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Integer programming models for colorings of mixed hypergraphs

Dumitru Lozovanu   Vitaly Voloshin

Abstract

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ consists of the vertex set X and two families of subsets: the family $\mathcal{C}$ of $\mathcal{C}$-edges and the family $\mathcal{D}$ of $\mathcal{D}$-edges. In a coloring, every $\mathcal{C}$-edge has at least two vertices of common color, while every $\mathcal{D}$-edge has at least two vertices of different colors. The largest (smallest) number of colors for which a coloring of a mixed hypergraph $\mathcal{H}$ using all the colors exists is called the upper (lower) chromatic number and is denoted $\bar{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$).

We consider integer programming models for colorings of mixed hypergraphs in order to show that algorithms for optimal colorings may be transformed and used for finding optimal solutions of the respective integer programming problems.

1 Mixed hypergraphs

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the *vertex set*, and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, called $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges*, respectively.

A *proper k-coloring* of a mixed hypergraph is a mapping from X into a set of k *colors* so that each $\mathcal{C}$-edge has at least two vertices of a **common** color and each $\mathcal{D}$-edge has at least two vertices of **different** colors. That means that in every coloring no $\mathcal{C}$-edge is polychromatic (i.e. no $\mathcal{C}$-edge has all the colors different) and no $\mathcal{D}$-edge is monochromatic. A mixed hypergraph is *k-colorable* if it has a coloring with at most k colors. If $\mathcal{H}$ admits no coloring then it is called *uncolorable*. A *strict k-coloring* of a mixed hypergraph is a proper coloring using all k colors.

The minimum number of colors in a coloring of $\mathcal{H}$ is its *lower chromatic number* $\chi(\mathcal{H})$; the maximum number of colors in a strict coloring is its *upper chromatic number* $\bar{\chi}(\mathcal{H})$.

Classical coloring theory of hypergraphs with edge set $\mathcal{E}$ [1] is the special case where the family of $\mathcal{C}$-edges is empty and we color the mixed hypergraph $(X, \emptyset, \mathcal{E})$.

Coloring of mixed hypergraphs is a new topic introduced in [3, 4].

2 Integer programming models

There are several ways to formulate the colorability problem for mixed hypergraphs as an integer programming problem.

Consider $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $X = \{x_1, x_2, \dots, x_n\}$, $n \geq 1$, $\mathcal{C} = \{C_1, C_2, \dots, C_l\}$, $l \geq 1$, and $\mathcal{D} = \{D_1, D_2, \dots, D_m\}$, $m \geq 1$. Let us have n colors. and $\mathcal{A} = (\alpha_{ij})$ be a (0,1)-matrix allocating vertices to colors in such a way that

$$\alpha_{ij} = \begin{cases} 1 & \text{if vertex } x_i \text{ is colored with color} j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the condition that each vertex receives exactly one color is:

$$\sum_{j=1}^{n} \alpha_{ij} = 1, \quad i = \overline{1, n}. \tag{1}$$

The constraints for proper coloring of $\mathcal{D}$-edges are the following:

$$\sum_{i \in D_k} \alpha_{ij} \leq |D_k| - 1, \quad \forall D_k \in \mathcal{D},\ j = \overline{1, n}. \tag{2}$$

The left side of the inequality represents the number of vertices in a $\mathcal{D}$-edge D_k colored with the color j. The right side assures that at least two vertices of D_k have different colors.

The constraints for proper coloring of $\mathcal{C}$-edges have the form:

$$\sum_{j=1}^{n} \max_{i \in C_k} \alpha_{ij} \leq |C_k| - 1, \quad \forall C_k \in \mathcal{C}. \tag{3}$$

The left side of the inequality equals to the number of different colors used in the $\mathcal{C}$-edge C_k since

$$\max_{i \in C_k} \alpha_{ij} = \begin{cases} 1 & \text{if the color } j \text{ is used in the coloring of vertex } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

The right side provides that at least two vertices of each $\mathcal{C}$-edge have a common color.

In this way, the proper coloring of a mixed hypergraph $\mathcal{H}$ with at most n colors is completely described by the relations 1-3 where $\alpha_{ij} \in \{0, 1\}$.

Integer programming model for the problem of finding the lower chromatic number may be formulated in the following way:

$$\text{minimize } z = \sum_{j=1}^{n} \max_{i \in X} \alpha_{ij}, \tag{4}$$

subject to the following constraints:

$$\begin{cases} \sum_{j=1}^{n} \alpha_{ij} = 1, \quad i = \overline{1, n}; \\ \sum_{i \in D_k} \alpha_{ij} \leq |D_k| - 1, \quad \forall D_k \in \mathcal{D}, \ j = \overline{1, n}; \\ \sum_{j=1}^{n} \max_{i \in C_k} \alpha_{ij} \leq |C_k| - 1, \quad \forall C_k \in \mathcal{C}; \\ \alpha_{ij} \in \{0, 1\}. \end{cases} \tag{5}$$

The last problem reduces to the following integer linear programming problem:

$$\text{minimize } z' = \sum_{j=1}^{n} \eta_j$$

subject to the constraints:

$$\begin{cases} \sum_{j=1}^{n} \alpha_{ij} = 1, \quad i = \overline{1,n}; \\ \sum_{i \in D_k} \alpha_{ij} \le |D_k| - 1, \quad \forall D_k \in \mathcal{D},\ j = \overline{1,n}; \\ \sum_{j=1}^{n} \eta_{C_k j} \le |C_k| - 1, \quad \forall C_k \in \mathcal{C}; \\ \alpha_{ij} \le \eta_{C_k j}, \quad \forall C_k \in \mathcal{C},\ i = \overline{1,n},\ j = \overline{1,n}; \\ \alpha_{ij} \le \eta_j,\ \ i = \overline{1,n},\ j = \overline{1,n}, \\ \alpha_{ij} \in \{0,1\}, \quad \eta_j, \eta_{C_k j} \in \{0,1\}. \end{cases} \tag{7}$$

Theorem 1 *If* $\alpha^*_{ij},\ i = \overline{1,n},\ j = \overline{1,n}$ *make the optimal solution of the problem (4),(5), then* $\alpha^*_{ij},\ i = \overline{1,n},\ j = \overline{1,n},\ \eta^*_{C_k j} = \max_{i \in C_k} \alpha^*_{ij},\ C_k \in \mathcal{C},\ j = \overline{1,n};\ \eta^*_j = \max_{i \in X} \alpha^*_{ij},\ j = \overline{1,n}$ *make the optimal solution of the problem (6), (7). If* $\alpha^\circ_{ij},\ i = \overline{1,n},\ j = \overline{1,n},\ \eta^\circ_{C_k j},\ C_k \in \mathcal{C}, j = \overline{1,n},\ \eta^\circ_j, j = \overline{1,n}$ *make the optimal solution of the problem (6), (7), then* $\alpha^\circ_{ij},\ i = \overline{1,n},\ j = \overline{1,n}$ *make the optimal solution of the problem (4), (5).*

Proof. Let $\alpha^*_{ij},\ i = \overline{1,n},\ j = \overline{1,n}$ is optimal solution of the problem (3), (4). Then $\alpha^*_{ij},\ i = \overline{1,n},\ j = \overline{1,n},\ \eta^*_{C_k j} = \max_{i \in C_k} \alpha^*_{ij},\ C_k \in \mathcal{C},\ j = \overline{1,n};\ \eta^*_j = \max_{i \in X} \alpha^*_{ij},\ j = \overline{1,n}$ satisfies the conditions (7). Consequently if $\alpha^\circ_{ij},\ i = \overline{1,n},\ j = \overline{1,n},\ \eta^\circ_{C_k j}, C_k \in \mathcal{C},\ j = \overline{1,n},\ \eta^\circ_j,\ j = \overline{1,n}$ is the optimal solution of the problem (6), (7), then

$$\overline{z_0} = \sum_{j=1}^{n} \eta_j^\circ \leq \sum_{j=1}^{n} \max \alpha_{ij}^* = z^*.$$

Let us show that it is impossible to have the strict inequality in this relation. If that would be so, then it would mean that α_{ij}^*, $i = \overline{1,n}$, $j = \overline{1,n}$ do not make the optimal solution of the problem (3), (4), because α_{ij}° satisfies (5). The last is implied by:

$$\begin{cases} \sum_{j=1}^{n} \alpha_{ij}^\circ = 1, \quad i = \overline{1,n}; \\ \sum_{i \in D_k} \alpha_{ij}^\circ \leq |D_k| - 1, \ D_k \in \mathcal{D}, \ j = \overline{1,n}; \\ \sum_{j=1}^{n} \max_{i \in C_k} \alpha_{ij}^\circ \leq \sum_{i=1}^{n} \eta_{C_k j}^\circ \leq |C_k| - 1, \quad \forall C_k \in \mathcal{C}; \\ \alpha_{ij}^\circ \in \{0,1\}, \quad \eta_{C_k j} \in \{0,1\}. \end{cases}$$

The value of the goal function for this solution is

$$z_0 = \sum_{j=1}^{n} \max \alpha_{ij}^\circ \leq \sum_{j=1}^{n} \eta_j^\circ < \sum_{j=1}^{n} \max \alpha_{ij}^* = z^*$$

This contradicts the fact that α_{ij}^*, $i = \overline{1,n}$, $j = \overline{1,n}$ make the optimal solution of the problem (3), (4).

The second part of the theorem is proved by analogy. Let α_{ij}°, $i = \overline{1,n}$, $j = \overline{1,n}$ $\eta_{C_k j}^\circ$, $C_k \in \mathcal{C}$, $j = \overline{1,n}$, η_j°, $j = \overline{1,n}$ make the optimal solution of the problem (6), (7). Then it is easy to check that numbers α_{ij}°, $i = \overline{1,n}$, $j = \overline{1,n}$ satisfy the conditions (3) (see (8)). Therefore, if we assume the contrary, i.e. that α_{ij}°, $i = \overline{1,n}$, $j = \overline{1,n}$ do not make the optimal solution of the problem (4). $(\sum_{j=1}^{n} \alpha_{ij}^\circ > \sum_{j=1}^{n} \max \alpha_{ij}^*)$, then we obtain the contradiction to the fact that $\alpha^\circ_{ij}, \eta^\circ_{C_k j}, \eta^\circ_j$ is not the optimal solution of the problem (6), (7), because

$$\sum_{j=1}^{n} \eta^*_{C_k j} = \sum_{j=1}^{n} \max_{i \in C_k} \alpha^*_{ij}, < \sum_{j=1}^{n} \max_{i \in C_k} \alpha^\circ_{ij}.$$

The theorem is proved. □

Now we consider the problem: does a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ admit at least one strict coloring with p colors, where $p \leq n$?

Theorem 2 *Mixed hypergraph* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *is strictly colorable with* p *colors if and only if the following system admits an integer solution:*

$$\begin{cases} \sum_{j=1}^{p} \alpha_{ij} = 1, \ i = \overline{1, n}; \\ \sum_{i \in X} \alpha_{ij} \geq 1, \ j = \overline{1, p}; \\ \sum_{i \in D_k} \alpha_{ij} \leq |D_k| - 1, \ j = \overline{1, p}; \\ \sum_{j=1}^{p} \eta_{C_k j} \leq |C_k| - 1, \quad \forall C_k \in \mathcal{C}; \\ \alpha_{ij} \leq \eta_{C_k j}, \ i \in C_k, \ \forall C_k \in \mathcal{C}, j = \overline{1, p}; \\ \alpha_{ij} \in \{0, 1\}, \quad \eta_{C_k j} \in \{0, 1\}. \end{cases} \qquad (8)$$

Proof. The condition that every vertex $x_i \in X$ is colored, is expressed by

$$\sum_{j=1}^{p} \alpha_{ij} = 1, \quad i = \overline{1, n}.$$

The condition that every color from the set of p colors $\{1, 2, \ldots, p\}$ is used (the coloring is strict), is expressed by:

$$\sum_{i \in X} \alpha_{ij} \geq 1, j = \overline{1,p}$$

We don't care here about which p colors from the set $\{1, 2, \ldots, n\}$ colors are really used: if some strict coloring with the other subset of colors exists then we can permute the colors and obtain the strict coloring with the first p colors.
The inequalities

$$\sum_{i \in D_k} \alpha_{ij} \leq |D_k| - 1, \ j = \overline{1,p}$$

express that all $\mathcal{D}$-edges are colored properly. The proper coloring of $\mathcal{C}$-edges is written in the following way:

$$\sum_{j=1}^{p} \max_{i \in C_k} \alpha_{ij} \leq |C_k| - 1, \quad \forall C_k \in \mathcal{C}.$$

Therefore the strict coloring using p colors has the following constraints:

$$\begin{cases} \sum_{j=1}^{p} \alpha_{ij} = 1, \ i = \overline{1,n}; \\ \sum_{i \in X} \alpha_{ij} \geq 1, \ j = \overline{1,p}; \\ \sum_{i \in D_k} \alpha_{ij} \leq |D_k| - 1, \ j = \overline{1,p}; \\ \sum_{j=1}^{p} \max_{i \in C_k} \alpha_{ij} \leq |C_k| - 1, \quad \forall C_k \in \mathcal{C}; \\ \alpha_{ij} \in \{0, 1\}. \end{cases}$$

It is easy to check that this system has the solution if and only if the system (8) has one. $\square$

If follows from the theorem 2 that the problem of finding the upper chromatic number is reduced to the problem of finding the maximal number p for which (8) has the solution.

3 Fractional colorings

The following model is taken from [2]. We show that not only the colorability but also the upper and lower chromatic numbers of a mixed hypergraph can be simultaneously determined by the solutions of an integer programming problem.

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph, where $X = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$, $\mathcal{C} = \{C_1, C_2, \ldots, C_l\}$, $l \geq 1$, and $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$, $m \geq 1$.

A set $S \subset X$ is called *$\mathcal{D}$-stable* if it contains no D-edge; and S is called *$\mathcal{C}$-stable* if it contains no $\mathcal{C}$-edge as a subset. We denote by $\mathcal{S}_\mathcal{C}$ and $\mathcal{S}_\mathcal{D}$ the collection of all $\mathcal{C}$–stable and all $\mathcal{D}$-stable sets of $\mathcal{H}$, respectively.

By definition, a mapping $c : X \to \{1, 2, \ldots, p\}$ is a coloring of $\mathcal{H}$ if and only if every $S \subset X$ satisfies the following two requirements:

(i) if S is monochromatic, then $S \in \mathcal{S}_\mathcal{D}$, and

(ii) if S is polychromatic, then $S \in \mathcal{S}_\mathcal{C}$.

It will be convenient for our purpose to review colorings from another point, namely as vertex partitions into stable sets satisfying condition (*ii*). Thus we consider a more general coloring/covering concept, assigning stable sets to *real weights* in the half-open interval $(0, 1]$ as follows.

A *fractional coloring* of $\mathcal{H}$ with t colors is a collection $\mathcal{S} = \{S_1, \ldots, S_t\} \subseteq \mathcal{S}_\mathcal{D}$ of t distinct $\mathcal{D}$-stable sets together with a *weight function*

$$w : \mathcal{S} \to (0, 1]$$

satisfying the following properties:

- For each vertex $x \in X$,

$$\sum_{\substack{S_i \in \mathcal{S} \\ x \in S_i}} w(S_i) = 1,$$

- For each $\mathcal{C}$-edge $C \in \mathcal{C}$,

$$\sum_{\substack{S_i \in \mathcal{S} \\ C \cap S_i \neq \emptyset}} w(S_i) \leq |C| - 1.$$

It is convenient to *extend the domain of* w to the entire $\mathcal{S}_{\mathcal{D}}$, by defining

$$w(S_i) = 0 \quad \forall S_i \in \mathcal{S}_{\mathcal{D}} \setminus \mathcal{S}.$$

Then the extended w on $\mathcal{S}_{\mathcal{D}}$ and its restriction to $\mathcal{S}$ can be considered equivalent, without ambiguity. The latter becomes important only in contexts where the number of colors assigned to fractional weights is relevant.

The *value* of a fractional coloring $(\mathcal{S}, w)$ is defined as

$$w(\mathcal{S}) = \sum_{i=1}^{t} w(S_i).$$

The quantities

$$\chi^*(\mathcal{H}) = \min_{(\mathcal{S},w)} w(\mathcal{S})$$

and

$$\bar{\chi}^*(\mathcal{H}) = \max_{(\mathcal{S},w)} w(\mathcal{S})$$

are called the *fractional lower chromatic number* and the *fractional upper chromatic number* of $\mathcal{H}$, respectively, where the corresponding minimum or maximum is taken over all t and all feasible fractional t-colorings $(\mathcal{S}, w)$.

It can be seen that the problem of determining χ^* and $\bar{\chi}^*$ can be solved by linear programming on an $|\mathcal{S}_{\mathcal{D}}|$-dimensional polyhedron defined by constraints over $|X| + |\mathcal{H}_{\mathcal{C}}|$.

Observe that the minimum and maximum values of the objective function coincide with $\chi(\mathcal{H})$ and $\bar{\chi}(\mathcal{H})$, respectively, if we restrict the range of the weight function w to the integers 0,1.

Unfortunately, $|\mathcal{S}_\mathcal{D}|$ can be exponential in $|X|$ even if $\mathcal{H}_\mathcal{D}$ is ‘nicely structured’ and has a polynomial number of *maximal* stable sets. Therefore, further structural investigations may be needed in order to compute $\chi^*(\mathcal{H})$ and $\bar{\chi}^*(\mathcal{H})$ efficiently.

At last, it is worth mentioning that fractional colorings may exist even in the case when a mixed hypergraph is uncolorable [2].

4 Generalizations

The colorability problem for mixed hipergraph can be formulated in a more general form. Let us consider that each $\mathcal{C}$-edge $C_k \in \mathcal{C}$ has at least s_k vertices of a **common** color and each $\mathcal{D}$-edge $D_k \in \mathcal{D}$ has at least r_k vertices of **different** colors. Then the integer programming model for the colorability problem with p colors has the following form:

$$\begin{cases} \sum\limits_{j=1}^{p} \alpha_{ij} = 1, \ i = \overline{1,n}; \\ \sum\limits_{i \in X} \alpha_{ij} \geq 1, \ j = \overline{1,p}; \\ \sum\limits_{i \in D_k} \alpha_{ij} \leq |D_k| - r_k + 1, \ j = \overline{1,p}, \quad \forall_k \in \mathcal{D}; \\ \sum\limits_{j=1}^{p} \max\limits_{i \in C_k} \alpha_{ij} \leq |C_k| - s_k + 1, \ i = \overline{1,n}, \quad \forall C_k \in \mathcal{C}; \\ \alpha_{ij} \in \{0,1\}, \quad \forall i = \overline{1,n}, \quad \forall j = \overline{1,p}. \end{cases}$$

If we consider the problem when the $\mathcal{C}$-edge $C_k \in \mathcal{C}$ has exactly s_k vertices of a **common** color the constraint $\sum\limits_{j=1}^{p} \max\limits_{i \in C_k} \alpha_{ij} \leq |C_k| - s_k + 1$

must be replaced by $\sum_{j=1}^{p} \max_{i \in C_k} \alpha_{ij} = |C_k| - s_k + 1$. The problem in the case when the $\mathcal{D}$-edge $D_k \in \mathcal{D}$ has exactly r_k vertices of **different** colors, the constraint $\sum_{i \in D_k} \alpha_{ij} \leq |D_k| - r_k + 1$ must be replaced by $\sum_{i \in D_k} \alpha_{ij} = |D_k| - r_k + 1$.

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D.Lozovanu, V.Voloshin,
Institute of Mathematics
and Computer Science,
Academy of Sciences of Moldova
5, Academiei str., Kishinev,
MD2028, Moldova
phone: 73–35–83
e–mail: *lozovanu@math.md*
voloshin@math.md

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