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Triple Roman domination subdivision number in graphs

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Abstract

For a graph $G = (V, E)$, a triple Roman domination function is a function $f : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ having the property that for any vertex $v \in V(G)$, if $f(v) < 3$, then $f(\mathrm{AN}[v]) \geq |\mathrm{AN}(v)|+3$, where $\mathrm{AN}(v) = \{w \in N(v) \mid f(w) \geq 1\}$ and $\mathrm{AN}[v] = \mathrm{AN}(v) \cup \{v\}$. The weight of a triple Roman dominating function f is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The triple Roman domination number of G, denoted by $\gamma_{[3R]}(G)$, equals the minimum weight of a triple Roman dominating function on G. *The triple Roman domination subdivision number* $\mathrm{sd}_{\gamma_{[3R]}}(G)$ of a graph G is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the triple Roman domination number. In this paper, we first show that the decision problem associated with $\mathrm{sd}_{\gamma_{[3R]}}(G)$ is NP-hard and then establish upper bounds on the triple Roman domination subdivision number for arbitrary graphs.

Keywords: Triple Roman domination number, Triple Roman domination subdivision number.

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1 Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) =$

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$\cup_{v\in S}N(v)$ and the *closed neighborhood* of S is $N[S] = N(S) \cup S$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *distance* between two vertices u and v is the length of a shortest path joining them. We denote by $N_2(v)$ the set of vertices at distance 2 from the vertex v and put $d_2(v) = |N_2(v)|$. For a more thorough treatment of domination parameters and for terminology not present here, see [17].

A *[k]-Roman domination function* ([k]-RDF) is a function $f : V(G) \longrightarrow \{0, 1, 2, 3, \ldots, k+1\}$ such that for any vertex $v \in V(G)$, if $f(v) < k$, then $f(\mathrm{N}[v]) \geq |\mathrm{AN}(v)| + k$, where $\mathrm{AN}(v) = \{w \in N(v) \mid f(w) \geq 1\}$. The weight of a [k]-RDF is the value $\omega(f) = \sum_{v\in V(G)} f(v)$. The *[k]-Roman domination number* $\gamma_{[kR]}(G)$ is the minimum weight of a [k]-RDF of G. A [k]-Roman domination function $f : V(G) \longrightarrow \{0, 1, 2, 3, \ldots, k+1\}$ can be represented by the order partition $(V_0^f, V_1^f, V_2^f, V_3^f, \ldots, V_{k+1}^f)$ of V, where $V_i^f = \{v \in V(G) \mid f(v) = i\}$ for $i \in \{0, 1, 2, 3, \ldots, k+1\}$. The [k]-Roman domination number was introduced by Abdollahzadeh Ahangar et al. in [1]. The case $k = 1$ is Roman domination which was introduced by Cockyne et al. in [13], the case $k = 2$ is the double Roman domination which was investigated in [9], the case $k = 3$ is *triple Roman domination number* (TRD-number) and has been studied in [1, 2, 16], and the case $k = 4$ is the *Quadruple Roman domination* and has been investigated in [4]. The literature on Roman domination and its variants has been detailed in two chapters of the books and a surveys paper (see [10–12]). Here, we restrict our attention to triple Roman domination number.

The *triple Roman domination subdivision number* $\mathrm{sd}_{\gamma_{[3R]}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the triple Roman domination number of G.

The domination subdivision number was first introduced in Velmmal's thesis [19], and since then many papers has been published in domination subdivision parameters, see for instance [3, 5–8, 14, 18].

If $G_1, G_2, \ldots, G_s$ are the components of G, then $\gamma_{[3R]}(G) = \sum_{i=1}^{s} \gamma_{[3R]}(G_i)$ and $\mathrm{sd}_{\gamma_{[3R]}}(G) = \min\{\mathrm{sd}_{\gamma_{[3R]}}(G_i) \mid 1 \leq i \leq s\}$. Hence, it is sufficient to study $\mathrm{sd}_{\gamma_{[3R]}}(G)$ for connected graphs. Since the triple Roman domination subdivision number of the graph K_2 does not change when its only edge is subdivided, in the study of the triple Roman domination subdivision number, we must assume that the graph has order at least 3.

In this paper, we first show that the decision problem associated with $\mathrm{sd}_{\gamma_{[3R]}}(G)$ is NP-hard and then establish upper bounds on the triple Roman domination subdivision number for bipartite graphs.

We make use of the following results in this paper.

Proposition A. [2] *In a triple Roman dominating function of weight $\gamma_{[3R]}(G)$, no vertex needs to be assigned the value 1.*

Proposition B. [2] *There is no connected graph of order n such that $\gamma_{[3R]}(G) = 5$.*

Proposition C. *Let G be a connected graph of order n.*

(i) If $n \geq 2$, then $\gamma_{[3R]}(G) = 4$ if and only if $\Delta(G) = n - 1$.

(ii) If $n \geq 4$, then $\gamma_{[3R]}(G) = 6$ if and only if there are two non adjacent vertices in $V(G)$ with degree $\Delta(G) = n - 2$.

Proposition D. [1] *For $n \geq 2$,*

$$\gamma_{[3R]}(P_n) = \begin{cases} 4\lfloor \frac{n}{3} \rfloor & \text{if } n \equiv 0 \pmod 3 \\ 4\lfloor \frac{n}{3} \rfloor + 3 & \text{if } n \equiv 1 \pmod 3 \\ 4\lfloor \frac{n}{3} \rfloor + 4 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proposition E. [1] For $n \geq 3$,

$$\gamma_{[3R]}(C_n) = \begin{cases} \lceil \frac{4n}{3} \rceil & \text{if either } n = 4, 5, 7, 10 \text{ or } n \equiv 0 \pmod 3 \\ \lceil \frac{4n}{3} \rceil + 1 & \text{if } n \neq 4, 5, 7, 10 \text{ and } n \equiv 1, 2 \pmod 3. \end{cases}$$

As the results of Propositions D and E, we have:

Corollary 1. For $n \geq 3$,

$$\mathrm{sd}_{\gamma_{[3R]}}(P_n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 3 \\ 2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Corollary 2. For $n \geq 3$,

$$\mathrm{sd}_{\gamma_{[3R]}}(C_n) = \begin{cases} 1 & \text{if either } n = 5 \text{ or } n \equiv 0, 1 \pmod 3 \\ 2 & \text{if } n \equiv 2 \pmod 3 \ n \neq 5. \end{cases}$$

The proofs of the following observations are straightforward and therefore omitted.

Observation 3. *If K_n is the complete graph of order n and $n \geq 3$, then* $\mathrm{sd}_{\gamma_{[3R]}}(K_n) = 1$.

Observation 4. *If $K_{n,m}$ is the complete bipartite graph and $m, n \geq 3$, then* $\mathrm{sd}_{\gamma_{[3R]}}(K_{n,m}) = 2$.

2 Some preliminary results

In this section, we present some upper bounds on $\mathrm{sd}_{\gamma_{[3R]}}(G)$ in terms of the vertex degree and the minimum degree of G. Our first result shows that subdividing an edge does not decrease the triple Roman domination number.

Lemma 1. Let G be a simple connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdivision the edge e, then $\gamma_{[3R]}(G') \geq \gamma_{[3R]}(G)$.

Proof. Let x be the subdivision vertex and let f be a $\gamma_{[3R]}(G')$-function. Since f is a TRDF on G', we have that $f(u) + f(v) + f(x) \geq 4$. Let $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ be a function defined by $g(u) = \min\{4, f(u) + f(x)\}$ and $g(z) = f(z)$ whenever $z \in V(G) \setminus \{u\}$. Notice that g is a TRDF on G and $\omega(g) \leq \omega(f)$. Hence $\gamma_{[3R]}(G') \geq \gamma_{[3R]}(G)$, which completes the proof. □

We proceed with three propositions giving some sufficient conditions for a graph to having small triple Roman domination subdivision number.

Proposition 1. If G contains a strong support vertex, then $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$.

Proof. Let u, v be two leaves adjacent to w and let G' be obtained from G by subdividing the edge uw with vertex x. Let f be a TRDF on G', then $f(u) + f(v) + f(w) + f(x) \geq 7$. Define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(w) = 4$, $g(u) = g(v) = 0$ and $g(z) = f(z)$ for each $z \in V(G) \setminus \{u, v, w\}$. Clearly, g is a TRDF of G with $\omega(g) < \omega(f)$ and hence $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$. □

Proposition 2. *Let G be a connected graph of order $n \geq 3$. If $\gamma_{[3R]}(G) = 4, 6$, then $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$.*

Proof. First assume $\gamma_{[3R]}(G) = 4$. By Proposition C, we have $\Delta(G) = n-1$. Suppose v is a vertex with maximum degree $\Delta(G)$ and $w \in N(v)$. Let G' be obtained from G by subdivision the edge vw with vertex x. Then $\Delta(G') < n$ and so $\gamma_{[3R]}(G') > 4$ by Proposition C, which implies $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$.

Now assume $\gamma_{[3R]}(G) = 6$, then $G = \overline{K_2} \vee H$ by Proposition C. Let $V(\overline{K_2}) = \{u, w\}$ and $e = uv \in E(G)$, where $v \in V(H)$. Let G' be obtained from G by subdividing the edge e with vertex x. If $\gamma_{[3R]}(G') = 6$, then, as above, $G' = \overline{K_2} \vee H'$ in where $V(K_2) = \{a, b\}$. If $x = a$ ($x = b$ is similar), then since $N_{G'}(x) = \{u, v\}$ and $w \notin N_{G'}(x), w = b$. Hence $w \in N_G(u)$ which is a contradiction. Thus $x \notin \{a, b\}$, this implies that $a, b \in N_{G'}(x) = \{u, v\}$, hence $\{a, b\} = \{u, v\}$ and so w and u are adjacent in G, a contradiction.Therefore $\gamma_{[3R]}(G) < \gamma_{[3R]}(G')$, implying that $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$. This completes the proof. □

By Proposition 2, we have:

Corollary 5. *Let G be a simple connected graph of order $n \geq 3$. If $\mathrm{sd}_{\gamma_{[3R]}}(G) \geq 2$, then $\gamma_{[3R]}(G) \geq 7$.*

Proposition 3. *For every connected graph G of order $n \geq 3$ with $\delta(G) = 1$, $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 2$.*

Proof. Let v be a support vertex of $G, u \in V(G)$ be the leaf adjacent to v and $w \in N(v) - \{u\}$. Assume G' be obtained from G by subdividing the edges uv, vw with vertices x, y, respectively. Let f be a $\gamma_{[3R]}(G')$-function. Without loss of generality, assume that $f(z) \neq 1$ for all vertices $z \in V(G')$ by Proposition A. It is easy to see that $f(u) + f(x) + f(v) + f(y) \geq 4$. If $f(w) \neq 0$, then the function $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ defined by $g(v) = 0$, $g(u) = 3$ and $g(x) = f(x)$ otherwise, is clearly TRDF of G of weight less than $\omega(f)$. If $f(w) = 0$, then $f(u) + f(x) + f(v) + f(y) \geq 7$. Define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(v) = 4, g(u) = 0$ and $g(z) = f(z)$ for each $z \in V(G) \setminus \{u, v\}$. Observe that g is a TRDF on G with $\omega(g) < \omega(f)$. Hence $sd_{\gamma_{[3R]}}(G) \leq 2$. $\square$

Next result is an immediate consequence of Proposition 3 and Corollary 1.

Theorem 6. *For every tree T of order at least 3, $\mathrm{sd}_{\gamma_{[3R]}}(T) \leq 2$. Furthermore, this bound is sharp for the paths of order n for which $n \equiv 2 \pmod 3$.*

3 Complexity of the triple Roman domination subdivision problem

In this section, we will show that the triple Roman domination subdivision number problem in bipartite graphs is NP-hard. We first state the problem as the following decision problem.

Triple Roman domination subdivision number problem (TRDSN):
Instance: *A nonempty graph G and a positive integer k.*
Question: *Is $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq k$?*

Following Garey and Johnson's techniques for proving NP-completeness given in [15], we prove our results by describing a polynomial transformation from the well-known NP-complete problem: **3-SAT**. To state 3-SAT, we recall some terms.

Let U be a set of Boolean variables. A truth assignment for U is a mapping $t : U \longrightarrow \{T, F\}$. If $t(u) = T$, then u is said to be "true" under t; if $t(u) = F$, then u is said to be "false" under t. If u is a variable in U, then u and $\bar{u}$ are *literals* over U. The literal u is true under t if and only if the variable $\bar{u}$ is false under t; the literal $\bar{u}$ is true if and only if the variable u is false.

A *clause* over U is a set of literals over U. It represents the disjunction of these literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathscr{C}$ of clauses over U is satisfiable if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment is called a *satisfying truth assignment* for $\mathscr{C}$. The 3-SAT is specified as follows.

3-satisfiability problem (3-SAT):

Instance: A collection $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|Cj| = 3$ for $j = 1, 2, ..., m$.

Question: *Is there a truth assignment for U that satisfies all the clauses in $\mathscr{C}$?*

Theorem 7. ([15] Theorem 3.1). *3-SAT is NP-complete.*

Theorem 8. *3RSN is NP-hard even for bipartite graphs.*

Proof. The transformation is from 3-SAT. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT. We will construct a bipartite graph G and choose an integer k such that $\mathscr{C}$ be satisfiable if and only if $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq k$. We construct such a graph G as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a complete bipartite graph $H_i = K_{3,5}$ with bipartite sets $X = \{x_i, y_i, z_i\}$ and $Y = \{v_i, u_i, w_i, \overline{u}_i, r_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$, associate a single vertex c_j and add the edge set $c_j u_i$ if $u_i \in C_j$ and $c_j \bar{u}_i$ if $\bar{u}_i \in C_j$. Finally, add a graph H with vertex set $V(H) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ and $E(H) = \{s_1 s_2, s_2 s_3, s_3 s_4, s_4 s_5, s_5 s_6, s_6 s_7\}$, join s_1 and s_7 to each vertex c_j with $1 \leq j \leq m$ and set $k = 1$.

Figure 1 shows that on the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}, C_2 = \{\bar{u}_1, u_2, u_4\}, C_3 = \{\bar{u}_2, u_3, u_4\}$. To prove that this is indeed a transformation, we only need to show that $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$ if and only if there is a truth assignment for

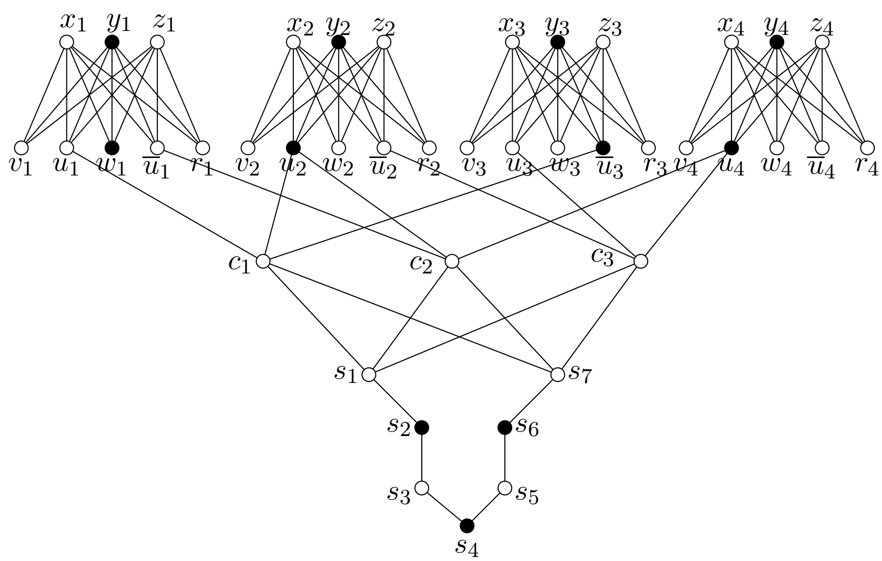

Figure 1. An instance of the triple Roman subdivision number problem resulting from an instance of 3-SAT. Here $k = 1$ and $\gamma_{[3R]}(G) = 43$, where the bold vertex p means there is a TRDF f with $f(p) = 4$, with the exception of s_4, where $f(s_4) = 3$.

U that satisfies all clauses in $\mathscr{C}$. This aim can be obtained by proving the following four claims.

Claim 1. $\gamma_{[3R]}(G) \geq 8n + 11$. Moreover, if $\gamma_{[3R]}(G) = 8n + 11$, then for any $\gamma_{[3R]}$-function f on G and for each $1 \leq i \leq n, f(H_i) = 8, f(V(H)) = 11, \max\{f(s_1), f(s_7)\} \leq 2, \min\{f(s_1), f(s_7)\} = 0$ and $f(c_j) = 0$ for each $1 \leq j \leq m$.

Proof. Let f be a $\gamma_{[3R]}$-function of G. Without loss of generality assume that $f(z) \neq 1$ for all vertices $z \in V(G)$ by Proposition A. For each $i = 1, 2, \ldots, n$, it is also clear that $f(V(H_i)) \geq 8$ and $f(N_G[H]) \geq 11$, implying that $\gamma_{[3R]}(G) \geq 8n + 11$.

Now suppose that $\gamma_{[3R]}(G) = 8n + 11$. Since $H_i := K_{m,n}$ with $m, n \geq 3$, $f(V(H_i)) = 8$. We show that $f(s_1) + f(s_7) < 5$ and

$\max\{f(s_1), f(s_7)\} \neq 4$. To contradiction, let $f(s_1) + f(s_7) \geq 5$. If $f(s_1) = 4$ ($f(s_7) = 4$ is similar), then since $\sum_{i=3}^{5} f(s_i) \geq 4$ and $f(s_6) + f(s_7) \geq 4$, we have $\gamma_{[3R]}(G) \geq 8n + 12$, a contradiction. Thus, $\max\{f(s_1), f(s_7)\} \neq 4$ and $f(s_1) + f(s_7) \leq 6$. If $f(s_1) + f(s_7) = 6$, then $f(s_1) = f(s_7) = 3$. This implies that $\sum_{i=2}^{6} f(s_i) \geq 6$ and so $\gamma_{[3R]}(G) \geq 8n+12$, which is a contradiction. If $f(s_1)+f(s_7) = 5$, without loss of generality, let $f(s_1) = 2$ and $f(s_7) = 3$ ($f(s_1) = 3, f(s_7) = 2$ is similar), then $f(N[s_1]) + f(s_3) + f(N[s_5]) \geq 9$. Thus, $\gamma_{[3R]}(G) \geq 8n + 12$, which is a contradiction again. Therefore, $f(H_i) = 8$ for each $1 \leq i \leq n, \max\{f(s_1), f(s_7)\} \leq 2, \min\{f(s_1), f(s_7)\} = 0$. Now assume $f(c_j) \neq 0$ for some $j \in \{1, 2, ..., n\}$. Since $\max\{f(s_1), f(s_7)\} \leq 2$ and $\min\{f(s_1), f(s_7)\} = 0$, we have $f(s_1) = f(s_7) = 0, f(s_1) = 2$ and $f(s_7) = 2$ or $f(s_1) = 2$ and $f(s_7) = 0$. First assume that $f(s_1) = f(s_7) = 0$. If $f(s_2) = 0$ or $f(s_6) = 0$, then $\sum_{j+1}^{m} f(c_j) \geq 4$ and $\sum_{i=2}^{6} f(s_i) \geq 8$. Hence $\gamma_{[3R]}(G) \geq 8n + 12$, a contradiction. If $f(s_2) \neq 0$ and $f(s_6) \neq 0$, then $\sum_{j+1}^{m} f(c_j) \geq 2$ and $\sum_{i=2}^{6} f(s_i) \geq 10$, which is a contradiction. Now, assume that $f(s_1) = 0$ and $f(s_7) = 2$ or $f(s_1) = f(s_7) = 2$. Then, similar as above, it is easy to see that $\gamma_{[3R]}(G) \geq 8n + 12$, a contradiction. Therefore, $f(c_j) = 0$ for each $1 \leq j \leq m$, as desired. □

Claim 2. $\mathscr{C}$ is satisfiable if and only if $\gamma_{dR}(G) = 8n + 11$.

Proof. Suppose that $\gamma_{[3R]}(G) = 8n + 11$ and let f be a $\gamma_{[3R]}$-function of G. By Claim 1, $f(H_i) = 8$ for each $i = 1, 2, \dots n$, and since H_i is a complete bipartite graph, at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 4 for each i. Define a mapping $t : U \longrightarrow \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if either } f(u_i) = 4 \text{ or } f(u_i), f(\overline{u_i}) \neq 4 \\ F & \text{if } \ f(\bar{u}_i) = 4. \end{cases} \quad i = 1, \dots, n \qquad (1)$$

Suppose that $\gamma_{[3R]}(G) = 8n+11$ and let f be a $\gamma_{[3R]}$-function of G. By Claim 1, $f(H_i) = 8$ for each $i = 1, 2, \dots .n$, and since H_i is a complete bipartite graph, at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 4 for each i. We now show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$ with $1 \leq j \leq m$.

By Claim 1, we have $f(s_1) + f(s_7) < 5, \max\{f(s_1), f(s_7)\} \leq 2, \min\{f(s_1), f(s_7)\} = 0$ and $f(c_j) = 0$ for each $1 \leq j \leq m$. Besides, since c_j is not adjacent to s_i for $i = 2, \ldots, 6$, then there exists some i with $1 \leq i \leq n$ such that c_j is adjacent to u_i or $\bar{u}_i$. Suppose that c_j is adjacent to u_i, where $f(u_i) = 4$. Since u_i is adjacent to c_j in G, the literal u_i is in the clause C_j by the construction of G. Since $f(u_i) = 4$, it follows that $t(u_i) = T$ by (1), which implies that the clause C_j is satisfied by t. Suppose that c_j is adjacent to $\bar{u}_i$, where $f(\bar{u}_i) = 4$. Since $\bar{u}_i$ is adjacent to c_j in G, the literal $\bar{u}_i$ is in the clause C_j. Since $f(\bar{u}_i) = 4$, it follows that $t(u_i) = F$ by (1). Thus, t assigns $\bar{u}_i$ the truth value T, that is, t satisfies the clause C_j. By the arbitrariness of j with $1 \leq j \leq m$, we show that t satisfies all the clauses in $\mathscr{C}$, so $\mathscr{C}$ is satisfiable.

Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \longrightarrow \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. Create a function f on $V(G)$ as follows: if $t(u_i) = T$, then let $f(u_i) = 4$, and if $t(u_i) = F$, then let $f(\bar{u}_i) = 4$. Let $f(y_i) = f(s_2) = f(s_6) = 4$ for each $1 \leq i \leq n, f(s_4) = 3$ and the remaining vertices of G assigned a 0 under f. Clearly, $f(G) = 8n + 11$. Since t is a satisfying truth assignment for $\mathscr{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex p with $f(p) = 4$. Since c_j is adjacent to each literal in C_j by the construction of G, thus f is a TRDF of G, and so $\gamma_{[3R]}(G) \leq f(G) = 8n + 11$. Hence $\gamma_{[3R]}(G) = 8n + 11$, by Claim 1. □

Claim 3. Let G' be obtained from G by subdividing any edge e of $E(G)$, then $\gamma_{[3R]}(G') \leq 8n + 12$.

Proof. Let $e = uv \in E(G)$ and let G' be obtained from G by subdividing the edge e with vertex w. If $e \in \{s_1s_2, s_2s_3\}$, consider the function $f : V(G') \longrightarrow \{0, 1, 2, 3, 4\}$ defined by $f(s_1) = f(s_3) = f(s_6) = f(x_i) = f(v_i) = 4$ for each $1 \leq i \leq n$ and $f(x) = 0$ for all other $x \in V(G')$. If

$e \in \{s_4s_5, s_5s_6\}$, consider the function $f : V(G') \longrightarrow \{0,1,2,3,4\}$ defined by $f(s_1) = f(w) = f(s_6) = f(x_i) = f(v_i) = 4$ for each $1 \le i \le n$ and $f(x) = 0$ for all other $x \in V(G')$. If $e = s_6s_7$, consider the function $f : V(G') \longrightarrow \{0,1,2,3,4\}$ defined by $f(s_1) = f(s_4) = f(w) = f(x_i) = f(v_i) = 4$ for each $1 \le i \le n$ and $f(x) = 0$ for all other $x \in V(G')$. If $e = s_6s_7$, consider the function $f : V(G') \longrightarrow \{0,1,2,3,4\}$ defined by $f(s_2) = f(s_4) = f(w) = f(x_i) = f(v_i) = 4$ for each $1 \le i \le n$ and $f(x) = 0$ for all other $x \in V(G')$. If $e \in \{s_1c_j, s_7c_j$, for each $j = 1,2,\ldots,m\}$, consider the function $f : V(G') \longrightarrow \{0,1,2,3,4\}$ defined by $f(s_1) = f(s_4) = f(s_7) = f(x_i) = f(v_i) = 4$ for each $1 \le i \le n$ and $f(x) = 0$ for all other $x \in V(G')$. If $e \in \{c_ju_i$, for each $1 \le i \le n$, $1 \le j \le m\}$ or $e \in \{c_j\overline{u}_i$, for each $1 \le i \le n$, $1 \le j \le m\}$, consider the function $f : V(G') \longrightarrow \{0,1,2,3,4\}$ defined by $f(s_1) = f(s_4) = f(s_7) = f(x_i) = f(u_i) = 4$ or $f(s_1) = f(s_4) = f(s_7) = f(x_i) = f(\bar{u}_i) = 4$, respectively, for each $1 \le i \le n$ and $f(x) = 0$ for all other $x \in V(G')$. If $e = uv$ such that $u \in \{x_i, y_i, z_i\}$ and $v \in \{v_i, u_i, w_i, \overline{u}_i, r_i\}$, consider the function $f : V(G') \longrightarrow \{0,1,2,3,4\}$ defined by $f(s_1) = f(s_4) = f(s_7) = f(u) = f(v) = 4$ and $f(x) = 0$ for all other $x \in V(G')$, then in each case, f is a TRDF on G' with $\omega(f) = 8n + 12$. Therefore, $\gamma_{[3R]}(G') \le 8n + 12$. □

Claim 4. $\gamma_{[3R]}(G) = 8n + 11$ *if and only if* $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$.

Proof. Assume $\gamma_{[3R]}(G) = 8n + 11$. Let G' be obtained from G by subdivision the edge $e = s_1s_2$ with vertex w.

Suppose for a contradiction that $\gamma_{[3R]}(G) = \gamma_{[3R]}(G')$. Let f' be a $\gamma_{[3R]}$-function of G'. It is easy to see that $f'(V(H) \cup \{w\}) \ge 12$. On the other hand, since $f'(H_i) \ge 8$ for each $1 \le i \le n$, we have $\gamma_{[3R]}(G') = \omega(f') \ge 8n + 12$. This implies that $\gamma_{[3R]}(G) \ge 12$, which is a contradiction. Therefore, $\gamma_{[3R]}(G) < \gamma_{[3R]}(G')$ and so $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$.

Conversely, assume that $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$. Let G' be obtained from G by subdivision the edge e such that $\gamma_{[3R]}(G) < \gamma_{[3R]}(G')$. By Claim 3, we have $\gamma_{[3R]}(G') \le 8n + 12$. Now the result follows by Claim 1. □

By Claims 2 and 4, we prove that $\mathrm{sd}_{\gamma_{[3R]}}(G) = 1$ if and only if there is a truth assignment for U that satisfies all clauses in $\mathscr{C}$. Since the construction of the triple Roman subdivision number instance is straightforward from a 3-satisfiability instance, the size of the triple Roman subdivision number instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial reduction and the proof is complete. $\square$

4 Bounds in terms of order and maximum degree

In this section, we present some upper bounds on $\mathrm{sd}_{\gamma_{[3R]}}(G)$ in terms of the vertex degree and the minimum degree of G.

Theorem 9. *Let G be a connected graph. If $v \in V(G)$ has the degree at least two, then $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq \deg(v)$.*

Proof. Let $t = \deg(v)$ and $N(v) = \{v_1, v_2, \ldots, v_t\}$ and G' be obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_t$ with vertices $x_1, x_2, \ldots, x_t$, respectively. Let f be a $\gamma_{[3R]}(G')$-function. Without loss of generality, assume that $f(z) \neq 1$ for all vertices $z \in V(G')$ by Proposition A. If $\Sigma_{i=1}^{t} f(x_i) + f(v) \geq 5$, then define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(v) = 4$ and $g(z) = f(z)$ for $z \in V(G) \setminus \{v\}$. Clearly, g is a TRDF of G with $\omega(g) < \omega(f)$ and hence $sd_{[3R]}(G) \leq \deg(v)$. We assume that $\Sigma_{i=1}^{t} f(x_i) + f(v) \leq 4$. (Note that f is a TRDF on G' and so $f(N[v]) \geq 3$). If $f(v) = 2$, then there exists some x_i, say without loss of generality, x_1, such that $f(x_1) = 2$ and $\sum_{i=2}^{t} f(x_i) = 0$. Hence $f(v_i) \geq 3$ for each $2 \leq i \leq t$. Define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(v) = 3$ and $g(z) = f(z)$ for $z \in V(G) \setminus \{v\}$. If $f(v) = 3$, then $\sum_{i=1}^{t} f(x_i) = 0$ and $f(v_i) \geq 2$ for each $1 \leq i \leq t$. Define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(v) = 2$ and $g(z) = f(z)$ for $z \in V(G) \setminus \{v\}$. Finally, assume that $f(v) = 4$. Hence $\sum_{i=1}^{t} f(x_i) = 0$. Define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(v) = 3$ and $g(z) = f(z)$ for $z \in V(G) \setminus \{v\}$. Clearly, in each case, g is a TRDF of G with $\omega(g) < \omega(f)$ and so $\gamma_{[3R]}(G) \leq \omega(g) < \omega(f) \leq \gamma_{[3R]}(G')$.

Now assume that $f(v) = 0$, then to be triple Roman dominate the vertex v, we must have $f(x_i) = 4$ for some $1 \leq i \leq t$, say $i = 1$.

Thus, $f(x_j) = 0$, and thus, $f(v_j) = 4$ for all $2 \leq j \leq t$. Define $g : V(G) \longrightarrow \{0,1,2,3,4\}$ by $g(v_1) = \min\{f(v_1)+3, 4\}$ and $g(z) = f(z)$ for $z \in V(G) \setminus \{v_1\}$. Since $t \geq 2$, clearly g is a TRDF of G with $\omega(g) < \omega(f)$. Thus, $\gamma_{[3R]}(G) \leq \omega(g) < \omega(f)$, and this implies that $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq \deg(v)$. □

A consequence of Theorem 9 is that $\mathrm{sd}_{\gamma_{[3R]}}(G)$ is defined for every connected graph G of order $n \geq 3$. In addition:

Corollary 10. *For every connected graph G with $\delta \geq 2$, $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq \delta$.*

It is well known that every planar graph contains at least one vertex of degree at most five. Thus, the following result is an immediate consequence of Corollary 10.

Corollary 11. *For every planar graph G, $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 5$.*

In the sequel, we present an upper bound on the triple Roman domination subdivision number in terms of δ_2. We make use of the following lemmas in the proof of Theorem 12.

Lemma 2. *Let G be a connected graph of order $n \geq 3$ and let G have a vertex $v \in V(G)$ which is contained in a triangle uvw such that $N(u) \cup N(w) \subseteq N[v]$. Then $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 3$.*

Proof. Let G' be obtained from G by subdivision the edges vu, vw, uw with vertices x, y, z, respectively. Let f be a $\gamma_{[3R]}(G')$-function. Without loss of generality, assume that $f(z) \neq 1$ for all vertices $z \in V(G')$ by Proposition A. We claim that $f(v)+f(u)+f(w)+f(x)+f(y)+f(z) \geq 6$. If $0 \notin \{f(x), f(y), f(z)\}$, then the claim is directly correct. Hence, we assume that $0 \in \{f(x), f(y), f(z)\}$, say without loss of generality, $f(x) = 0$, then $f(u) + f(v) \geq 4$. Now, if $f(y) \neq 0$ or $f(z) \neq 0$, then $f(v) + f(u) + f(w) + f(x) + f(y) + f(z) \geq 6$, as desired. Hence, we assume that $f(y) = f(z) = 0$, therefore, to triple Roman dominate x, y and z we must have $f(u) + f(v) + f(w) \geq 8$. Now define the function $g : V(G) \longrightarrow \{0,1,2,3,4\}$ by $g(v) = 4$, $g(u) = g(w) = 0$ and $g(s) = f(s)$ for each $s \in V(G) \setminus \{v, u, w\}$. Obviously g is a TRDF of G of weight less than $\gamma_{[3R]}(G)$. This completes the proof. □

Lemma 3. *Let G be a connected graph of order $n \geq 3$ and let G have a vertex $v \in V(G)$ which is contained in a triangle uvw such that $N(u) \subseteq N[v]$ and $N(w) \setminus N[v] \neq \emptyset$. Then*

$$\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 3 + |N(w) \setminus N[v]|.$$

Proof. Let $N(w) \setminus N[v] = \{w_1, w_2, \ldots, w_k\}$ and let G' be obtained from G by subdividing the edges vu, vw, uw with x, y, z, respectively, and for each $1 \leq i \leq k$ the edge ww_i with vertex z_i. Assume g is a $\gamma_{[3R]}(G')$-function. Similar as the proof of Lemma 2, we have $g(v) + g(u) + g(w) + g(x) + g(y) + g(z) \geq 6$. Define $h : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $h(v) = 4, h(u) = h(w) = 0, h(w_i) = \min\{g(w_i) + g(z_i), 4\}$ for each $1 \leq i \leq k$ and $h(s) = g(s)$ for each $s \in V(G) \setminus \{v, u, w, w_1, \ldots, w_k\}$. It is easy to see that h is a TRDF of G of weight less than $\gamma_{[3R]}(G')$ and the proof is complete. □

Lemma 4. *Let G be a simple connected graph of order $n \geq 3$ and v a vertex of degree at least 2 of G such that*

(i) $N(y) \setminus N[v] \neq \emptyset$ *for each* $y \in N(v)$,

(ii) *there exists a pair* α, β *of vertices in* $N(v)$ *such that* $(N(\alpha) \cap N(\beta)) \setminus N[v] = \emptyset$,

Then $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 3 + |N_2(v)|$.

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_{\deg(v)}\}$. Without loss of generality, assume that $\alpha = v_1$ and $\beta = v_2$. Moreover, we will assume that the pair α, β is chosen first among the adjacent vertices in $N(v)$. Hence, if $\alpha\beta \in E(G)$, then we assume also that $v_1v_2 \in E(G)$. In addition, let $S = \{v_1, v_2, \ldots, v_k\}$ be one of the largest subset of $N(v)$ containing v_1, v_2 and such that every pair α, β of vertices of S satisfies (ii). According to item (i), let $N(v_i) \setminus N[v] = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{l_i}}\}$ for each $i \in \{1, 2, \ldots, k\}$. Now consider the graph G_1 obtained from G by subdividing the edges vv_1 and vv_2 with new vertices x_1 and x_2, respectively, and for all $i \in \{1, 2, \ldots, k\}$, each of the edges $v_iv_{i_j}, 1 \leq j \leq l_i$, with a new vertex v^{i_j}. We put $T_i = \{v^{i_j} \mid 1 \leq j \leq l_i\}$ and $T = \cup_{1 \leq i \leq k} T_i$. Furthermore, if v_1 and v_2 are adjacent, then we also subdivide the edge v_1v_2 with a vertex u. Let f be a $\gamma_{[3R]}(G_1)$-function. Without loss of generality, assume that $f(z) \neq 1$ for all vertices $z \in V(G_1)$ by Proposition A. Assume first that $v_1v_2 \in E(G)$. Then, as seen in the proof of Lemma 2, we have $f(v)+f(v_1)+f(v_2)+f(x_1)+f(x_2)+f(u) \geq 6$. Define $g : V(G) \to \{0,1,2,3,4\}$ by $g(v) = 4$, $g(v_1) = g(v_2) = 0$, $g(v_{i_j}) = \min\{f(v_{i_j}) + f(v^{i_j}), 4\}$ for all $i \in \{1,2,\dots,k\}$ and all $j \in \{1,2,\dots,l_i\}$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

Assume now that $v_1v_2 \notin E(G)$. By the choice of v_1, v_2, we conclude that S is an independent set. Also to dominate x_1, x_2, we must have $f(v)+f(x_1)+f(x_2)+f(v_1)+f(v_2) \geq 4$. If $f(v)+f(x_1)+f(x_2)+\sum_{i=1}^{k} f(v_i) \geq 5$, then the function g defined on $V(G)$ by $g(v) = 4$, $g(v_i) = 0$ for $i \in \{1,2,\dots,k\}, g(v_{i_j}) = \min\{f(v_{i_j}) + f(v^{i_j}), 4\}$ for all $i \in \{1,2,\dots,k\}$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Thus, we may assume that $f(v)+f(x_1)+f(x_2)+\sum_{i=1}^{k} f(v_i) = 4$. Hence, $f(v) = 0$ or $f(v) = 4$.

If $f(v) = 0$, then $f(x_1)+f(x_2)+f(v_1)+f(v_2) \geq 6$, which is a contradiction. Thus, $f(v) = 4$ and so $f(x_1)+f(x_2)+\sum_{i=1}^{k} f(v_i) = 0$.

If $\sum_{j=1}^{l_i} f(v^{i_j}) \geq 5$ for some $1 \leq i \leq k$, say $i = 1$, then the function g defined on $V(G)$ by $g(v_1) = 4$, $g(v_{i_j}) = \min\{f(v_{i_j}) + f(v^{i_j}), 4\}$ all $i \in \{2,3,\dots,k\}$ and all $j \in \{1,2,\dots,l_i\}$, and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Hence, suppose that $\sum_{j=1}^{l_i} f(v^{i_j}) \leq 4$, for each $i \in \{1,2,\dots,k\}$. If $2 \leq f(v^{i_j}) \leq 4$ for some $i \in \{1,2,\dots,k\}$ and for some $j \in \{1.2,\dots,l_i\}$, say $i = j = 1$, then define g by

$$g(v_{1_1}) = \begin{cases} \min\{4, 3+f(v_{1_1})\} & \text{if} \quad f(v^{1_1}) = 4 \\ \min\{4, 2+f(v_{1_1})\} & \text{if} \quad f(v^{1_1}) = 3 \\ \min\{4, 1+f(v_{1_1})\} & \text{if} \quad f(v^{1_1}) = 2, \end{cases}$$

$g(v_{i_j}) = \min\{4, f(v_{i_j}) + f(v^{i_j})\}$ when $i_j \neq 1_1$ and $g(x) = f(x)$ otherwise. Clearly, g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Hence, we can assume that $f(v^{i_j}) = 0$ for each i and j. This implies that $f(v_{i_j}) = 4$ for every i and j. In this case, we can define the function g

on $V(G)$ by $g(v) = 3$ and $g(x) = f(x)$ otherwise. Clearly, g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

In each of the situation we saw, graph G has a TRDF of weight less than $\gamma_{[3R]}(G_1)$. Moreover, since G_1 was obtained by inserting at most $3+|T| \leq 3+|N_2(v)|$ new vertices, we obtained $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 3+|N_2(v)|$. This completes the proof. $\square$

Lemma 5. *Let G be a simple connected graph of order $n \geq 3$ and v be a vertex of degree at least 2 in G such that*

(i) $N(y) \setminus N[v] \neq \emptyset$ for each $y \in N(v)$

(ii) for every pair of vertices α, β in $N(v)$, $(N(\alpha) \cap N(\beta)) \setminus N[v] \neq \emptyset$.

Then $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 3 + |N_2(v)|$.

Proof. The result follows from Theorem 9 if $\deg(v) \leq 3 + |N_2(v)|$. Hence, assume that $\deg(v) \geq 4 + |N_2(v)|$. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and let $M = N(v_1) - N[v] = \{w_1, w_2, \ldots, w_p\}$. It follows from the hypothesis that each $y \in N(v) \setminus \{v_1\}$ has a neighbor in M. Let T be one of the largest subsets of $N(v) \setminus \{v_1\}$ such that for each $T_1 \subseteq T, |N(T_1) \setminus (N[v] \cup M)| \geq |T_1|$. By the definition of $T, |N_2(v)| \geq |M| + |T|$. Moreover, every vertex u in $U = N(v) \setminus (T \cup \{v_1\})$, has a neighbor in M and $N(u) \setminus N[v] \subseteq M \cup N(T)$. Also M dominates $N(v)$ (by item (ii)). We deduced from $4 + |M| + |T| \leq 4 + |N_2(v)| \leq \deg(v) = |T| + 1 + |U|$ that $|U| \geq 4$. If $T \neq \emptyset$, then without loss of generality, let $T = \{v_2, v_3, \ldots, v_s\}$.

Let G_1 be the graph obtained from G by subdividing the edges v_1w_j with new vertices y_j for all $j \in \{1, 2, \ldots, p\}$ and vv_i with vertices x_i for $1 \leq i \leq s+2$ (when $T \neq \emptyset$) or $1 \leq i \leq 3$ (when $T = \emptyset$). Hence, $|M| + |T| + 3$ edges of G are subdivided. Let f be a $\gamma_{[3R]}(G_1)$-function. Without loss of generality, assume that $f(z) \neq 1$ for all vertices $z \in V(G_1)$ by Proposition A. If $f(v)+f(v_1)+\sum_{i=1}^{s+2} f(x_i) \geq 5$, then define $g : V(G) \longrightarrow \{0, 1, 2, 3, 4\}$ by $g(v) = 4$, $g(v_1) = 0$, $g(w_j) = \min\{4, f(w_j)+f(y_j) \mid 1 \leq j \leq p\}$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Hence, we assume that

$$f(v) + f(v_1) + \sum_{i=1}^{s+2} f(x_i) \leq 4. \tag{2}$$

If $f(v) = 3$, then it follows from (2) that $f(x_1) = f(v_1) = 0$ which is a contradiction. Consider the following three cases depending on the values of v under f.

Case 1. $f(v) = 4$.
It follows from (2) that $f(v_1) = \sum_{i=1}^{s+2} f(x_i) = 0$. Assume first that $\sum_{i=1}^{p} f(y_i) \geq 5$, then the function $g : V(G) \to \{0, 1, 2, 3\}$ defined by $g(v_1) = 4$, and $g(x) = f(x)$ otherwise, is a TRDF of weight less than $\gamma_{[3R]}(G_1)$. Assume now that $\sum_{i=1}^{p} f(y_i) = 4$. If $f(y_j) = 4$ for some $j \in \{1, 2, \ldots, p\}$, say $j = 1$, then $f(y_j) = 0$ for all $j \in \{2, 3, \ldots, p\}$ and the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ defined by $f(w_1) = \min\{4, 3 + f(w_1)\}$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Now assume that $f(y_j) = f(y'_j) = 2$ for some $j, j' \in \{1, 2, \ldots, p\}$, say $j = 1$ and $j' = 2$, then $f(y_j) = 0$ for all $j \in \{3, \ldots, p\}$. Moreover, the definition of f implies that $f(w_1) \geq 2$ and $f(w_2) \geq 2$. Define the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(w_1) = g(w_2) = 3$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Suppose now that $\sum_{i=1}^{p} f(y_i) = 3$. Thus, there exists some $j \in \{1, 2, \ldots, p\}$, say $j = 1$, such that $f(y_1) = 3$ and $f(y_j) = 0$ for all $j \in \{2, 3, \ldots, p\}$. Then the function g defined on $V(G)$ by $g(w_1) = \min\{3, f(w_1) + 2\}$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Now assume that $\sum_{i=1}^{p} f(y_i) = 2$, then $f(y_i) = 2$ for some $i \in \{1, 2, \ldots, k\}$, say $i = 1$, so by the definition of f, $f(w_1) \geq 2$. Then the function g defined on $V(G)$ by $g(w_1) = \min\{3, f(w_1) + 1\}$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Finally, assume that $\sum_{i=1}^{p} f(y_i) = 0$, the $f(w_i) = 4$ for each $1 \leq j \leq p$. Since every vertex of U has a neighbor in M, define $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(v) = 3$ and $g(x) = f(x)$ otherwise. Since every vertex of U has a neighbor in M, we deduced that g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

Case 2. $f(v) = 0$.

To dominate x_1, we must have $f(x_1) + f(v_1) \geq 3$. On the other hand, $f(x_1) + f(v_1) \leq 4$ by (2). If $\sum_{i=1}^{p} f(y_i) \geq 2$, then define function $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(v_1) = 4$ and $g(x) = f(x)$ otherwise. Clearly, g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Now let $\sum_{i=1}^{p} f(y_i) = 0$. If $f(v_1) = 4$, then $f(x_1) = 0$ and the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ defined by $g(v_1) = 3$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

Now let $3 \leq f(x_1) \leq 4$, then $f(v_1) = f(x_i) = 0$ and so $f(v_i) = f(w_j) = 4$ for each $2 \leq i \leq s + 2$ and $1 \leq j \leq p$, respectively. Define $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(v) = g(v_1) = 4, g(v_{s+1}) = g(v_{s+2}) = 0$ and $g(x) = f(x)$ otherwise. Since for each $s + 1 \leq i$, v_i has a neighbor in $M = \{w_1, w_2, \ldots, w_p\}$ and $N(v_i) \setminus N[v] \subseteq M \cup N(T)$, we deduced that g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

Now let $f(x_1) = 2$ and $f(v_1) = 2$. Since $\sum_{j=1}^{p} f(y_j) = \sum_{i=2}^{s+2} f(x_i) = 0$, $f(w_j) \geq 3$ and $f(v_i) = 4$ for each $1 \leq j \leq p$ and $2 \leq i \leq s + 2$, respectively. Define $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(v_1) = 3$ and $g(x) = f(x)$ otherwise. Clearly, g is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

Case 3. $f(v) = 2$.

To dominate x_1, we must have $f(x_1) + f(v_1) \geq 2$. On the other hand, $f(x_1) + f(v_1) \leq 2$ by (2). It implies that $f(x_1) + f(v_1) = 2$ and so $f(x_1) = 2$, $f(v_1) = 0$. If $\sum_{i=1}^{p} f(y_i) \geq 3$, then define $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(v_1) = 4$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Assume now that $\sum_{i=1}^{p} f(y_i) = 2$, then $f(y_j) = 2$ for some j, say $j = 1$. Thus, $f(w_1) \geq 2$ and $f(y_j) = 0$ for each $2 \leq j \leq p$. Define $g : V(G) \to \{0, 1, 2, 3, 4\}$, by $g(w_1) = \min\{f(w_1) + 1, 4\}, f(v) = 3$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$. Finally, suppose that $\sum_{i=1}^{p} f(y_i) = 0$, then $f(w_j) = 4$ for each $1 \leq j \leq p$. Define $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(v) = 3$ and $g(x) = f(x)$ otherwise, is a TRDF of G of weight less than $\gamma_{[3R]}(G_1)$.

In either case, $\gamma_{[3R]}(G) < \gamma_{[3R]}(G_1)$, and this completes the proof. $\square$

We are now ready to prove our main result.

Theorem 12. *Let G be a simple connected graph of order $n \geq 3$. Then*

$$sd_{\gamma_{[3R]}(G)} \leq 3 + \min\{d_2(v) \mid v \in V \text{ and } \deg(v) \geq 2\}.$$

Proof. If G is a star $K_{1,n-1}$, then $sd_{\gamma_{[3R]}(G)} = 1$, and the result is valid. Hence, assume that G is different from a star. If G has a leaf, then by Proposition 3, the result is also valid. Now, assume that G is a graph with $\delta(G) \geq 2$. According to Lemmas 2, 3, 4, and 5, we have

$$sd_{\gamma_{[3R]}(G)} \leq 3 + \min\{d_2(v) \mid v \in V \text{ and } \deg(v) \geq 2\}.$$

$\square$

The following example of graphs shows that the bound in Theorem 12 is better than the one in Theorem 9. Consider $r \geq 3$ copies of the complete graph K_n with $n \geq 7$, and let x_i be a vertex of the i-th copy of K_n. Let G be the connected graph obtained from the r copies of K_n by adding edges between vertices x_i's so that they induce cycle C_r. One can easily check that by Theorem 9, $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq n-1$, while by Theorem 12, we have $\mathrm{sd}_{\gamma_{[3R]}}(G) \leq 3 + \deg_2(v) = 5$ for any vertex $v \notin \{x_1, \ldots, x_r\}$.

Let $\delta_2(G) = \min\{\deg_2(v) \mid v \in V \text{ and } \deg(v) \geq 2\}$ and observe that for every vertex v of degree at least two, $\delta_2(G) \leq |N_2(v)| \leq n - \Delta - 1$.

The following two Corollaries are immediate consequences of Theorem 12.

Corollary 13. For any connected graph G with $\delta(G) \geq 2$, $sd_{\gamma_{[3R]}(G)} \leq \delta_2(G) + 3$.

We observe that for a vertex v of degree Δ, $|N_2(v)| \leq n - \Delta - 1$ and thus we obtain the following result.

Corollary 14. Let G be a connected graph of order $n \geq 3$. Then $sd_{\gamma_{[3R]}(G)} \leq n - \Delta + 2$.

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