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On the bondage, strong and weak bondage numbers in Complementary Prism Graphs

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Abstract

Let $G = (V(G), E(G))$ be a simple undirected graph of order n, and let $S \subseteq V(G)$. If every vertex in $V(G) - S$ is adjacent to at least one vertex in S, then the set S is called a *dominating set*. The *domination number* of G is the minimum cardinality taken over all sets of S, and it is denoted by $\gamma(G)$. Recently, the effect of one or more edges deletion on the domination number has been examined in many papers. Let $F \subseteq E(G)$. The *bondage number* $b(G)$ of G is the minimum cardinality taken over all sets of F such that $\gamma(G-F) > \gamma(G)$. In the literature, a lot of domination and bondage parameters have been defined depending on different properties. In this paper, we investigate the *bondage, strong and weak bondage numbers* of complementary prism graphs of some well-known graph families.

Keywords: Connectivity, Domination number, Strong and weak domination numbers, Bondage number, Strong and weak bondage numbers, Complementary prism graphs.

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1 Introduction

Graph theory has become an important mathematical tool in many different sciences. For example, the domination number is an important graph parameter, and it has many different application areas [15]. In the near past, some papers published about how the domination varies when there are changes in edges or vertices by adding or removing. This is important because vertices in the domination set can be considered transmitters that cover a wide variety of communication links. The loss of certain links may make the transmitter set a non-dominating set. i.e., the communications between some links can be disrupted by a wrecker. Consider that a wrecker does not know which vertices in the network serve as transmitters but knows that those vertices form a minimum domination set in the network. What is the minimum number of connections that the wrecker must disrupt such that at least a new transmitter is needed to connect with all sites? With this in mind, the concept of bondage has begun to be studied in graph theory.

Let G be a simple undirected graph without loops and multiple edges with vertex set $V(G)$ and edge set $E(G)$. The order of G is the number of vertices in G. The degree of a vertex $v \in V(G)$ is the number of edges incident to v and it is denoted by $deg(v)$. Let $S \subseteq V(G)$. If every vertex in $V(G) - S$ is adjacent to at least one vertex in S, then the set S is called a *dominating set*. The *domination number* of G is the minimum cardinality over all domination set of G, and it is denoted by $\gamma(G)$.

The following question about domination number is very important: what is the minimum number of links that must be removed so that the domination number increases? Bauer et al. [7] has given the answer of this question. They have defined the *bondage number* for the vulnerability of a graph. The *bondage number* $b(G)$ of G is defined as the minimum cardinality among all subsets of edges $F \subseteq E(G)$ for which $\gamma(G-F) > \gamma(G)$ [11]. There are different parameters depending upon the domination number such as the reinforcement number [13], the average lower bondage number [18], the average lower reinforcement number [19], the residual domination number [20] and the link residual domination number [21]. Furthermore, different papers about the domination and bondage numbers can be seen in [2],[4],[5],[11],[18],[19].

The concept of a *strong dominating set* (sd-set) has been introduced by Sampathkumar and Pushpalatha [9]. Let $u, v \in V(G)$. A set $S \subseteq V(G)$ is a strong dominating set of G if every vertex u in $V(G) - S$ is adjacent to the vertex v in S such that $deg(v) \geq deg(u)$ and $(u, v) \in E(G)$. The *strong domination number* $\gamma_s(G)$ is the minimum cardinality over all strong dominating set of G. The strong bondage number $b_s(G)$ of G is defined as the minimum cardinality among all subsets of edges $F \subseteq E(G)$ for which $\gamma_s(G-F) > \gamma_s(G)$. This concept has been introduced by J. Ghoshal et al. [12].

A set $S \subseteq V(G)$ is a *weak dominating set* (wd-set) of G if every vertex u in $V(G)-S$ is adjacent to the vertex v in S such that $deg(v) \leq deg(u)$ and $(u,v) \in E(G)$. The *weak domination number* $\gamma_w(G)$ is the minimum cardinality over all strong dominating set of G. The *weak bondage number* $b_w(G)$ of G is defined as the minimum cardinality among all subsets of edges $F \subseteq E(G)$ for which that $\gamma_w(G-F) > \gamma_w(G)$ [9].

There have been applications of *strong* and *weak domination* in specific practical situations. For example, in a road network, where certain locations are related, the degree of vertex v is the number of roads that meet at v. If $deg(u) \geq deg(v)$, then the traffic at u is more severe than that at v, and vice versa. If traffic between u and v is considered, predilection should be given to the vehicles going from u to v. Thus, *u strongly dominates v* and *v weakly dominates u*.

Complementary prism graphs have been introduced by Haynes et al. [17]. Let $\overline{G}$ be a complemantary graph of a graph G. The complementary prism is denoted by $G\overline{G}$. It is a graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. The vertex $\overline{v}$ denotes the vertex v in the copy of $\overline{G}$, and it is defined for each $v \in V(G)$ [16], [17]. Many well-known graphs may be actualized as complementary prism graphs. For example, the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$. Another example, the Petersen graph is the complementary prism $C_5\overline{C}_5$ (see [17]).

Throughout this paper, minimum degree, maximum degree, vertex set and edge set of the graph G are denoted by $\delta(G)$, $\Delta(G)$, V and E, respectively [8]. Similarly, the vertex set and the edge set of the graph $\overline{G}$ are denoted by $\overline{V}$ and $\overline{E}$, respectively [8]. Furthermore, e_{uv} denotes the edges between the vertices u and v, $N(u)$ denotes the neighborhood of the vertex u.

The paper proceeds as follows. In Section 2, basic results of literature on the *strong-weak bondage number* of some special graphs are presented. Some results of the *bondage*, *strong* and *weak bondage numbers* for complementary prisms are given in Section 3. Finally, the conclusion of paper is given in Section 4.

2 General Bounds on Strong and Weak Bondage Numbers

In [14], sharp bounds were obtained for $b(G)$, $b_s(G)$ and $b_w(G)$. Furthermore, the exact values were determined for several classes of graphs such as K_n, C_n, P_n, $W_{1,n}$, $K_{m,n}$. In this section, we will review some of the known results.

Theorem 1 ([14]). *If G is a nonempty graph with a unique minimum dominating set, then $b(G) = 1$.*

Theorem 2 ([7], [11]). *If G is a nonempty graph, then $b(G) \leq min_{uv \in E(G)}(deg(u) + deg(v) - 1)$.*

Theorem 3 ([6]). *If G has edge connectivity k, then $b(G) \leq \Delta(G) + k - 1$.*

Theorem 4 ([14]). *If T is a nontrivial tree, then $b_s(T) \leq 3$ and $b_w(T) \leq \Delta(T)$.*

Theorem 5 ([14]). *If any vertex of tree T is adjacent with two or more end-vertices, then $b_s(T) = 1$.*

3 Exact Values for $b(G\overline{G})$, $b_s(G\overline{G})$ and $b_w(G\overline{G})$

We begin this subsection by determining the bondage, strong and weak bondage of the complementary prism $G\overline{G}$ when G is a specified family of graphs, such as the star graph $K_{1,n}$, the complete graph K_n, the path graph P_n, the cycle graph C_n, the wheel graph $W_{1,n}$, the complete bipartite graph $K_{m,n}$, the graph tK_2 and the graph $K_n o K_1$. The graph $K_n o K_1$ is obtained by adding a pendant vertex v that is $deg(v) = 1$ to each vertex of the graph K_n. In order to understand the proofs of the theorems given in this section more easily, the set of vertices belonging to the graph G in $G\overline{G}$ graph is shown as V and the set of vertices belonging to the $\overline{G}$ graph are shown as $\overline{V}$. Furthermore, when the edges set of $G\overline{G}$ is divided into $E(G\overline{G}) = E_1(G\overline{G}) \cup E_2(G\overline{G}) \cup E_3(G\overline{G})$, the edge sets here are respectively expressed as the set of edges of the graph G, the set of edges combining the graph G with the $\overline{G}$ graph, and the set of edges of the $\overline{G}$ graph.

Theorem 6. *If $G = K_{1,n}$, then $b(G\overline{G}) = b_s(G\overline{G}) = b_w(G\overline{G}) = 1$.*

Proof. Let $G = K_{1,n}$ and u be the center vertex of the graph G.
For $b(G\overline{G})$ and $b_s(G\overline{G})$; vertices u and $\overline{u}$ have to be in $\gamma(G\overline{G})$ and $\gamma_s(G\overline{G})$-strong dominating sets, where $\overline{v} \in \overline{V} - \{\overline{u}\}$. $\gamma(G\overline{G} - e_{u\overline{u}}) = \gamma_s(G\overline{G} - e_{u\overline{u}}) = \gamma(G\overline{G}) + |\{\overline{u}\}| = \gamma_s(G\overline{G}) + |\{\overline{u}\}|$ when an edge $e_{u\overline{u}} \in E(G\overline{G})$ is removed from the graph $G\overline{G}$, also it is easy to see that $b(G\overline{G}) = b_s(G\overline{G}) = 1$.

Now let's calculate the $b_w(G\overline{G})$ value of the graph. It is easily seen that the weak domination number of graph G is $\gamma_w(G\overline{G}) = n + 1$ from [1]. There are n vertices of degree 2 and a vertex of degree 1, say $\overline{u}$, which is adjacent to center vertex in $\gamma_w(G\overline{G})$-weak dominating set. This dominating set is unique. If an edge $e_{v\overline{v}} \in E(G\overline{G})$ is removed from the graph $G\overline{G}$, then the vertex $\overline{v}$ is not weakly dominated, where the vertex v is degree of 2 and so the vertex $\overline{v}$ must be in $\gamma_w(G\overline{G})$-weak dominating set. It can be easily seen that, $\gamma_w(G\overline{G} - e_{v\overline{v}}) = \gamma_w(G\overline{G}) + |\{\overline{v}\}|$ and it follows that $b_w(G\overline{G}) = 1$. □

Theorem 7. *If $G = K_n$, then $b(G\overline{G}) = b_s(G\overline{G}) = n$ and $b_w(G\overline{G}) = 1$.*

Proof. Let $G = K_n$. The vertices of graph $G\overline{G}$ are of two kinds: vertices of degree $n+1$ and one, respectively. The vertices of degree one will be referred to as pendant vertices and vertices of degree $n+1$ – as support vertices.

Let's calculate the bondage and strong bondage numbers of the graph. Since each vertex $\overline{v}$ in $\overline{V}$ $deg(\overline{v}) = 1$, $\gamma(G\overline{G})$-dominating sets and $\gamma_s(G\overline{G})$-strong dominating sets have to contain vertex set V. There are two ways for deleting the edges to increase the domination number of the graph $G\overline{G}$:

(i) If the n edges between V and $\overline{V}$ are removed from the graph $G\overline{G}$, then the rest of the graph $G\overline{G}$ is all independent pendant vertices and support vertices which consist of two complete graphs K_n. There are all pendant vertices and one support vertex in $\gamma(G\overline{G})$-dominating set and $\gamma_s(G\overline{G})$-strong dominating set. Therefore, we obtain $b(G\overline{G}) = b_s(G\overline{G}) = n$.
(ii) If one of the support vertices is isolated, then the domination number increases by one. Thus, n edges attached to any support vertex are removed.

From (i) and (ii), we have $b(G\overline{G}) = b_s(G\overline{G}) = n$.

Now let's calculate the weak bondage number of the graph. $\gamma_w(G\overline{G})$-weak dominating set must contain all vertices of $\overline{V}$. If an edge $e_{u\overline{u}} \in E(G\overline{G})$ for $\exists u \in V$ is removed from $G\overline{G}$, then the vertex $\overline{u}$ becomes the isolated vertex. Furthermore, since the degree of the vertex u is less than degrees of all vertices of the $V - \{u\}$, then $V - \{u\}$ set does not weakly dominate the vertex u. So, the vertex u must be in $\gamma_w(G\overline{G})$ -weak dominating set. It can be easily seen that $\gamma_w(G\overline{G} - e_{u\overline{u}}) > \gamma_w(G\overline{G})$ and we have $b_w(G) = 1$. □

Theorem 8. *If $G = P_n$ for $n > 5$ and $k > 1$, then*

i) $b(G\overline{G}) = \begin{cases} 1, & n=3k, \\ 2, & otherwise \end{cases}$

ii) $b_s(G\overline{G}) = \begin{cases} 3, & n=3k, \\ 2, & otherwise \end{cases}$

iii) $b_w(G\overline{G}) = \begin{cases} 2, & n=3k+1, \\ 1, & otherwise. \end{cases}$

Proof. In three different bondage measure proofs, three cases are examined according to $n \ mod \ 3$.

Proof of $b(G\overline{G})$ is obtained by three cases.
Case 1. If $n = 3k$, then the graph G consists of k-copies of P_3. The dominance number increases when an edge between two 3-degree vertices of any P_3 graph in the $G\overline{G}$ graph is deleted. Since $n = 3k$, $\gamma(G\overline{G})$-dominating set is unique. Therefore, we have $b(G\overline{G}) = 1$.

Case 2. $n = 3k + 1$, then there are many $\gamma(G\overline{G})$-dominating sets. Furthermore, it is easy to see that $b(G\overline{G}) > 1$. Let the vertex u be any end vertex with degree two. Let $S(u)$ be the set of edges connected to the vertex of u. When $S(u)$ is removed from the graph $G\overline{G}$, the remaining structure contains graph $P_{3k}\overline{P_{3k}}$. So, we have $\gamma(G\overline{G}) = \lceil (n + 3)/3 \rceil = k + 1$ from [1]. Thus, $\gamma(G\overline{G} - S(u)) = \gamma(P_{3k}\overline{P_{3k}}) + 1 = \lceil (n + 3)/3 \rceil + 1 = k + 2$. Then, we obtain $b(G\overline{G}) = 2$.

Case 3. $n = 3k + 2$, then the proof is made similar to Case 2.

Proof is completed by Case 1, Case 2 and Case 3.

For $\forall u \in V(G)$ and $\forall \bar{u} \in V(\bar{G})$, $d_{G\bar{G}}(\bar{u}) > d_{G\bar{G}}(u)$. Thus, $\gamma_s(G\overline{G})$-strong dominating set contains any two vertices of $\overline{G}$ to strong dominate all vertices of the graph $\overline{G}$. It is easily seen that $b_s(G\overline{G}) > 1$. The graph $G\overline{G}$ has more than one $\gamma_s(G\overline{G})$-strong dominating sets. The proof of $b_s(G\overline{G})$ is obtained by three cases.

Case 1. Let $n = 3k$. Since $\delta(G\overline{G}) = 2$, when any two edges are deleted from the $G\overline{G}$ graph, the strong dominance number of the graph does not change. Therefore, $b_s(G\overline{G}) > 1$. Let vertices u and v be end vertices of the graph G and P_4 induced subgraph of the graph G without $\{u, v\}$-vertices. When the edges of this P_4 graph are removed, it is easy seen that $\gamma_s(G\overline{G} - E(P_4)) > \gamma_s(G\overline{G})$. So, we have $b_s(G\overline{G}) = 3$.

Case 2. Let $n = 3k + 1$. By [1], we have $\gamma_s(G\overline{G}) = k + 2$. Let u and v be end vertices of the graph $G\overline{G}$. Let $S(u)$ be the set of edges connected to the vertex of u. When $S(u)$ is removed from the graph $G\overline{G}$, the remaining structure contains graph $P_{3k}\overline{P_{3k}}$. As the vertex u is an isolated vertex, it is in $\gamma_s(G\overline{G})$-strong dominating set. Furthermore, the strong domination number of remaining graph is equal to strong domination number of graph $P_{3k}\overline{P_{3k}}$. Thus, $\gamma_s(P_{3k}\overline{P_{3k}}) = \lceil (3k+4)/3 \rceil = k+2$ and vertex $\overline{u}$ is strong dominated by $\gamma_s(P_{3k}\overline{P_{3k}})$-strong dominating set. Finally, $\gamma_s(G\overline{G} - S(u)) > \gamma_s(G\overline{G})$. Now, we have $b_s(G\overline{G}) = 2$, since $\gamma_s(G\overline{G} - S(u)) > \gamma_s(G\overline{G})$.

Case 3. Let $n = 3k + 2$. When the same edges are deleted as in Case 2, the remaining structure contains the graph $P_{3k+1}\overline{P_{3k+1}}$. The rest of the proof is made similar to Case 2. Therefore, we obtain $b_s(G\overline{G}) = 2$.

Proof is completed by Case1, Case2 and Case3.

The proof of $b_w(G\overline{G})$ is obtained by two cases.
Case 1. Let $n = 3k+1$. It can be easily seen that $\gamma_w(G\overline{G}) = \gamma_w(G\overline{G} - e)$, where e is an edge of the graph $G\overline{G}$. Therefore, $b_w(G\overline{G}) > 1$. Let u be an end vertex of the graph G. Similarly, let $v \in N(u)$ and $m \in N(v)$ in G. When the edges e_{uv} and e_{vm} are removed from the graph $G\overline{G}$, the vertices u, v and m must be in $\gamma_w(G\overline{G})$-weak dominating set from definition of weak domination set. Furthermore, the other end vertex of the graph G must be also in $\gamma_w(G\overline{G})$-weak dominating set. Since $\overline{v}$ is weakly dominate to $\overline{V} - \{\overline{u}, \overline{m}\}$, this vertex must be in $\gamma_w(G\overline{G})$-weak dominating set. There are $(n-6)$ vertices that are not weakly dominated such that these vertices are formed as graph P_{n-6}. So, $\gamma_w(G\overline{G}) = \gamma_w(P_{n-6}) + 5$ and $\gamma_w(G\overline{G}) = \lceil (n-6)/3 \rceil + 5 = \lceil (n+9)/3 \rceil$. Therefore, $\lceil (n+9)/3 \rceil > \lceil (n+6)/3 \rceil$, since $n = 3k+1$. Consequently, we have $b_w(G\overline{G}) = 2$.

Case 2. For $n = 3k$ and $n = 3k+2$, let u be end vertex of the graph G and $v \in N(u)$, where $N(u)$ be the neighborhood of vertex u in the graph G. When an edge e_{uv} is removed from the graph $G\overline{G}$, $\gamma_w(G\overline{G})$-weak dominating set must contain vertices u and v and also the other end vertex of the graph G. So, the remaining graph is P_{n-5}. The rest of the proof is similar to Case 1. So, we obtain $\gamma_w(G\overline{G}) = \lceil (n+10)/3 \rceil$. Then we have $b_w(G\overline{G}) = 1$, since $\lceil (n+10)/3 \rceil > \lceil (n+6)/3 \rceil = \gamma_w(G\overline{G})$. The proof is completed. □

Theorem 9. *If $G = C_n$ for $n > 5$ and $k > 1$, then*

i) $$b(G\overline{G}) = b_s(G\overline{G}) = \begin{cases} 5, & n=3k, \\ 3, & \text{otherwise.} \end{cases}$$

ii) $$b_w(G\overline{G}) = \begin{cases} 2, & n=3k, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. In three different bondage measure proofs, three cases are examined according to $n \ mod\ 3$.

Proof of $b_s(G\overline{G})$ is obtained by three cases.

When one or two edges are removed from $G\overline{G}$, strong domination number does not increase since $\gamma_s(G\overline{G})$-strong dominating set is more than one set. It is easy to see that $b_s(G\overline{G}) > 2$, since $\delta(G\overline{G}) = 2$.

Case 1. Let $n = 3k + 1$ and u and v be two adjacent vertices in graph G. When the edge e_{uv} is removed from the graph $G\overline{G}$, the remaining graph is $(P_{3k+1}\overline{P_{3k+1}} + \{e_{\overline{u}\,\overline{v}}\})$. Furthermore, it is easy to see that $\gamma_s(P_{3k+1}\overline{P_{3k+1}} + \{e_{\overline{u}\,\overline{v}}\}) = \gamma_s(P_{3k+1}\overline{P_{3k+1}})$. Finding the bondage number of $(P_{3k+1}\overline{P_{3k+1}} + \{e_{\overline{u}\,\overline{v}}\})$ is similar to finding the bondage number of $(P_{3k+1}\overline{P_{3k+1}})$ by Theorem 8. So, we have $b_s(G\overline{G}) = 1 + b_s(P_{3k+1}\overline{P_{3k+1}}) = 3$.

Case 2. Let $n = 3k + 2$. The proof is made similar to Case 1.

Case 3. Let $n = 3k$. The degrees of all vertices of graph G and graph $\overline{G}$ are 3 and $(n - 2)$, respectively. The edges of the graph $\overline{G}$ are not removed, since $n - 2 > 3$. It can be easily seen that $\gamma_s(G\overline{G})$-strong dominating set must contain some vertices from the graph G. In order to increase strong domination number of the graph $G\overline{G}$, we have two sub cases.

Subcase 1. Let's take any subgraph P_6 of the graph G. When all edges of the graph P_6 are removed, the remaining graph includes the graph P_{n-4} and 4-isolated vertices. Let x and y be any two isolated vertices. Then we have $\gamma_s(G\overline{G} - E(P_6)) = \gamma_s(P_{n-4}) + 2\gamma_s(K_1) + |\{\overline{x}, \overline{y}\}|) = \lceil (n - 4)/3 \rceil + 4 = \lceil (n + 8)/3 \rceil$. So, it can be easily seen that $\gamma_s(G\overline{G} - E(P_6)) > \gamma_s(G\overline{G})$, since $\gamma_s(G\overline{G}) = \lceil (n - 4)/3 \rceil$. Then we obtain $b_s(G\overline{G}) = 5$.

Subcase 2. $\gamma_s(G\overline{G})$-strong dominating set must contain any two vertices from the graph $G\overline{G}$. According to this situation, we must remove some edges. These are three edges from $E_1(G\overline{G})$ and two edges from $E_2(G\overline{G})$. Let S be the set of these edges, so $|S| = 5$. The remaining graph is graph P_{n-2} and two isolated vertices, when edges of S are removed. Furthermore, $\gamma_s(G\overline{G})$-strong dominating set must contain any two vertices in $\overline{V}$. These vertices strong dominate all vertices of $\overline{V}$, since the degree of all vertices of graph $\overline{G}$ is $(n - 2)$ [1]. Then, we have $\gamma_s(G\overline{G} - S) = \gamma_s(P_{n-2}) + 2 + \gamma_s(\overline{G})$. So, it can be easily seen that $\gamma_s(G\overline{G} - S) > \gamma_s(G\overline{G})$ since $\gamma_s(G\overline{G}) = \lceil (n + 4)/3 \rceil$. Finally, we obtain

$b_s(G\overline{G}) = 5$.

By Subcase 1 and Subcase 2, we have $b_s(G\overline{G}) = 5$ for $n = 3k$.

Thus, the proof of $b_s(G\overline{G})$ is completed by Case 1, Case 2 and Case 3.

Since the G and $\overline{G}$ graphs are regular graphs, $\gamma(G\overline{G})$-dominating set and $\gamma_s(G\overline{G})$-strong dominating set are the same set from the definition of domination and strong domination. Therefore, $b(G\overline{G}) = b_s(G\overline{G})$.

Proof of $b_w(G\overline{G})$ is obtained by two cases.

Case 1. If $n = 3k+1$ and $n = 3k+2$, then degrees of all vertices of the graphs G and $\overline{G}$ are 3 and $(n-2)$, respectively. Let u be any vertex of the graph G. Furthermore, $deg(u) = 2$ and $deg(\overline{u}) = n-3$. When the edge $e_{u\overline{u}} \in E(G\overline{G})$ is removed from the graph $G\overline{G}$, $\delta(G) = deg(u)$ and $\delta(\overline{G}) = deg(\overline{u})$. Thus, $\gamma_w(G\overline{G})$-weak dominating set must contain $\{u, \overline{u}\}$. $|D_u| = \gamma_w(P_{n-3}) = \lceil (n-3)/3 \rceil$, where D_u is weak dominating set of the graph $G - N(u)$. Moreover, the vertex $\overline{u}$ weak dominates all vertices of $\overline{V} - N(\overline{u})$. So, the remaining graph is $P_2 = \overline{G} - N(\overline{u})$ that is not weakly dominated. If $\gamma_w(G\overline{G})$-weak dominating set includes any vertex of the graph P_2, then $\gamma_w(G\overline{G}) = |D_u| + \{u, \overline{u}\} + 1 = \lceil (n-3)/3 \rceil + 2 + 1 = \lceil (n+6)/3 \rceil$. Thus, it can be easily seen that we have $b_w(G\overline{G}) = 1$ since $\lceil (n+6)/3 \rceil > \lceil (n+4)/3 \rceil$ when $n = 3k+1$ and $n = 3k+2$.

Case 2. If $n = 3k$, then the domination number does not increase when any edge is removed from the graph $G\overline{G}$. So, $b_w(G\overline{G}) > 1$. Let vertex v be neighbor of vertex u in the graph G for $\exists u \in V$. $\gamma_w(G\overline{G})$-weak dominating set must contain vertices u, v and $\overline{u}$ when the edges $e_{u\overline{u}}$ and e_{uv} are removed from the graph $G\overline{G}$. Then, there are $(n-4)$ vertices with degree three in the graph G and one vertex from the graph $\overline{G}$, where these vertices are not weakly dominated. So, there are $\lceil (n-4)/3 \rceil + 4 = \lceil (n+8)/3 \rceil$ vertices in $\gamma_w(G\overline{G})$-weak dominating set. Then we have $b_w(G\overline{G}) = 2$ since $\lceil (n+8)/3 \rceil > \lceil (n+4)/3 \rceil$ for $n = 3k$.

The proof is completed. □

Remark 1. *The $\gamma_s(G\overline{G})$-strong dominating sets and the $\gamma(G\overline{G})$-dominating sets are the same since graphs G and $\overline{G}$ are regular. So,* *the value of the bondage number and strong bondage number are the same if $G = C_n$.*

Theorem 10. *If $G = tK_2$ and $(t > 2)$, then $b(G\overline{G}) = b_s(G\overline{G}) = 2$ and $b_w(G\overline{G}) = 1$.*

Proof. For $b(G\overline{G})$ and $b_s(G\overline{G})$, it is easy to see that any edge e is removed from the graph $G\overline{G}$, we have $\gamma(G\overline{G}) = \gamma(G\overline{G} - e)$ and $\gamma_s(G\overline{G}) = \gamma_s(G\overline{G} - e)$. So, $b(G\overline{G}) > 1$ and $b_s(G\overline{G}) > 1$. Let vertex u be a vertex of any graph K_2 and $v \in N(u)$ for the graph G. If the edge e_{uv} and $e_{u\overline{u}}$ are removed from the graph $G\overline{G}$, then $\gamma(G\overline{G} - \{e_{uv}, e_{u\overline{u}}\}) = \gamma(G\overline{G} - \{e_{uv}, e_{u\overline{u}}\}) = t + 2$. So, we obtain $b(G\overline{G}) = b_s(G\overline{G}) = 2$.

For $b_w(G\overline{G})$, we have $\gamma(G\overline{G}) = \gamma_s(G\overline{G}) = \gamma_w(G\overline{G}) = t + 1$ by [1]. Let u and v be two vertices of any graph K_2. $\gamma_w(G\overline{G})$-weak dominating set must contain vertices u and v, when the edge e_{uv} is removed from the graph $G\overline{G}$. Furthermore, $\gamma_w(G\overline{G})$-weak dominating set must contain a vertex of every remaining graph K_2. Moreover, all vertices of $\overline{V}$ are weakly dominated, when $\gamma_w(G\overline{G})$-weak dominating set contains vertex $\overline{u}$. So, it can be easily seen that $\gamma_w(G\overline{G} - e_{uv}) > \gamma_w(G\overline{G})$. Then we have $b_w(G\overline{G}) = 1$. □

Corollary 1. *If $G = tK_2$ and $t = 1$, then $\gamma(G\overline{G}) = \gamma_s(G\overline{G}) = \gamma_w(G\overline{G}) = 1$.*

Theorem 11. *If $G = tK_n$, then $b(G\overline{G}) = b_s(G\overline{G}) = n$ and $b_w(G\overline{G}) = 1$.*

Proof. For $b(G\overline{G})$ and $b_s(G\overline{G})$, we have $\gamma(G\overline{G}) = \gamma_s(G\overline{G}) = \gamma_w(G\overline{G}) = t+2$ by [1]. Let vertices u and v be two vertices of different two graphs K_n. $\{\overline{u}, \overline{v}\}$-set dominates (strong dominates) $\overline{V} \cup \{u, v\}$. There are two ways to increase domination (strong domination) number, since $\gamma(K_n) = \gamma_s(K_n) = 1$.

Case 1. If any vertex of any graph K_n is an isolated vertex, then $\gamma(G\overline{G}) = \gamma(G\overline{G}) = t + 3$. So, we have $b(G\overline{G}) = b_s(G\overline{G}) = n$.

Case 2. For the graph $G\overline{G}$ the degrees of the all vertices of the graph K_n are n. Let $u \in V(K_n)$. If the degree of all vertices of graph K_n

are $(n-3)$, then all vertices of the graph K_n are dominated by two vertices. The number of the edges of the graph K_n is $(n(n-1)/2)$. When the degree of the vertices of the graph K_n decreases, the number of the edges of the new graph is $(n(n-3)/2)$. Moreover, we obtain $(n(n-1)/2)-(n(n-3)/2)=n$ and $\gamma(G\overline{G})=\gamma(G\overline{G})=t+3$. Then we have $b(G\overline{G})=b_s(G\overline{G})=n$.

The proof is completed by Case 1 and Case 2.

For $b_w(G\overline{G})$, let u and v be two vertices of different two graphs K_n. $\{\overline{u},\overline{v}\}$-set weak dominates $\overline{V}$. If an edge is removed from any graph K_n, then $\gamma_w(G\overline{G})=t+3$. So, we have $b_w(G\overline{G})=n$. □

Theorem 12. *If $G=K_n o K_1$, then $b(G\overline{G})=b_s(G\overline{G})=\lceil n/2\rceil$ and $b_w(G\overline{G})=2$.*

Proof. It is easy to see that the proofs of $b(G\overline{G})$ and $b_s(G\overline{G})$ are similar to $b_s(K_n)$ in [4]. So, $b(G\overline{G})=b_s(G\overline{G})=\lceil n/2\rceil$.

For $b_w(G\overline{G})$, there are n vertices which are degree 2 and n vertices which are degree n in the subgraph G and $\overline{G}$ of $G\overline{G}$, respectively. These vertices are the smallest degree vertices of the graph $G\overline{G}$. Moreover, these vertices are independent from each other. So, $\gamma_w(G\overline{G})=2n$ by [1]. If any edge e is removed from the graph $G\overline{G}$, then $\gamma_w(G\overline{G})=\gamma_w(G\overline{G}-e)$. So, $b_w(G\overline{G})>1$. Let u be any vertex whose degree 2 and $v\in N(u)$ and $v\neq\overline{u}$. It can be easily seen that vertices u and $\overline{v}$ weakly dominate vertex u. If the edges $e_{uv}\in E(G\overline{G})$ and $e_{v\overline{v}}\in E(G\overline{G})$ are removed from the graph $G\overline{G}$, then $\gamma_w(G\overline{G}-\{e_{uv},_{v\overline{v}}\}=2n+1$. So, we obtain $b_w(G\overline{G})=2$. □

Theorem 13. *If $G=W_{1,n}$ and $(n>5)$, then*

$$b(G\overline{G})=b_s(G\overline{G})=1 \text{ and } b_w(G\overline{G})=\begin{cases} 3 & , n=3k;\\ 2 & , \text{otherwise.}\end{cases}$$

Proof. For $b(G\overline{G})$ and $b_s(G\overline{G})$, let u be center vertex in the graph G and let $v,z\in V(G)-\{u\}$. $\{u,\overline{v},\overline{z}\}$-set dominates (strong dominates) $V\cup\overline{V}$. It is easy to see that $\gamma(G\overline{G})=\gamma_s(G\overline{G})=3$. $\gamma(G\overline{G})$-dominating set ($\gamma_s(G\overline{G})$-strong dominating set) must contain the vertex u, when the edge $e_{u\overline{u}}$ is deleted from the graph $G\overline{G}$. Clearly, $\gamma(G\overline{G}-e_{u\overline{u}}) > \gamma(G\overline{G})$ $(\gamma_s(G\overline{G}-e_{u\overline{u}}) > \gamma_s(G\overline{G}))$. So, we have $b(G\overline{G}) = b_s(G\overline{G}) = 1$.

For $b_w(G\overline{G})$, the graph G consists of $K_1 + C_n$. $\gamma_w(G\overline{G})$-weak dominating set includes the vertex $\overline{u}$ and vertices of $\gamma_w(C_n\overline{C}_n)$-weak dominating set. It can be easily seen that the proof of $b_w(G\overline{G})$ is similar to the proof of $b_w(C_n\overline{C}_n)$. □

Theorem 14. *If $G = K_{m,n}$ and $(m \leq n)$, then*

i) $b(G\overline{G}) = \begin{cases} 2, & m{=}2 \text{ and } m < n, \\ m, & \text{otherwise} \end{cases}$

ii) $b_s(G\overline{G}) = \begin{cases} 1, & m < n, \\ m, & m{=}n \end{cases}$

iii) $b_w(G\overline{G}) = \begin{cases} 2, & m{=}n{=}3, \\ 1, & \text{otherwise.} \end{cases}$

Proof. Let G_1 and G_2 be a partite sets of the graph G, whose cardinality are m and n, respectively. Clearly, m vertices are of order $(n+1)$ and these vertices are independent from each other. Similar to n vertices of order $(m+1)$, these vertices are also independent from each other. In $\overline{G}$, complete graphs K_m and K_n are formed by vertices of G_1 and G_2 [1].

For $b_s(G\overline{G})$, we recall $\gamma_s(G\overline{G}) = m+1$ by [1]. We must examine in two cases for the proof of $b_s(G\overline{G})$. Let $v \in V(G_1)$ and $u \in V(G_2)$.

Case 1. If $m < n$, $\gamma_s(G\overline{G})$-strong dominating set must include the vertex $\overline{u}$ since it is not strong dominated, where the edge $e_{u\overline{u}}$ is removed from the graph $G\overline{G}$. So, we have $\gamma_s(G\overline{G}-e_{u\overline{u}}) > \gamma_s(G\overline{G})$ and $b_s(G\overline{G}) = 1$.

Case 2. If $m = n \geq 3$, the graphs K_m and K_n are m-regular. There are two Sub Cases for this situation.

Subcase 2.1. The proof is similar to the proof of Case 1 of Theorem 11.

Subcase 2.2. The proof is similar to the proof of Case 2 of Theorem 11.

By Case 1 and Case 2, we have $b_s(G\overline{G}) = m$.

For $b(G\overline{G})$, we must examine in two cases for the proof of $b(G\overline{G})$.
Case 1. If $m = 2$ and $m < n$, the domination number of the graph $G\overline{G}$ is 3. $\gamma(G\overline{G})$-dominating set includes two vertices of the G_1 and any vertex of the graph K_n. The domination number does not change, an edge is removed from the graph $G\overline{G}$. So, $b(G\overline{G}) > 1$. The domination number increases by 1, when all edges are removed between G_1 and $V(K_n)$. Then we have $b(G\overline{G}) = 2$.

Case 2. If $m > 2$ and ($m < n$ or $m = n$), $\gamma(G\overline{G})$-dominating set and $\gamma_s(G\overline{G})$-strong dominating set are the same, where ($m = n \geq 3$). Clearly, the proof is similar to proof of the $b_s(G\overline{G})$, where for ($m = n \geq 3$). So, we obtain $b(G\overline{G}) = m$.

By Case 1 and Case 2, we have $b(G\overline{G}) = m$.

For $b_w(G\overline{G})$, we must examine in two cases for the proof of $b_w(G\overline{G})$.
Case 1. If $m \leq n$, we recall $\gamma_w(G\overline{G}) = n + 1$ by [1]. $\gamma_w(G\overline{G})$-weak dominating set must contain the vertex $\overline{u}$ when the edge $e_{u\overline{u}}$ is removed from the graph $G\overline{G}$, where $u \in V(G_2)$. So, $\gamma_w(G\overline{G} - e_{u\overline{u}}) > \gamma(G\overline{G})$, then we have $b_w(G\overline{G}) = 1$.

Case 2. If $m = n$, then let $m = n \neq 3$. The degrees of all vertices of G_1 and G_2 are $(m + 1)$. Similarly, degrees of all vertices of the graphs K_m and K_n are m. Degrees of vertices, which are incident with the edge e, decrease by one when an edge e is removed from the graph K_m. $\gamma_w(G\overline{G})$-weak dominating set must contain these vertices. So, $\gamma_w(G\overline{G} - e) > \gamma(G\overline{G})$, then we have $b_w(G\overline{G}) = 1$. Let $m = n = 3$. The weak domination number does not increase, when any edge e is removed from the graph K_m. Therefore, if any edge between graphs K_m and G_1 is removed, then the weak domination number increases by one. So, we have $b_w(G\overline{G}) = 2$. $\square$

4 Conclusion

The characteristics of strong and weak dominating sets are not exhibited by the ordinary dominating sets and hence the problems of strong and weak bondage numbers for the graph are considerably harder than bondage number of that. In this paper, the results of the bondage number, strong bondage number and weak bondage number of the complementary prisms of several well-known graphs have been obtained. As a further study, many general results of bondage parameters of complementary prism of any given graph G may be obtained.

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