)!!}{n!}\operatorname{arctg}\frac{1}{5^n}.$

5.9. $\sum_{n=1}^{\infty}\frac{n^{\ln 2}}{(\ln 2)^n}.$

5.10. $\sum_{n=2}^{\infty}n\operatorname{tg}\frac{\pi}{2^n}.$

5.11. $\sum_{n=1}^{\infty}\frac{2^{n-1}}{n!+(n+2)!}.$

5.12. $\sum_{n=1}^{\infty}\frac{n!}{10^{n+1}}.$

5.13. $\sum_{n=1}^{\infty}\frac{(n!)^2}{2^{n^2}}.$

5.14. $\sum_{n=1}^{\infty}\frac{4^{n^2-1}}{3^{n^2}\sqrt{n}}.$

5.15. $\sum_{n=1}^{\infty} n^2 \sin \frac{\pi}{2^n}$.

5.16. $\sum_{n=1}^{\infty} \frac{(n+1)!}{2^n n!}$.

5.17. $\sum_{n=1}^{\infty} \frac{n^3}{(n+3)!}$.

5.18. $\sum_{n=1}^{\infty} \frac{3^n \sqrt[3]{n}}{(n+1)!}$.

5.19. $\sum_{n=1}^{\infty} \frac{(n+2)}{n!} \sin \frac{2}{5^n}$.

5.20. $\sum_{n=1}^{\infty} \frac{(n+1)!}{(n+1)^n}$.

6. Utilizând criteriul radical Cauchy, să se stabilească natura seriilor:

6.1. $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4n+3}\right)^{n^2}$.

6.2. $\sum_{n=1}^{\infty} \left(\frac{n+1}{7n+6}\right)^{n^2}$.

6.3. $\sum_{n=1}^{\infty} \left(\frac{2n-1}{n+2}\right)^{n^2}$.

6.4. $\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+2}\right)^{\frac{n}{2}}$.

6.5. $\sum_{n=1}^{\infty} n^3 \sin^n \frac{\pi}{2n}$.

6.6. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{n^n}$.

6.7. $\sum_{n=1}^{\infty} \frac{n3^n}{5^n}$.

6.8. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} \frac{1}{3^n}$.

6.9. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} \frac{1}{2^n}$.

6.10. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

6.11. $\sum_{n=1}^{\infty} \frac{2n^n}{(3n+1)^n}$.

6.12. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} 4^n$.

6.13. $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{\sqrt{n^3+3n+1}}$.

6.14. $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}+1}{\sqrt{n+1}+2}\right)$.

6.15. $\displaystyle\sum_{n=1}^{\infty} 3^{n-1}e^{-2n}$.

6.16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+1}{5n+6}\right)^{n^3}$.

6.17. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n^2+2n+1}{5n^2+3n+2}\right)^{n}$.

6.18. $\displaystyle\sum_{n=1}^{\infty} \left[2+(0,1)^{n-1}\right]$.

6.19. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\left(1+\frac{1}{n}\right)^{n^2}}$.

6.20. $\displaystyle\sum_{n=1}^{\infty} 2^{n+1}e^{-n}$.

7. Utilizând criteriul integral Cauchy, să se studieze natura următoarelor serii:

7.1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}},\ \alpha \in \mathbb{R}$.

7.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln^2(n+1)}$.

7.3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$.

7.4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$.

7.5. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n+1}}}{\sqrt{n+1}}$.

7.6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

7.7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(9n-1)\ln(9n-1)}$.

7.8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln^p n}$.

7.9. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p(\ln\ln n)^q}$.

7.10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}$.

8. Să se calculeze suma seriei cu exactitatea α:

8.1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n^3},\ \alpha=0,01.$

8.2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)^2},\ \alpha=0,01.$

8.3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!},\ \alpha=0,01.$

8.4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n},\ \alpha=0,01.$

8.5. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!3^n},\ \alpha=0,001.$

8.6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{(n+1)^n},\ \alpha=0,001.$

8.7. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{5^n},\ \alpha=0,0001.$

8.8. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)!!},\ \alpha=0,0001.$

8.9. $\displaystyle\sum_{n=0}^{\infty}\frac{\cos\pi n}{3^n(n+1)},\ \alpha=0,001.$

8.10. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n(n^2+3)},\ \alpha=0,01.$

9. Utilizând criteriul lui Leibniz, să se demonstreze natura seriilor:

9.1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$

9.2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln(n+1)}.$

9.3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n(n+1)}.$

9.4. $\displaystyle\sum_{n=3}^{\infty}\frac{(-1)^{n}}{(n+1)\ln n}.$

9.5. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n-1}{3n}.$

9.6. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}.$

9.7. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left[2+(0,1)^n\right].$

9.8. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n-3}{2n-1}.$

9.9. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}.$

9.10. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln^2 n}{n}.$

9.11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{\sqrt{n}}$.

9.12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)^{n+1}}{n^{n+2}}$.

9.13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n\sqrt{n}-1}$.

9.14. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{1}{n}$.

9.15. $\sum_{n=1}^{\infty}(-1)^{n+1}\operatorname{tg}\frac{2}{n}$.

9.16. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{\alpha}}, \ \alpha\in\mathbb{R}$.

9.17. $\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\cdot\frac{2^n+n^2}{3^n+n^3}$.

9.18. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$.

9.19. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}$.

9.20. $\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{2}+\pi n\right)}{n^3+1}$.

10. Folosind criteriul lui Dirichlet sau criteriul lui Abel, să se demonstreze convergenţa seriilor următoare:

10.1. $\sum_{n=1}^{\infty}\frac{\sin nx}{n}, \ x\in\mathbb{R}\backslash\{2k\pi, k\in\mathbb{Z}\}$.

10.2. $\sum_{n=1}^{\infty}\frac{\sin n\sin n^2}{\sqrt{n}}$.

10.3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}\operatorname{arctg} n$.

10.4. $\sum_{n=1}^{\infty}\int_0^1 x\cos(nx)\ dx$.

10.5. $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n^2x\right)\sin(nx), \ x\in\mathbb{R}$.

10.6. $\sum_{n=1}^{\infty}\frac{1}{n}\cos n\sin(nx), \ x\in\mathbb{R}$.

10.7. $\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(n^2x\right)\sin(nx), \ x\in\mathbb{R}$.

10.8. $\sum_{n=1}^{\infty}\frac{\sin n\alpha}{\ln\ln(n+2)}\cos\frac{1}{n}$.

10.9. Să se demonstreze, că dacă şirul numeric $\{a_n\}$ converge monoton la zero, atunci seria

$$\sum_{n=1}^{\infty} a_n \sin n\alpha$$

converge pentru orice $\alpha \in \mathbb{R}$, iar seria

$$\sum_{n=1}^{\infty} a_n \cos n\alpha$$

converge pentru orice $\alpha \in \mathbb{R}\backslash\{2\pi m,\ m \in \mathbb{Z}\}$.

11. Să se studieze convergenţa absolută sau semiconvergenţa seriilor următoare:

11.1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$.

11.2. $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{\ln\ln n}$.

11.3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{\pi}{\sqrt{n}}$.

11.4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$.

11.5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n\ln(n+1)\ln\ln(n+2)}$.

11.6. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt[3]{n}}{\sqrt{n-1}+2}$.

11.7. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)\sin 2n}{n^2-\ln n}$.

11.8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)}{n\sqrt{n+1}} \operatorname{tg} \frac{1}{\sqrt{n}}$.

11.9. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n \cos \frac{1}{n}}{\sqrt[4]{n}}$.

11.10. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^{\alpha}}$, $\alpha > 0$.