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Vector discrete problems: parametrization of an optimality principle and conditions of solvability in the class of algorithms involving linear convolution of criteria

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Abstract

An n–criteria problem with a finite set of vector valuations is considered. An optimality principle of this problem is given by an integer-valued parameter s, which is varied from 1 to $n-1$. At that, the majority and Pareto optimality principles correspond to the extreme values of the parameter. Sufficient conditions, under which the problem of finding efficient valuations corresponding to the parameter s is solvable by the linear convolution of criteria, are indicated.

1 Basic definitions and lemma

As usually [1], let a vector function

$$y = (y_1(x), y_2(x), ..., y_n(x)) : X \to \mathbf{R}^n, \ n \geq 2,$$

be defined on a set of alternatives X.

When choosing an optimal alternative from the set X it is enough to consider the set of feasible valuations

$$Y = \{y \in \mathbf{R^n} : \ y = y(x), \ x \in X\}.$$

Here $\mathbf{R^n}$ is the n–dimentional criteria space. We consider a vector problem

$$y \to \min_{y \in Y}$$

and suppose Y to be a finite set containing $|Y| \geq 2$ elements. This problem is naturally called discrete.

A mechanism of choosing an optimal valuation is usually based on a binary relation expressing "preference" of a valuation to another [1–6]. In its turn, any binary relation generates an optimality principle [6–8].

In this paper we continue our research of solvability conditions of vector discrete problems of finding the Pareto set in the class of algorithms involving linear convolution of criteria (see [9,10]). This time we consider the case of the parametrization of an optimality principle.

For a vector $z = (z_1, z_2, ..., z_n) \in \mathbf{R^n}$ we denote

$$(z)^+ = |\{i \in N_n : \ z_i > 0\}|,$$

$$(z)^- = |\{i \in N_n : \ z_i < 0\}|,$$

where $N_n = \{1, 2, ..., n\}$.

For any number $s \in N_{n-1}$ we define the binary relation

$$y' \prec_s y'' \Longleftrightarrow s(y' - y'')^+ < (y' - y'')^-$$

in the criteria space $\mathbf{R^n}$. By that, the valuation y' is preferred to the valuation y'' by the binary relation $\prec_s$ if and only if the number of criteria, by which y' is "preferred" to y'', is more than s times greater than the number of criteria, by which y'' is "preferred" to y'.

For any subscript $s \in N_{n-1}$ we also introduce the set of s–efficient valuations $C_s^n(Y)$ by setting

$$C_s^n(Y) = \{y \in Y : \ \gamma_s(y) = \emptyset\},$$

where $\gamma_s(y) = \{y' \in Y : \ y' \prec_s y\}$.

It is obvious that the set $C_s^n(Y)$ can be defined as follows:

$$y \in C_s^n(Y) \Longleftrightarrow \forall y' \in Y \ (s(y' - y)^+ \geq (y' - y)^-). \qquad (1)$$

Therefore $C_s^n(Y) \subseteq C_k^n(Y)$ for any $1 \leq s < k \leq n - 1$.

It is clear that the set $C_1^n(Y)$ coincides with the set of majority efficient valuations defined in [4,11–13]:

$$M^n(Y) = \{y \in Y : \ \mu(y) = \emptyset\},$$

where $\mu(y) = \{y' \in Y : \sum_{i=1}^{n} \text{sign}(y_i - y'_i) > 0\}$. The majority optimality principle realizes the procedure of making decision by the majority of voices.

It is easy to check that a valuation y is Pareto optimal if and only if the inequality $(y - y')^- \geq 1$ is true for any valuation $y' \neq y$. Hence, taking into account the obvious inequality $(y - y')^+ \leq n - 1$, we obtain that the set $C^n_{n-1}(Y)$ is the Pareto set defined as follows:

$$P^n(Y) = \{y \in Y : \pi(y) = \emptyset\},$$

where $\pi(y) = \{y' \in Y : y - y' \geq 0, \ y \neq y'\}$.

Thus the following lemma is valid.

Lemma. *For any number $n \geq 2$ the relations*

$$M^n(Y) = C^n_1(Y) \subseteq C^n_2(Y) \subseteq ... \subseteq C^n_{n-1}(Y) = P^n(Y)$$

hold.

From the lemma it follows that $M^2(Y) = P^2(Y)$.

So any parameter $s \in N_{n-1}$ defines the set of s–efficient valuations of a n–criteria discrete problem.

2 Example

The following example shows that the sets $C^n_1(Y), C^n_2(Y), ..., C^n_{n-1}(Y)$ can be nonempty and distinct, i.e. any $C^n_k(Y)$ can be a proper subset of the set $C^n_{k+1}(Y)$ for any number $k \in N_{n-2}$.

Example. Let $Y = \{y^{(1)}, y^{(2)}, ..., y^{(n-1)}\}, n \geq 3$, where

$$y^{(1)} = (1, 0, 0, ..., 0)$$

$$y^{(i)} = (0, 0, .., 0, \underbrace{n - i, n - i, ..., n - i}_{i \text{ times}}) \in \mathbf{R^n}, \ i = 2, 3, ..., n - 1;$$

i.e. the valuation $y^{(i)}$ is the $(n - i)$–th row of the following matrix of the dimension $(n - 1) \times n$:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & ... & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & ... & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & ... & 3 & 3 & 3 \\ & & & & ... & & & \\ 0 & 0 & 0 & 0 & ... & 0 & n-2 & n-2 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix}.$$

Let us show that the equality

$$C_s^n(Y) = \{y^{(1)}, y^{(2)}, ..., y^{(s)}\} \tag{2}$$

holds for any $s \in N_{n-1}$, i.e.

$$\emptyset \neq C_1^n(Y) \subset C_2^n(Y) \subset ... \subset C_{n-1}^n(Y).$$

We prove inequality (2) by induction.

First of all, equality (2) is evident for $s = 1$ since the relation

$$y^{(1)} \prec_1 y^{(i)},\ i = 2, 3, ..., n-1$$

is true.

So $C_1^n(Y) = \{y^{(1)}\}$.

Further on, suppose that (2) is valid for $s = k-1$. Then let us show that

$$C_k^n(Y) = \{y^{(1)}, y^{(2)}, ..., y^{(k)}\} = C_{k-1}^n(Y) \cup \{y^{(k)}\}.$$

On account of the lemma ($C_{k-1}^n(Y) \subseteq C_k^n(Y)$) and definition (1), it is sufficient to prove that the inequality

$$k(y^{(i)} - y^{(k)})^+ \geq (y^{(i)} - y^{(k)})^- \tag{3}$$

holds for any subscript $i \in N_{n-1}$.

Consider two cases.

Case 1. $i > k$. It is easy to see that

$$(y^{(i)} - y^{(k)})^- = k,\ (y^{(i)} - y^{(k)})^+ = i - k.$$

Consequently, inequality (3) holds.

Case 2. $i < k$. If $i = 1$, then

$$(y^{(1)} - y^{(k)})^- = k,\ (y^{(1)} - y^{(k)})^+ = 1.$$

Thus inequality (3) is true.

If $1 < i < k$, then it can be easily seen that

$$(y^{(i)} - y^{(k)})^- = k - i,\ (y^{(i)} - y^{(k)})^+ = i.$$

Consequently, inequality (3) is valid.

3 Solvability conditions

From now on put

$$\Lambda^n(Y) = \bigcup_{\lambda \in \Lambda_n} \Lambda^n(Y, \lambda),$$

$$\Lambda^n(Y, \lambda) = \arg\min\{\sum_{i=1}^{n} \lambda_i y_i :\ y \in Y\},$$

$$\Lambda_n = \{\lambda \in R^n :\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i > 0,\ i \in N_n\}.$$

The Pareto set $P^n(Y)$ is widely known [1] to contain the set $\Lambda^n(Y)$. The problem of finding the Pareto set is said to be solvable in the class of algorithms involving linear convolution of criteria if the inclusion

$$P^n(Y) \subseteq \Lambda^n(Y)$$

holds. The interest to the problem of solvability (see for instance [14–21]) can be explained by the fact that the inclusion above reveals the possibility to use scalar optimization methods in vector optimization.

Now we formulate and prove a sufficient solvability condition of the problem of finding the set $C_s^n(Y)$ of a vector discrete problem.

Set

$$\mathbf{R^n_+} = \{y \in Y :\ y_i \geq 0,\ i \in N_n\}.$$

From now on, for any vector $z = (z_1, z_2, ..., z_n) \in \mathbf{R^n}$ we denote

$$N_n^+(z) = \{i \in N_n : \ z_i > 0\},$$

$$N_n^-(z) = \{i \in N_n : \ z_i < 0\}.$$

$$N_n^0(z) = \{i \in N_n : \ z_i = 0\}.$$

Theorem. *Let* $Y \subset \mathbf{R^n_+},\ 2 \le |Y| < \infty,\ s \in N_{n-1}$ *and the formula*

$$\forall i \in N_n \ (y_i < y'_i \Longrightarrow (s+1)y_i \le y'_i) \tag{4}$$

holds for any valuations $y = (y_1, y_2, ..., y_n) \in Y$ *and* $y' = (y'_1, y'_2, ..., y'_n) \in Y$*. Then for any* $i \in N_s$ *the inclusion*

$$C_i^n(Y) \subseteq \Lambda^n(Y)$$

is true, i.e. the problem of finding the set of i–efficient solutions is solvable in the class of algorithms involving linear convolution of criteria

Proof. Let $s \in N_{n-1}$. According to the lemma, the theorem will be proved if we show that

$$C_s^n(Y) \subseteq \Lambda^n(Y).$$

Let $y = (y_1, y_2, ..., y_n) \in C_s^n(Y)$. Consider the vector λ with the coordinates

$$\lambda_i = \frac{L}{\zeta_i},\ i \in N_n, \tag{5}$$

where

$$L = \frac{1}{\sum\limits_{i=1}^{n} 1/\zeta_i},$$

$$\zeta_i = \begin{cases} y_i & \text{if } i \in N_n^+(y), \\ \gamma/s & \text{if } i \notin N_n^+(y), \end{cases}$$

$$\gamma = \min\{y'_i : \ y' \in Y,\ i \in N_n^+(y')\}. \tag{6}$$

It is easy to check that $\lambda \in \Lambda_n$ since the existence of γ is guaranteed by the conditions $|Y| \ge 2$ and $Y \subset \mathbf{R^n_+}$.

Let us show that the inequality

$$\sum_{i=1}^{n} \lambda_i y'_i \geq \sum_{i=1}^{n} \lambda_i y_i \tag{7}$$

holds for any $y' \in Y$. To do this, we partition the set Y into two disjoint subsets

$$Y_1 = \{y' \in Y : \ \forall i \in N_n \ (y_i \leq y'_i)\},$$

$$Y_2 = \{y' \in Y : \ \exists i \in N_n \ (y_i > y'_i)\}.$$

It is easy to see that inequality (7) holds for any valuation $y' \in Y_1$.

Let $y' \in Y_2$. Then it is evident that $N_n^-(y' - y) \neq \emptyset$, and the set $N_n^+(y' - y)$ is nonempty since $y \in C_s^n(Y)$. Therefore

$$\sum_{i=1}^{n} \lambda_i (y'_i - y_i) = \sum_{i \in N_n^-(y'-y)} \lambda_i (y'_i - y_i) + \sum_{j \in N_n^+(y'-y)} \lambda_j (y'_j - y_j). \tag{8}$$

On account of (5), we have

$$\sum_{i \in N_n^-(y'-y)} \lambda_i (y'_i - y_i) \geq - \sum_{i \in N_n^-(y'-y)} \lambda_i y_i = -L(y' - y)^-. \tag{9}$$

Let us estimate the second summand of the right part of (8).

Let $j \in N_n^+(y' - y)$. Then the following two cases are possible.

Case 1. $j \in N_n^+(y)$. By (4) we obtain $y'_j - y_j \geq s y_j$. Taking into account (1) and the obvious inequality $(y' - y)^+ > 0$, we deduce

$$\lambda_j (y'_j - y_j) \geq \lambda_j s y_j \geq L \frac{(y' - y)^-}{(y' - y)^+}. \tag{10}$$

Case 2. $j \notin N_n^+(y)$. Then $j \in N_n^0(y)$. By (1) we obtain

$$\lambda_j (y'_j - y_j) = \lambda_j y'_j = L s \frac{y'_j}{\gamma} \geq L \frac{y'_j (y' - y)^-}{\gamma (y' - y)^+}. \tag{11}$$

As $j \in N_n^+(y' - y)$ we have $y'_j > y_j = 0$. Consequently, by (6) the inequality $y'_j \geq \gamma$ holds. Thus, from (11) we obtain (10) once again. Therefore

$$\sum_{j \in N_n^+(y'-y)} \lambda_j (y'_j - y_j) \geq L(y' - y)^-.$$

From this by (8) and (9) we have

$$\sum_{i=1}^{n} \lambda_i (y'_i - y_i) \geq 0.$$

Thereby the inequality (7) holds for any valuation $y' \in Y_2$.

Summarizing what has been already proved, we see that $C_s^n(Y) \subseteq \Lambda^n(Y)$.

The theorem has been proved.

The following known results follow from the theorem.

Corollary 1 [9]**.** *Let $Y \subset \mathbf{R}_+^{\mathbf{n}}$ and the formula*

$$\forall y, y' \in Y \ \forall i \in N_n \ (y_i < y'_i \Longrightarrow n y_i \geq y'_i)$$

holds. Then $P^n(Y) \subseteq \Lambda^n(Y)$.

Corollary 2 [10]**.** *Let $Y \subset \mathbf{R}_+^{\mathbf{n}}$ and the formula*

$$\forall y, y' \in Y \ \forall i \in N_n \ (y_i < y'_i \Longrightarrow 2 y_i \geq y'_i)$$

be true. Then $M^n(Y) \subseteq \Lambda^n(Y)$.

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