Computer Science Journal of Moldova, vol.**16**, no.3(48), 2008

About Directed d-Convex Simple Graphs

Nadejda Sur   Sergiu Cataranciuc

Abstract

In this article we introduce a pseudo-metric on directed graphs, which forms there a family of convex sets. The graphs without d-convex sets, except empty set, sets of one vertex and set of all vertexes, are called d-convex simple. We give an iterative method of description of the set of all directed d-convex simple graphs. Then we research the structure of directed d-convex simple graphs and do this by using some new operations and new graphs. After that we show that the set of directed d-convex simple graphs contains all known undirected d-convex simple graphs.

1 An Iterative Method of Description for Directed d-Convex Simple Graphs.

We are going to study below directed graphs, without loops or multiple arcs. A directed graph $\vec{G} = (X, \vec{U})$ is called to be *strongly connected* graph, if for each two vertexes $x, y \in X$ there is at least one path (directed chain), from vertex x to vertex y and at least one path from vertex y to vertex x. Let $D = (x = z_1, z_2, \ldots, z_p = y)$ be a path from x to y. In this case we will say, also, that path D joins the vertexes x and y in the indicated ordering, which are called the *extremities* of D. The number p is called the *length* of the path

$$D = (x = z_1, z_2, \ldots, z_p = y)$$

and we will write $l(D) = p$.

Let $\mathcal{D}(x, y)$ be the family of all paths form $\vec{G}$ that joins the vertexes x and y. The length of the shortest path from $\mathcal{D}(x, y)$ will be called the *distance* between vertexes x and y and will be denote by $d(x,\ y)$. So, we have

$$d(x,\ y) = \min_{D \in \mathcal{D}(x,\ y)} \{l(D)\}.$$

In the case when between two vertexes $x,\ y \in X$ a path that joins them does not exist, it is considered that $d(x,\ y) = \infty$.

It is easy to see that the notion of distance, introduced by this way, does not respect the commutative propriety, i. e.

$$d(x,\ y) \neq d(y,\ x).$$

This distance is a function $d : X \times X \to \mathbb{N}$, that respects the properties:

1. $d(x,\ y) \geq 0$, for each two vertexes $x,\ y \in X$, and $d(x,\ y) = 0$ if and only if $x = y$;

2. $d(x,\ y) \leq d(x,\ z) + d(z,\ y)$, for each three vertexes $x,\ y,\ z \in X$.

We have, that the defined above distance function $d : X \times X \to \mathbb{N}$ is a *pseudo-metric* in directed graph $\vec{G}$.

The set $\langle \overrightarrow{x,\ y} \rangle = \{z \in X \mid d(x,\ z) + d(z,\ y) = d(x,\ y)\}$ is called *directed d-segment* from x to y. Obviously, the notion of directed segment $\langle \overrightarrow{x,\ y} \rangle$ has sense only if there is at least one path that joins x with y. From these considerations, we are going to study below only strongly connected directed graphs.

Definition 1.1 *The set $A \subset X$ is called to be* ***d-convex*** *set in the graph $\vec{G} = (X,\ \vec{U})$ if for each $x,\ y \in A$, considered in the indicated order, there is the relation $\langle \overrightarrow{x,\ y} \rangle \subset A$.*

We observe that each set A, $|A| = 0$ or $|A| = 1$, is d-convex.

It is easy to see that in the arbitrarily directed graph the sets $\langle \overrightarrow{x,\ y} \rangle$ and $\langle \overrightarrow{y,\ x} \rangle$ are not indispensably equal. Let A be an arbitrary set of vertexes from graph $\vec{G} = (X,\ \vec{U})$. According to definition 1.1, if A is a d-convex set, then both directed d-segments $\langle \overrightarrow{x,\ y} \rangle$ and $\langle \overrightarrow{y,\ x} \rangle$ belong to A. This implies that the union $\langle \overrightarrow{x,\ y} \rangle \cup \langle \overrightarrow{y,\ x} \rangle$, which contains all vertexes of at least one circuit that passes through vertexes x and y, also belongs to A. More, the set $\langle \overrightarrow{x,\ y} \rangle \cup \langle \overrightarrow{y,\ x} \rangle$ contains the vertexes of all circuits of minimal length that pass trough x and y.

Lemma 1.1 *If A and B are two d-convex sets of a directed graph $\vec{G} = (X,\ \vec{U})$, then the intersection $A \cap B$ is, also, a d-convex set in $\vec{G}$.*

Proof: Let $\vec{G} = (X, \vec{U})$ be a directed graph, and A and B be two d-convex sets of it. If $|A \cap B| \leq 1$, then according to definition 1.1 the assertion of lemma is true. Let us suppose that $|A \cap B| \geq 2$, and let $x,\ y$ be any two vertexes from $A \cap B$. Because A and B are d-convex simple sets, we have:

$$\langle \overrightarrow{x,\ y} \rangle\ \subseteq A,$$

$$\langle \overrightarrow{x,\ y} \rangle\ \subseteq B,$$

that implies relation $\langle \overrightarrow{x,\ y} \rangle\ \subseteq A \cap B$. So we have that $A \cap B$ is d-convex simple set in $\vec{G}$. $\square$

By analogy with classical model of d-convex hull notion [8], in case of directed graphs we have:

Definition 1.2 *Intersection of all d-convex sets of a directed graph $\vec{G} = (X,\ \vec{U})$, that contains a subset of vertexes $B \subseteq X$, is called **d-convex hull** of the set B and denoted by **d-conv(B)**.*

Obviously, if $B \subseteq X$ is already a d-convex set in $\vec{G}$, then $d-conv(B) = B$.

For an arbitrary subset of vertexes S of the graph $\vec{G}$, we will define the next operation

$$P(S) = \bigcup_{\forall x,\, y\, \in S} \langle \overrightarrow{x,\ y} \rangle.$$

Then d-convex hull of the set B can be iteratively described like follows:

$$B_0 = B,$$

$$B_1 = \bigcup_{\forall x,\, y\, \in\, B_0} \langle \overrightarrow{x,\ y} \rangle = P(B_0),$$

$$B_2 = \bigcup_{\forall x,\, y\, \in\, B_2} \langle \overrightarrow{x,\ y} \rangle = P(B_1) = P(P(B_0)) = P^2(B_0) \text{ and } P(B_1) \neq B_1,$$

$$\ldots,$$

$$B_{q-1} = \bigcup_{\forall x,\, y \,\in\, B_{q-2}} \langle\overrightarrow{x,\, y}\rangle = P(B_{q-2}) = P^{q-1}(B_0) \; and \; P(B_{q-2}) \neq B_{q-2},$$

$$B_q = \bigcup_{\forall x,\, y \,\in\, B_{q-1}} \langle\overrightarrow{x,\, y}\rangle = B_{q-1} = d - conv(B).$$

By this way, the construction of d-convex hull $d{-}conv(B)$ is reduced to construction of a sequence of subsets:

$$B = B_0 \subset B_1 \subset B_2 \subset \ldots \subset B_{q-1} = B_q,$$

where B_i, $1 \leq i \leq q$, is determined by using the P operation, described above. In case of infinite graphs, the iteratively constructed sequence could be also infinite. Then, d-convex hull of the set B is computed using relation:

$$d - conv(B) = \bigcup_{i=0}^{\infty} B_i.$$

By definition 1.2, it is easy to see that next relations are true:

1. $d - conv(\emptyset) = \emptyset$;
2. $d - conv(\{x\}) = \{x\}$;
3. $d - conv(X) = X$;
4. $A \subseteq d - conv(A)$;
5. $d - conv(d - conv(A)) = d - conv(A)$.

By the relations $1 - 5$, we can say that the notion of d-convexity for the directed graphs do not get out of the general axiomatic theory of the convexity [8].

In undirected graphs any subset of vertexes $A \neq X$, which induce a complete subgraph and the set of all vertexes X, is always d-convex. We can say the same thing about directed graphs. The sets that are always d- convex in any directed graph are the empty set, the sets with one vertex, the sets that induce a complete subgraph and the set of all vertexes X. This is because the introduced by this way notion of convexity in directed graphs is the extension of the notion of convexity in the undirected graphs, and this thing will be shown below. Further the mentioned sets will be called *trivial* d-convex sets in directed graphs.

Definition 1.3 *Directed and strongly connected graph* $\vec{G} = (X,\ \vec{U})$ *is called* ***d-convex simple*** *if it does not contain d-convex sets* $A \subset X$*, such that* $1 < |A| < |X|$.

From the definition it results that in a directed d-convex simple graph, between any two vertexes there can be only one of two possible arcs. Indeed, if for some two vertexes $x,\ y$ in a d-convex simple graph both arcs $(x,\ y)$ and $(y,\ x)$ exist, then the set $A = \{x,\ y\}$ will be d-convex, that contradicts to the definitions assertion of d-convex simple graph. This remark means that all directed d-convex simple graphs are *antisymmetric.*

Theorem 1.1 *Next assertions are equivalent:*

1. $\vec{G} = (X, \vec{U})$ *is directed d-convex simple graph;*
2. $d - conv(\{x,\ y\}) = X$ *for any two distinct vertexes* $x,\ y \in X$*;*
3. $d - conv(\{x,\ y\}) = X$ *for any two adjacent vertexes* $x,\ y \in X$.

Proof: $1 \rightarrow 2$. Let $\vec{G} = (X, \vec{U})$ be a directed d-convex simple graph. By definition 1.3 it does not contain d-convex sets with more than one vertex or less than the number of vertexes of X. It follows that it does not contain d-convex sets of cardinal two. This fact implies $d - conv(\{x,\ y\}) = X$, for any two distinct vertexes $x,\ y \in X$.

$2 \rightarrow 3$. Relation $d - conv(\{x,\ y\}) = X$ is true for any two distinct vertexes $x,\ y \in X$, so it is true for adjacent vertexes, too. It results that the assertion 3 is true.

$3 \rightarrow 1$. Let us suppose that for any two adjacent vertexes $x,\ y \in X$, the relation $d - conv(\{x,\ y\}) = X$ is true, but the directed graph $\vec{G}$ is not d-convex simple. This means that in $\vec{G}$ exists a d-convex set $A \subset X$ and $1 < |A| < |X|$. From $|A| > 1$ it results that in A there exist two vertexes $p,\ q$, such that $\langle\overrightarrow{p,\ q}\rangle\ \subset A$. This implies the existence of two adjacent vertexes $x,\ y$ in A. Because A is the d-convex simple set, there is the relation $d-conv(x,\ y)\subseteq A$. On the other hand, from condition 3 we have $d-conv(x,\ y)=X$. It follows that $A=X$, that contradicts the supposition $1<|A|<|X|$. □

Now we consider the next class of directed graphs $\mathfrak{D}$, that is defined recursively as follows:

I. In the class $\mathfrak{D}$ there are all graphs $\vec{G}_0=(X_{\vec{G}_0},\ \vec{U}_{\vec{G}_0})$, where:
$X_{\vec{G}_0}=\{x_1,\ x_2,\ \dots,\ x_n\},\ n\geq 3$,
$\vec{U}_{\vec{G}_0}=\{(x_n,\ x_1)\}\cup\{(x_i,\ x_{i+1})|\ i=1,\ 2,\ \dots,\ n-1\}$,
i. e. $\vec{G}_0$ is an elementary circuit with n vertexes;

II. From the graph $\vec{G}_{i-1}$, $(i\geq 1)$ we construct the graph $\vec{G}_i=(X_{\vec{G}_i},\ \vec{U}_{\vec{G}_i})$, where:
$X_{\vec{G}_i}=X_{\vec{G}_{i-1}}\cup\{y_1,\ y_2,\ \dots,\ y_m\},\ m\geq 1$,
$\vec{U}_{\vec{G}_i}=\vec{U}_{\vec{G}_{i-1}}\ \cup\ \{(a,\ b)\mid a,\ b\ \in\ X_{\vec{G}_i}$, not both are in $X_{\vec{G}_{i-1}}$, and such that conditions a), b) are satisfied $\}$;

a) For each vertex y_i, there exist two distinct vertexes $p,\ q\ \in X_{\vec{G}_i}$, such that $y_i\ \in\ d-conv(\{p,\ q\})$;

b) For any two adjacent vertexes $a,\ b\ \in\ X_{\vec{G}_i}$, there exist two distinct vertexes $p,\ q\ \in\ X_{\vec{G}_{i-1}}$, such that the following relations are satisfied:

1. $p,\ q\ \in\ d-conv(\{a,\ b\})$;
2. $d_{\vec{G}_{i-1}}(p,\ q)=d_{\vec{G}_i}(p,\ q)$;
3. $d_{\vec{G}_{i-1}}(q,\ p)=d_{\vec{G}_i}(q,\ p)$;

III. In $\mathfrak{D}$ there are no other graphs, except the graphs iteratively described in I. and II.

The class of directed graphs $\mathfrak{D}$ is a union of graphs families, recursively obtained, according to the procedures described above:

$$\mathfrak{D}=\mathfrak{D}_0\cup\mathfrak{D}_1\cup\ldots\cup\mathfrak{D}_i\ldots,$$

where $\mathfrak{D}_0$ represents all graphs $\vec{G}_0 = (X_{\vec{G}_0},\ \vec{U}_{\vec{G}_0})$, described at step I of construction of class $\mathfrak{D}$, and $\mathfrak{D}_i,\ i \geq 1$, represent the family of all graphs that are obtained from $\mathfrak{D}_{i-1}$, applying the operation II.

Theorem 1.2 *All graphs of the class $\mathfrak{D}$ are d-convex simple graphs.*

Proof: We will prove that $\mathfrak{D}_0,\ \mathfrak{D}_1,\ \ldots,\ \mathfrak{D}_i,\ \ldots$ are families of directed d-convex simple graphs, using mathematical induction method by the index $i = 0,\ 1,\ 2,\ \ldots$ of the family of graphs.

It is easy to see, that if $i = 0$, then $\vec{G}_0 = (X_{\vec{G}_0},\ \vec{U}_{\vec{G}_0})$ from $\mathfrak{D}_0$ is d-convex simple, because it is an elementary circuit with $n \geq 3$ vertexes.

Let us prove that $\mathfrak{D}_i,\ i \geq 1$, is a family of directed d-convex simple graphs, in conditions that all graphs from $\mathfrak{D}_0,\ \mathfrak{D}_1,\ \ldots,\ \mathfrak{D}_{i-1}$ have already this property. We choose a graph $\vec{G}_i \in \mathfrak{D}_i$ with $n \geq 3$ vertexes. Let $\vec{G}_i = (X_{\vec{G}_i},\ \vec{U}_{\vec{G}_i})$ be obtained from $\vec{G}_{i-1} \in \mathfrak{D}_{i-1}$ as the result of application of the operation II of construction of the class $\mathfrak{D}$. According to theorem 1.1 for this, it is necessary to prove that for any two adjacent vertexes $a,\ b \in X_{\vec{G}_i}$ there is the relation $d - conv_{\vec{G}_i}(\{a,\ b\}) = X_{\vec{G}_i}$. Let us suppose that between a and b there exists the arc $(a,\ b) \in \vec{U}_{\vec{G}_i}$. According to the condition II b) of the class $\mathfrak{D}$ description, there exist two distinct vertexes $p,\ q \in X_{\vec{G}_{i-1}}$, such that $p,\ q \in d - conv_{\vec{G}_i}(\{a,\ b\})$. This implies the relation

$$d - conv_{\vec{G}_i}(\{p,\ q\}) \subset d - conv_{\vec{G}_i}(\{a,\ b\}).$$

The fact that, by induction, $\vec{G}_{i-1}$ is d-convex simple, and in process of construction of the graph $\vec{G}_i$, new arcs were not added to vertexes from $\vec{G}_{i-1}$, and the distance between p and q has kept unchanged in $\vec{G}_i$ (see condition II b)), results in:

$$X_{\vec{G}_{i-1}} \subset d - conv_{\vec{G}_i}(\{p,\ q\}) \subset d - conv_{\vec{G}_i}(\{a,\ b\}).$$

From condition II a) we obtain:

$$X_{\vec{G}_{i-1}} \cup \{y_1,\ y_2,\ \ldots,\ y_m\} = X_{\vec{G}_i} \subset d - conv_{\vec{G}_i}(\{a,\ b\}).$$

Reverse inclusion is obvious and so we have $X_{\vec{G}_i} = d - conv_{\vec{G}_i}(\{a,\ b\})$. This means that directed graph $\vec{G}_i$ is d-convex simple. So $\mathfrak{D}_i$ is a family of directed d-convex simple graphs. □

Theorem 1.3 *A directed strongly connected graph* $\vec{G} = (X,\ \vec{U})$, $|X| \geq 3$ *is d-convex simple if and only if* $\vec{G} \in \mathfrak{D}$.

Proof:

Necessity: Let $\vec{G} = (X,\ \vec{U})$ be a directed d-convex simple graph. Thus $|X| \geq 3$ results that in $\vec{G}$ there exist two distinct vertexes u and v. Because $\vec{G}$ is a strongly connected graph then there exists at least one circuit that passes trough these vertexes. So, we obtained that in $\vec{G}$ there exists at least one elementary circuit. We consider thus elementary circuit of minimal length. We also observe that if $C = [z_0, z_1, \ldots, z_p, z_{p+1} = z_0]$ is a circuit of minimal length in graph $\vec{G}$, then the subgraph generated by the set of vertexes $\{z_0, z_1, \ldots, z_p\}$ is isomorphic with C. It follows that in $\vec{G}$ do not exist arcs that join any two vertexes $z_i,\ z_j$, where $|i - j| > 1$. We will denote by $\vec{G}_0 = (X_{\vec{G}_0},\ \vec{U}_{\vec{G}_0})$ the subgraph generated by the set of vertexes $\{z_0,\ z_1,\ \ldots,\ z_p\}$, and by $\vec{G}_1$ denote the graph $\vec{G}$ itself. It is easy to verify that $\vec{G}_1$ is obtained from $\vec{G}_0$ by adding the sets $X \backslash X_{\vec{G}_0}$ of vertexes and $\vec{U} \backslash \vec{U}_{\vec{G}_0}$ of arcs, according to condition II of description of class $\mathfrak{D}$. It follows that $\vec{G} \in \mathfrak{D}$.

Sufficiency: It results from theorem 1.2. □

2 Operations Over Directed d-Convex Simple Graphs.

Let $\vec{G}_1 = (X_1,\ \vec{U}_1)$ and $\vec{G}_2 = (X_2,\ \vec{U}_2)$ be two directed graphs, where we choose by one pair of nonadjacent vertexes: $x_1,\ x_2$ in $\vec{G}_1$ and $y_1,\ y_2$ in $\vec{G}_2$. By analogy with [2, 4, 5] we will denote by $M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2)$ the graph obtained from $\vec{G}_1$ and $\vec{G}_2$ as the result of joining the vertexes x_1 with y_1 and x_2 with y_2. For the graph $\vec{G} = M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2)$ the following relations are true:

$$|X_{\vec{G}}| = |X_1| + |X_2| - 2,$$

$$\vec{U}_{\vec{G}} = \vec{U}_1 \cup \vec{U}_2.$$

In order to simplify notations we will write $\vec{G} = M(\vec{G}_1,\ \vec{G}_2)$ if the pairs of vertexes that take part in generation of new graph are known.

Let $\vec{G} = (X,\ \vec{U})$ be a directed graph and x be any vertex from X. We denote by

$$\Gamma^+(x) = \{y \in X \mid (x,\ y) \in \vec{U}\} \text{ and}$$

$$\Gamma^-(x) = \{y \in X \mid (y,\ x) \in \vec{U}\}$$

the set of *successors* of the vertex x and the set of *predecessors* of it respectively. The vertex which does not have predecessors is called *source* vertex, and the vertex which does not have successors - *destination* vertex.

We denote by $\mathcal{P}(p,\ q;\ r)$, $r > 0$, the directed graph where a source vertex p, and a destination vertex q are fixed and which satisfies the conditions:

1. in $\mathcal{P}$ there exist paths that join vertex p with vertex q;
2. any vertex and any arc from $\mathcal{P}$ belong to at least one path that joins p with q;
3. all paths that join vertexes p and q are of the same length $r > 0$;
4. other vertexes or arcs in $\mathcal{P}$ do not exist.

Theorem 2.1 *For any two graphs $\mathcal{P}_1(p_1,\ q_1;\ r_1)$ and $\mathcal{P}_2(p_2,\ q_2;\ r_2)$, the graph*

$$\vec{G} = M^{p_1=q_2}_{q_1=p_2}(\mathcal{P}_1,\ \mathcal{P}_2)$$

is d-convex simple.

Proof: In the graph $\vec{G}$ all elementary circuits are of the same length, equal to $r_1 + r_2$, and they pass through vertexes $p_1 = q_2,\ q_1 = p_2$ and any vertex of graph $\vec{G}$ belongs to at least one circuit of this type. It follows that $\vec{G}$ is strongly connected graph and for any two vertexes $x,\ y$ from $\vec{G}$ we have:

$$p_1 = q_2,\ q_1 = p_2\ \in\ \langle\overrightarrow{x,\ y}\rangle \cup \langle\overrightarrow{y,\ x}\rangle,$$

but $d - conv(\{p_1 = q_2,\ q_1 = p_2\}) = X_{\vec{G}}$. From this it results that $\vec{G}$ is d-convex simple graph.

Theorem is proved. □

Let $\mathcal{P}_1(p_1,\ q_1;\ r_1),\ \mathcal{P}_2(p_2,\ q_2;\ r_2),\ \ldots,\ \mathcal{P}_s(p_s,\ q_s;\ r_s)$ be s-directed graphs. We denote by:

$$M_{q_s=p_1}^{q_1=p_2,\ q_2=p_3,\ \ldots,\ q_{s-1}=p_s}(\mathcal{P}_1,\ \mathcal{P}_2,\ \ldots,\ \mathcal{P}_s)$$

the graph obtained from $\mathcal{P}_1,\ \mathcal{P}_2,\ \ldots,\ \mathcal{P}_s$ as the result of joining the vertexes q_1 with p_2, q_2 with p_3, $\ldots$, q_{s-1} with p_s and q_s with p_1. For the graph $\vec{G} = M_{q_s=p_1}^{q_1=p_2,\ q_2=p_3,\ \ldots,\ q_{s-1}=p_s}(\mathcal{P}_1,\ \mathcal{P}_2,\ \ldots,\ \mathcal{P}_s)$ the following relations are true:

$$|X_{\vec{G}}| = |X_{\mathcal{P}_1}| + |X_{\mathcal{P}_2}| + \ldots + |X_{\mathcal{P}_s}| - s,$$

$$\vec{U}_{\vec{G}} = \vec{U}_{\mathcal{P}_1} \cup \vec{U}_{\mathcal{P}_2} \cup \ldots \cup \vec{U}_{\mathcal{P}_s}.$$

Corollary 2.1 *For any $s \geq 2$ directed graphs*

$$\mathcal{P}_1(p_1,\ q_1;\ r_1),\ \mathcal{P}_2(p_2,\ q_2;\ r_2),\ \ldots,\ \mathcal{P}_s(p_s,\ q_s;\ r_s),$$

the graph $\vec{G} = M_{q_s=p_1}^{q_1=p_2,\ q_2=p_3,\ \ldots,\ q_{s-1}=p_s}(\mathcal{P}_1,\ \mathcal{P}_2,\ \ldots,\ \mathcal{P}_s)$ is d-convex simple.

Proof: From graphs $\mathcal{P}_2(p_2,\ q_2;\ r_2),\ \mathcal{P}_3(p_3,\ q_3;\ r_3),\ \ldots,\ \mathcal{P}_s(p_s,\ q_s;\ r_s)$ we build a new graph $\tilde{\mathcal{P}}(p_2,\ q_s;\ r_2 + r_3 + \ldots + r_s)$ by joining the vertexes q_2 with p_3, $\ldots$, q_{s-1} with p_s. Then graph $\vec{G} = M_{q_s=p_1}^{q_1=p_2,\ q_2=p_3,\ \ldots,\ q_{s-1}=p_s}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s)$ coincides with graph $M_{q_s=p_1}^{q_1=p_2}(\mathcal{P}_1, \tilde{\mathcal{P}})$. From theorem 2.1 the last graph is d-convex simple. This means the graph $\vec{G}$ is d-convex simple, too. □

Theorem 2.2 *If $\vec{H} = (X_{\vec{H}},\ \vec{U}_{\vec{H}})$ is a directed d-convex simple graph, where there exists a pair of vertexes $x_1,\ x_2$, such that $d_{\vec{H}}(x_1,\ x_2) = r > 1$, then for any graph $\mathcal{P}(p,\ q;\ r)$ the graph $\vec{G} = M^{x_1=p}_{x_2=q}(\vec{H},\ \mathcal{P})$ is d-convex simple.*

Proof: By theorem 1.2, in order to prove the assertion of this theorem it is sufficient to show, that the graph $\vec{G}$ can be obtained from the graph $\vec{H}$ with respect of condition II of description of the class $\mathfrak{D}$. Let us consider $\vec{G}_{i-1} = \vec{H}$ and $\vec{G}_i = \vec{G}$. From construction of graph $\vec{G} = M^{x_1=p}_{x_2=q}(\vec{H},\ \mathcal{P})$ and condition $r > 1$, it results that the vertexes of graph $\vec{H}$ were not joined with new arcs. More than that:

a) for any $y_j \in X_{\mathcal{P}}$ there exist vertexes $x_1 = p,\ x_2 = q \in X_{\vec{H}}$, such that: $y_j \in d-conv(\{p,\ q\}) = d-conv(\{x_1,\ x_2\})$;

b) for any two adjacent vertexes $a,\ b \in X_{\vec{G}}$, there exist two distinct vertexes $x_1,\ x_2 \in X_{\vec{H}}$, such that the following relations are true:

 0. $x_1 = p,\ x_2 = q$;
 1. $x_1,\ x_2 \in d-conv(\{a,\ b\})$;
 2. $d_{\vec{H}}(x_1,\ x_2) = d_{\vec{G}}(x_1,\ x_2) = d_{\vec{G}}(p,\ q) = r$;
 3. $d_{\vec{G}}(q,\ p) = d_{\vec{H}}(q,\ p) = d_{\vec{H}}(x_2,\ x_1)$.

It follows that $\vec{G} = M^{x_1=p}_{x_2=q}(\vec{H},\ \mathcal{P}) \in \mathfrak{D}$. According to theorem 1.2, this means the $\vec{G}$ is d-convex simple graph.

The theorem is proved. □

Theorem 2.3 *If $\vec{G}_1$ and $\vec{G}_2$ are two d-convex simple graphs, where there exist the nonadjacent vertexes $x_1,\ x_2 \in \vec{G}_1$ and $y_1,\ y_2 \in \vec{G}_2$, which satisfy conditions $d_{\vec{G}_1}(x_1,\ x_2) = d_{\vec{G}_2}(y_1,\ y_2)$ and $d_{\vec{G}_1}(x_2,\ x_1) = d_{\vec{G}_2}(y_2,\ y_1)$, then the graph $\vec{G} = M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2)$ is also d-convex simple.*

Proof: We are going to prove this theorem using the same approach as in the last theorem. We consider that $\vec{G}_{i-1} = \vec{G}_1$ and $\vec{G}_i = \vec{G}$, and will show that $\vec{G}$ can be obtained from $\vec{G}_1$ with respect of condition II of description of the class $\mathfrak{D}$. From the condition that the selected pairs of vertexes are nonadjacent and construction of the graph $\vec{G} = M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2)$, it results that the vertexes of graph $\vec{G}_1$ were not joined with new arcs. More than that, the following relations are true:

a) for any $y_j \in X_{\vec{G}_2}$, there exist vertexes $x_1 = y_1,\ x_2 = y_2 \in X_{\vec{G}_1}$, such that: $y_j \in d - conv(\{y_1,\ y_2\}) = d - conv(\{x_1,\ x_2\})$;

b) for any two adjacent vertexes $a,\ b \in X_{\vec{G}}$, there exist two distinct vertexes $x_1,\ x_2 \in X_{\vec{G}_1}$, such that:

0. $x_1 = y_1,\ x_2 = y_2$;
1. $x_1,\ x_2 \in d - conv(\{a,\ b\})$;
2. $d_{\vec{G}_1}(x_1,\ x_2) = d_{\vec{G}_2}(y_1,\ y_2) = d_{\vec{G}}(y_1,\ y_2)$;
3. $d_{\vec{G}_1}(x_2,\ x_1) = d_{\vec{G}_2}(y_2,\ y_1) = d_{\vec{G}}(y_2,\ y_1)$.

It follows that $\vec{G} = M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2) \in \mathfrak{D}$.

According to theorem 1.2, this means the graph $\vec{G}$ is d-convex simple graph. □

From the theorem 2.3 it results that the operation M, introduced above and being applied to the pairs of vertexes, that are at the same distances in the different d-convex simple graphs, is an *algebraic operation* on the set of the directed d-convex simple graphs $\mathfrak{D}$.

Definition 2.1 *Two vertexes u and v of a graph $\vec{G} = (X,\ \vec{U})$ are called to be* ***copies*** *vertexes in $\vec{G}$ if there are equalities:*

$$\Gamma^+(u) = \Gamma^+(v) \text{ and } \Gamma^-(u) = \Gamma^-(v).$$

In case of

$$\Gamma^+(u) = \Gamma^-(v) \text{ and } \Gamma^-(u) = \Gamma^+(v),$$

the vertexes u and v are called ***anti-copies****.*

Let us observe, that if $\vec{G} = (X,\ \vec{U})$ is a directed and strongly connected graph, where there exists a pair of vertexes anti-copies u and v, then $d_{\vec{G}}(u,\ v) = d_{\vec{G}}(v,\ u) = 2$. Indeed, from the fact that the graph $\vec{G}$ is strongly connected it results that any vertex has as predecessors as successors. So, no one of the sets $\Gamma^+(u)$, $\Gamma^+(v)$, $\Gamma^-(u)$, $\Gamma^-(v)$ is empty. Since u and v are vertexes anti-copies and the relations $\Gamma^+(u) = \Gamma^-(v)$ and $\Gamma^-(u) = \Gamma^+(v)$ are true, then it results:

a) u and v are not adjacent, i. e. neither u nor v is from the mentioned sets (otherwise the graph $\vec{G}$ will contain loops);

b) in $\vec{G}$ exist paths of length two that join u with v and v with u.

From a) we obtain that $d_{\vec{G}}(u,\ v)$ and $d_{\vec{G}}(v,\ u)$ are not equal to 1, and from b) we have that $d_{\vec{G}}(u,\ v) = d_{\vec{G}}(v,\ u) = 2$. So, circuit of minimal length that contains the vertexes u and v is of length 4.

From the above and theorem 2.3 we have:

Corollary 2.2 *If $\vec{G}_1$ and $\vec{G}_2$ are two d-convex simple graphs, where there exist the vertexes anti-copies $x_1,\ x_2 \in \vec{G}_1$ and $y_1,\ y_2 \in \vec{G}_2$, then the graph $\vec{G} = M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2)$ is also d-convex simple.*

Let us observe that if $\vec{G} = (X,\ \vec{U})$ is a directed and strongly connected graph, where there exists a pair of vertexes copies u and v, then $d_{\vec{G}}(u,\ v) = d_{\vec{G}}(v,\ u)$. Indeed, if we suppose that, the following inequality is true:

$$d_{\vec{G}}(u,\ v) < d_{\vec{G}}(v,\ u),$$

and $[u,\ x_1,\ x_2,\ \ldots,\ x_r,\ v]$ is one path of minimal length that joins the vertex u with vertex v, then, since u and v are vertexes copies we have $\Gamma^+(u) = \Gamma^+(v)$, that implies $(v,\ x_1) \in \vec{U}$, and $\Gamma^-(u) = \Gamma^-(v)$, that implies $(x_r,\ u) \in \vec{U}$. It follows that in $\vec{G}$ there exists the path $[v,\ x_1,\ x_2,\ \ldots,\ x_r,\ u]$, the fact that contradicts the assumption that the distance from v to u is longer than the distance from u to v. So, assumption is false and $d_{\vec{G}}(u,\ v) = d_{\vec{G}}(v,\ u)$.

From the above and theorem 2.3 we have:

Corollary 2.3 *If $\vec{G}_1$ and $\vec{G}_2$ are two d-convex simple graphs, where there exist the vertexes copies $x_1,\ x_2 \in \vec{G}_1$ and $y_1,\ y_2 \in \vec{G}_2$, which satisfy the condition $d_{\vec{G}_1}(x_1,\ x_2) = d_{\vec{G}_2}(y_1,\ y_2)$, then the graph $\vec{G} = M^{x_1=y_1}_{x_2=y_2}(\vec{G}_1,\ \vec{G}_2)$ is also d-convex simple.*

Now, let $\vec{G} = (X, \vec{U})$ be a d-convex simple graph and v any vertex of $\vec{G}$. We form the graph $\vec{G}^{++}$, which is obtained from the graph $\vec{G}$ by adding one vertex copy for v, which is denoted by $\tilde{v}$.

Theorem 2.4 *If $\vec{G} = (X, \vec{U})$, $X \geq 3$, is a d-convex simple graph, then $\vec{G}^{++}$ is also d-convex simple graph.*

Proof: Let $\vec{G} = (X, \vec{U})$ be a d-convex simple graph. By the theorem 1.3 this graph is from $\mathfrak{D}$. We choose an arbitrary vertex $v \in X$ and, according to the above, form the graph $\vec{G}^{++}$. Let $x,\ y$ be two vertexes from $X^{++}_{\vec{G}}$ and let $d - conv_{\vec{G}^{++}}(\{x,\ y\})$ be d-convex hull of these vertexes. It is easy to see that if $v \in d - conv_{\vec{G}^{++}}(\{x,\ y\})$ then we immediately have $\tilde{v} \in d - conv_{\vec{G}^{++}}(\{x,\ y\})$, the reverse assertion is also true. More than that, if $x,\ y$ are two different vertexes from $X_{\vec{G}^{++}}$ and they are different from the vertexes $v,\ \tilde{v}$, then $v \in d - conv_{\vec{G}^{++}}(\{x,\ y\})$, because graph $\vec{G}$ is d-convex simple. It follows that $\tilde{v} \in d - conv_{\vec{G}^{++}}(\{x,\ y\})$. But the set $\langle\overrightarrow{v,\ \tilde{v}}\rangle_{\vec{G}^{++}} \cup \langle\overrightarrow{\tilde{v},\ v}\rangle_{\vec{G}^{++}}$ contains all vertexes of at least one circuit of minimal length that pass trough vertex v, because the graph $\vec{G}$ is d-convex simple and this determine it to be strongly connected. This circuit contains vertexes from $\vec{G}$, which had kept the same distances among them in the graph $\vec{G}^{++}$ like in $\vec{G}$. The d-convex hull of these last vertexes contains all vertexes from $\vec{G}$. From this it results:

$$d - conv_{\vec{G}^{++}}(\{v,\ \tilde{v}\}) = X_{\vec{G}^{++}}.$$

By this way we obtained that d-convex hull of any two distinct vertexes from $\vec{G}^{++}$, which are different from v and $\tilde{v}$, contains all vertexes from $\vec{G}^{++}$.

Let y be any vertex from $\vec{G}^{++}$, different from vertexes v and $\tilde{v}$. D-convex hull of the set $\{y,\ v\}$ in $\vec{G}$ contains all vertexes of at least one circuit of minimal length that pass through v and y, which contains vertexes from $\vec{G}$, which had kept the same distances in the graph $\vec{G}^{++}$. So, $d - conv_{\vec{G}^{++}}(\{v,\ y\}) = X_{\vec{G}^{++}}$. And because the vertexes v and $\tilde{v}$ are vertexes copies then there are the equalities $\langle\overrightarrow{v,\ y}\rangle = \langle\overrightarrow{\tilde{v},\ y}\rangle$ and $\langle\overrightarrow{y,\ v}\rangle = \langle\overrightarrow{y,\ \tilde{v}}\rangle$, from where it results

$$d - conv_{\vec{G}^{++}}(\{v,\ y\}) = d - conv_{\vec{G}^{++}}(\{\tilde{v},\ y\}) = X_{\vec{G}^{++}}.$$

This means that the graph $\vec{G}^{++}$ is d-convex simple. □

From the theorem 2.4 it results that in directed d-convex simple graphs, like in undirected d-convex simple graphs, we can multiply any vertex as many times as we need, and the obtained graphs will be also d-convex simple.

Let $\vec{G} = (X,\ \vec{U})$ be a directed d-convex simple graph, where there exist three vertexes copies v_1, v_2 and v_3. We denote by $\vec{G}^{--}$ the graph that is obtained by the graph $\vec{G}$ as the result of elimination of one of them, for example, the vertex v_3.

Theorem 2.5 *If $\vec{G}$ is d-convex simple graph and v_1, v_2 and v_3 are three vertexes copies of it, then the graph $\vec{G}^{--}$, where one of them is missing, is also d-convex simple graph.*

Proof: Let $\vec{G} = (X,\ \vec{U})$ be a directed d-convex simple graph, where there exist three vertexes copies v_1, v_2 and v_3, and let $\vec{G}^{--}$ be the graph that is obtained by the graph $\vec{G}$ as the result of elimination of one of them, for example, of vertex v_3. Obviously, from the fact that the vertexes v_1, v_2 and v_3 are vertexes copies in $\vec{G}$, it results that any d-segment that contains one of these three vertexes, will immediately contain the others as well. This property is true for the vertexes v_1 and v_2 in graph $\vec{G}^{--}$, too. From the same considerations, for any two vertexes $x,\ y \in X_{\vec{G}^{--}}$ the following equality holds:

$$d_{\vec{G}}(x,\ y) = d_{\vec{G}^{--}}(x,\ y). \qquad (*)$$

More than that, since $\vec{G}$ is d-convex simple, then in $\vec{G}$ there are equalities $\langle\overrightarrow{v_1,\ v_2}\rangle = \langle\overrightarrow{v_1,\ v_3}\rangle$ and $\langle\overrightarrow{v_2,\ v_1}\rangle = \langle\overrightarrow{v_3,\ v_1}\rangle$, from where it results:

$$d - conv_{\vec{G}}(\{v_1,\ v_2\}) = d - conv_{\vec{G}}(\{v_1,\ v_3\}) = X_{\vec{G}}.$$

But the last implies:

$$d - conv_{\vec{G}^{--}}(\{v_1,\ v_2\}) = X_{\vec{G}^{--}}. \qquad (**)$$

Now, we choose two distinct vertexes $a,\ b \in X_{\vec{G}^{--}}$. Because initial graph $\vec{G}$ is d-convex simple, it follows, that d-convex hull of the set $\{a,\ b\}$ in this graph can be built by the sequence of sets:

$$B_0 = \{a,\ b\},\ B_1 = P(B_0),\ B_2 = P(B_1),\ \ldots,$$

$$B_i = P(B_{i-1}),\ \ldots,\ d - conv(\{a,\ b\}) = X_{\vec{G}}.$$

We consider that, for example, B_i is the first set of this sequence, which contains the vertex v_3. Obviously, the vertexes v_1 and v_2 belong to this set, too.

Because of the relation (*), for the set $\{a,\ b\}$ in the graph $\vec{G}^{--}$ we can build the sequence:

$$B_0^{--} = \{a,\ b\},\ B_1^{--} = P(B_0^{--}),\ B_2^{--} = P(B_1^{--}),\ \ldots,\ B_i^{--} =$$

$$= P(B_{i-1}^{--}),\ \ldots,$$

such that $B_j = B_j^{--}$, for all $0 \leq j \leq i-1$. In these conditions we obtain that $B_i \backslash B_i^{--} = \{v_3\}$, but the vertexes v_1 and v_2 belong to the set B_i^{--}, that means, according to the (**), that $d - conv_{\vec{G}^{--}}(\{a,\ b\}) = X_{\vec{G}^{--}}$. □

From the theorem 2.5 it results that in the directed d-convex simple graphs, like in the undirected d-convex simple graphs, we can eliminate the vertexes copies of the vertex v, keeping only one copy for v, and the obtained graph will be d-convex simple, too.

Let $\vec{G} = (X,\ \vec{U})$ be a directed graph. We form the graph $\vec{G}^t = (X,\ \vec{U}^t)$ that is obtained from $\vec{G}$ by redirecting of all arcs of it. It is easy to see, that the adjacent matrix of the graph $\vec{G}^t$ is the transpose of the adjacent matrix of the graph $\vec{G}$.

Theorem 2.6 *If $\vec{G}$ is d-convex simple graph, then the graph $\vec{G}^t$ is also d-convex simple graph.*

Proof: Let $\vec{G} = (X,\ \vec{U})$ be a d-convex simple graph, and $\vec{G}^t = (X,\ \vec{U}^t)$ the graph formed like it is described above. From the construction it results that for any two distinct vertexes x and y from X the following equalities are true:

$$\langle\overrightarrow{x,\ y}\rangle_{\vec{G}} = \langle\overrightarrow{y,\ x}\rangle_{\vec{G}^t} \text{ and } \langle\overrightarrow{y,\ x}\rangle_{\vec{G}} = \langle\overrightarrow{x,\ y}\rangle_{\vec{G}^t}.$$

It follows that we have $d - conv_{\vec{G}}(\{x,\ y\}) = d - conv_{\vec{G}^t}(\{y,\ x\})$. But the initial graph $\vec{G}$ is d-convex simple and the order of elements is not important in sets. It results that $\vec{G}^t$ is d-convex simple, too. □

Let $\vec{G} = (X,\ \vec{U})$, $|X| \geq 4$ be a directed graph. We choose in $\vec{G}$ four vertexes x_1, x_2, y_1 and y_2, that are nonadjacent two by two in $\vec{G}$. We denote by $W^{x_1=y_1}_{x_2=y_2}(\vec{G})$ the graph obtained from $\vec{G}$ as the result of joining the vertexes x_1 with y_1 and x_2 with y_2.

Theorem 2.7 *If $\vec{G} = (X,\ \vec{U})$, $|X| \geq 4$ is d-convex simple graph, where there exist four vertexes x_1, x_2, y_1 and y_2, which satisfy conditions:*

1. *the vertexes x_1, x_2, y_1 and y_2 are nonadjacent two by two;*
2. $d(x_1,\ x_2) = d(y_1,\ y_2),\ d(x_2,\ x_1) = d(y_2,\ y_1)$;
3. $\min\{d(x_1,\ y_1),\ d(y_1,\ x_1),\ d(x_2,\ y_2),\ d(y_2,\ x_2)\} \geq d(x_1,\ x_2) + d(x_2,\ x_1)$;
4. $\min\{d(x_1,\ y_2),\ d(y_2,\ x_1),\ d(x_2,\ y_1),\ d(y_1,\ x_2)\} \geq d(x_1,\ x_2) + d(x_2,\ x_1)$,

then the graph $\vec{H} = W^{x_1=y_1}_{x_2=y_2}(\vec{G})$ is also d-convex simple.

Proof: Let $\vec{G} = (X,\ \vec{U})$ be a directed d-convex simple graph that satisfies the conditions of the theorem. Let $\vec{H} = W^{x_1=y_1}_{x_2=y_2}(\vec{G})$ be the graph constructed as it is shown above. We remind that in this work we study the directed graphs, without loops or multiple arcs. Because the initial graph $\vec{G}$ is d-convex simple, and the set of selected vertexes $\{x_1,\ x_2,\ y_1,\ y_2\}$ satisfies conditions 1 of the theorem, it results that in graph $\vec{H}$ there are no loops or multiple arcs, too. More, from the last we have that for any two vertexes $s,\ t \in X_{\vec{H}}$ there exist in $\vec{H}$ at most one of the two possible arcs $s,\ t$ and $t,\ s$.

Let us suppose that $\vec{H}$ is not d-convex simple. This means that in this graph there exists a subset of d-convex vertexes B, $1 < |B| < |X_{\vec{H}}|$. Thus, initial graph $\vec{G}$ is d-convex simple, we have that d-convex hull of the set B in the graph $\vec{G}$ is $d - conv_{\vec{G}}(B) = X_{\vec{G}}$. This d-convex hull can be formed in $\vec{G}$ as follows:

$$B_0 = B,\ B_1 = P_{\vec{G}}(B_0),\ B_2 = P_{\vec{G}}(B_1),\ \ldots,\ B_i = d - conv_{\vec{G}}(B) = X_{\vec{G}}.$$

Let us form the d-convex hull of the set B in the graph $\vec{H}$. By using the same operation we have:

$$A_0 = B,\ A_1 = P_{\vec{H}}(A_0),\ A_2 = P_{\vec{H}}(A_1),\ \ldots,\ A_i = P_{\vec{H}}(A_{i-1}) \ldots$$

From conditions 2, 3 and 4 of the theorem, we can have that:

$$B_j \backslash \{x_1,\ x_2,\ y_1,\ y_2\} = A_j \backslash \{x_1 = y_1,\ x_2 = y_2\},\ \forall j \in \mathbb{N}.$$

The last implies that for $j = i$ we have

$$X_{\vec{G}} \backslash \{x_1,\ x_2,\ y_1,\ y_2\} = A_j \backslash \{x_1 = y_1,\ x_2 = y_2\}.$$

But the vertexes $x_1,\ x_2,\ y_1,\ y_2$ are in $B_i = X_{\vec{G}}$, it results that $x_1 = y_1, x_2 = y_2 \in A_i$. But $[X_{\vec{G}} \backslash \{x_1, x_2, y_1, y_2\}] \cup \{x_1 = y_1, x_2 = y_2\} = X_{\vec{H}}$. So, we have

$$X_{\vec{H}} \subseteq A_i \subseteq d - conv_{\vec{H}}(B),$$

that contradicts the assertion that $d - conv_{\vec{H}}(B) \neq X_{\vec{H}}$. The assumption is false, the graph $\vec{H}$ is d-convex simple. □

Corollary 2.4 *If* $\vec{G} = (X,\ \vec{U})$, $|X| \geq 4$ *is d-convex simple graph, where there exist two pairs of vertexes anti-copies* $x_1,\ x_2$ *and* $y_1,\ y_2$, *which satisfy conditions:*

a. $\min\{d(x_1,\ y_1),\ d(y_1,\ x_1),\ d(x_2,\ y_2),\ d(y_2,\ x_2)\} \geq 4$;

b. $\min\{d(x_1,\ y_2),\ d(y_2,\ x_1),\ d(x_2,\ y_1),\ d(y_1,\ x_2)\} \geq 4$, *then the graph* $\vec{H} = W^{x_1=y_1}_{x_2=y_2}(\vec{G})$ *is also d-convex simple graph.*

Corollary 2.5 *If* $\vec{G} = (X,\ \vec{U})$, $|X| \geq 4$ *is d-convex simple graph, where there exist two pairs of vertexes copies* x_1, x_2 *and* y_1, y_2, *which satisfy conditions:*

a. $d(x_1,\ x_2) = d(y_1,\ y_2)$;

b. $\min\{d(x_1,\ y_1),\ d(y_1,\ x_1),\ d(x_2,\ y_2),\ d(y_2,\ x_2)\} \geq 2 \cdot d(x_1,\ x_2)$;

c. $\min\{d(x_1,\ y_2),\ d(y_2,\ x_1),\ d(x_2,\ y_1),\ d(y_1,\ x_2)\} \geq 2 \cdot d(x_1,\ x_2)$,

then the graph $\vec{H} = W^{x_1=y_1}_{x_2=y_2}(\vec{G})$ *is also d-convex simple graph.*

Further, we will use the notions of chain and cycle for directed graphs, defined in [9]. The number of arcs that belong to a chain (cycle) is called *length* of it. For example, the chain $l = (x_{i_1},\ x_{i_2},\ \ldots,\ x_{i_t})$ has length equal to $t-1$. For simplicity, further, the cycle of length three, will be called *triangle*. The directed graph $\vec{G} = (X,\ \vec{U})$ is called *weakly connected*, if any two vertexes of it are joined by a chain.

We define for the directed graphs a special operation, denoted by L_2, which, to tell the truth, is defined analogically to the case of undirected graphs [7, 8]. Let us denote by $X = \{x_1,\ x_2,\ \ldots,\ x_n\}$ the set of vertexes of the directed graph $\vec{G} = (X,\ \vec{U})$. Let $\vec{G}_1$, $\vec{G}_2$ be two copies of the graph $\vec{G}$ with the sets of vertexes $X_{\vec{G}_1} = \{x^1_1,\ x^1_2,\ \ldots,\ x^1_n\}$ and $X_{\vec{G}_2} = \{x^2_1,\ x^2_2,\ \ldots,\ x^2_n\}$ respectively. The vertexes x^1_j and x^2_j, $j = 1,\ 2,\ \ldots,\ n$, are called correspondent vertexes to the vertex x_j, $1 \leq j \leq n$, in conditions when $(x_i, x_k) \in \vec{U}$ if and only if $(x^p_i, x^p_k) \in \vec{U}_{\vec{G}_p}$, where $p = 1,\ 2$. By $L_2(\vec{G})$ we define the graph that is obtained from $\vec{G}_1$ and $\vec{G}_2$ by adding the following arcs: for any vertex x^1_j, $1 \leq j \leq n$, from $\vec{G}_1$, we add arcs to all vertexes which are adjacent to the vertex x^2_j in $\vec{G}_2$, only that they are of the opposite directions. It is obvious that if $|X| = n$ and $|\vec{U}| = r$, then the graph $L_2(\vec{G})$ will have $2n$ vertexes and $4r$ arcs. More than that, in the $L_2(\vec{G})$, the vertexes x^1_j and x^2_j are vertexes anti-copies, for any $j = 1,\ 2,\ \ldots,\ n$.

Theorem 2.8 *If $\vec{G}$ is a directed, weakly connected, antisymmetric graph, without triangles, then the graph $L_2(\vec{G})$ is d-convex simple graph.*

Proof: We will prove first that if the graph $\vec{G}$ is a graph that satisfies the conditions of the theorem, then the graph $L_2(\vec{G})$ is strongly connected. Let u and v are two vertexes from $L_2(\vec{G})$. There are possible two cases: both of these vertexes belong to one of the two copies of graph $\vec{G}$, $\vec{G}_1$ and $\vec{G}_2$; or, for example, u is from $\vec{G}_1$ and v is from $\vec{G}_2$.

a. Let us suppose that both vertexes u and v are, for example, from $\vec{G}_1$. The fact that the graph $\vec{G}$ is weakly connected results that in $\vec{G}_1$ there exists the chain $l = (u = x^1_{i_1},\, x^1_{i_2},\, \ldots,\, x^1_{i_t} = v)$, that joins the vertexes u and v. The arcs, that joined the vertexes $x^1_{i_k}$ and $x^1_{i_{k+1}}$, $1 \le k \le t-1$, are of the arbitrary direction (from the vertex $x^1_{i_k}$ to the $x^1_{i_{k+1}}$, or reverse). From this chain we will build a path from u to v in the graph $L_2(\vec{G})$. Let the arc $(x^1_{i_{s+1}},\, x^1_{i_s})$ be the first from this chain that is directed from v to u. Then in our path from $L_2(\vec{G})$ this arc will be replaced with next three arcs: $(x^1_{i_s},\, x^2_{i_{s+1}})$, $(x^2_{i_{s+2}},\, x^2_{i_s})$, $(x^2_{i_s},\, x^1_{i_{s+1}})$, where $x^2_{i_s}$, $x^2_{i_{s+1}}$ are the vertexes anti-copies corresponding to $x^1_{i_s}$, $x^1_{i_{s+1}}$ respectively. We will do the same thing with all arcs from l, similar to arc $(x^1_{i_{s+1}},\, x^1_{i_s})$, and finally we obtain the wanted path from u to v in $L_2(\vec{G})$.

b. Let us suppose now that u is from $\vec{G}_1$ and v is from $\vec{G}_2$. By analogy with the case a we can, first, build a path from u to $\tilde{v} \in X_{\vec{G}_1}$, that is anti-copy of the vertex v from graph $\vec{G}_1$. Let $(x,\, \tilde{v})$ be the last arc from this path. We extend the path from u to $\tilde{v}$ with arcs $(\tilde{v},\, \tilde{x})$, $(\tilde{x},\, v)$, where $\tilde{x} \in X_{\vec{G}_2}$ is the anti-copy of vertex x, and obtain by this way a path in $L_2(\vec{G})$ from u to v.

From investigated cases a and b it results that for any pair of ordered vertexes $(u,\, v)$ from $L_2(\vec{G})$ there is a path from u to v. It follows that $L_2(\vec{G})$ is a strongly connected graph.

Let us prove, now, that graph $L_2(\vec{G})$ is d-convex simple. Let u and v be two adjacent vertexes in $L_2(\vec{G})$ and $\tilde{u}$ and $\tilde{v}$ be anti-copies of these vertexes respectively (the existence of the anti-copies $\tilde{u}$ and $\tilde{v}$ result from the construction of the graph $L_2(\vec{G})$). From the fact that the graph $\vec{G}$ is antisymmetric, without triangles, and from the construction of the graph $L_1(\vec{G})$ we obtain that the graph $L_2(\vec{G})$ is antisymmetric and without triangles, too. So it follows that the cycles and circuits of minimal length in this graph have at least four arcs. Such a circuit of minimal length is determined by vertexes u, v, $\tilde{u}$ and $\tilde{v}$. We obtain that d-convex hull of any two adjacent vertexes u, v contains at least one pair of vertexes anti-copies. Let z and $\tilde{z}$ be a pair of vertexes anti-copies from d-convex hull of any two adjacent vertexes $a, \ b \in X_{L_2(\vec{G})}$. We form in $L_2(\vec{G})$ the sequence of sets:

$$B_0 = \{z, \ \tilde{z}\} \subset d - conv(\{a, \ b\}), \ B_1 = P(B_0), \ \ldots, \ d - conv(B_0).$$

From the construction of the graph $L_2(\vec{G})$, and the condition of theorem that the graph $\vec{G}$ is weakly connected, the construction of the set B_{i+1} from the set B_i is always followed by addition of at least one pair of the new vertexes copies. This means, that in the last we will cover all the vertexes, and the set $d - conv(B_0)$ will coincide with $X_{L_2(\vec{G})}$. So we have that $L_2(\vec{G})$ is a d-convex simple graph. $\square$

3 Relation between Directed d-Convex Simple Graphs and Undirected d-Convex Simple Graphs.

Let $G = (X, \ U)$ be an undirected graph. This means that it is an directed graph, completely symmetric, where each edges $u = (x, y) \in U$ are considered as two arcs $(x, \ y)$ and $(y, \ x)$. Let us eliminate from each edge of the graph G one and only one of these two arcs. The obtained graph is antisymmetric, and it will be called *direction* graph of the initial graph G, and denote by $\vec{G}$. Of course, in dependence of what arcs are eliminated, for the graph G, we can obtain several its direction graphs. In this section we are going to show, that for any undirected, d-convex simple graph, the structure of which is known, there is at least one directed d-convex simple graph, that corresponds to that undirected, and the correspondence we will search in the set of direction graphs of the initial undirected graph.

Let us consider the class $\mathcal{A}$ of the undirected d-convex simple graphs from [7].

Theorem 3.1 *If G is an undirected d-convex simple graph from the class $\mathcal{A}$ then there exists at least one direction of the G, that is a directed d-convex simple graph.*

Proof: We are going to prove this theorem, by giving a way of construction of one directed d-convex simple graph, we need.

Let $G \in \mathcal{A}$ be an undirected d-convex simple graph. Then from [7] we have that $G = L(\Gamma, \Gamma_0)$, where Γ is a connected graph without triangles and Γ_0 is its atom. In order to construct Γ_0 first we have to determine the sets:

$$S = \{x \in X \mid \forall y \in X \Rightarrow \Delta(x) \nsubseteq \Delta(y)\};$$

$$R = \{x \in X \backslash S \mid \forall y \in X \Rightarrow \Delta(x) \not\subset \Delta(y)\},$$

Then, for any $x \in R$ we form the set $R(x) = \{x\} \cup \{y \in R \mid \Delta(x) = \Delta(y)\}$. By this way the set R is divided in classes of equivalence. Γ_0 is an induced subgraph of graph G, the set of vertexes of which is formed from the set S and by one vertex from each class of equivalence. It is easy to see from construction that the graph Γ_0 is also a connected graph, without triangles (see [7]).

Let us consider any direction of the graph Γ_0 and denote it by $\vec{\Gamma}_0$. The graph $\vec{\Gamma}_0$ is a directed, weakly connected, antisymmetric graph, without triangles. According to the theorem 2.8 the graph $L_2(\vec{\Gamma}_0)$ is a directed d-convex simple graph. Any vertex of the graph $L_2(\vec{\Gamma}_0)$ can be multiplied, as many times as we need, according to the theorem 2.4. Let us build now a new graph, denoted by $\vec{G}$, which is formed from the graph $L_2(\vec{\Gamma}_0)$ where those vertexes were multiplied, that satisfy the following condition: for any vertex from R, there will be a correspondence vertex in the graph $\vec{G}$. The new graph $\vec{G}$ is a directed d-convex simple graph, and it is a direction of the initial graph G. □

From the last proof, we have that if G is undirected, d-convex simple graph and $\vec{G}$ is its direction d-convex simple graph, then in $\vec{G}$ any vertex has a anti-copy. So, the pairs of vertexes copies from G, have become pairs of vertexes anti-copies in $\vec{G}$. The last means that these graphs can participate in operations M and W with any graph which has a pair of vertexes anti-copies.

The directed and d-convex simple graph from the fig.1 is a direction from the graph J_1 from [2, 3], denoted $\vec{J_1}$. It has the property that if we add a vertex anti-copy to any its vertex, the new graph will be also d-convex simple.

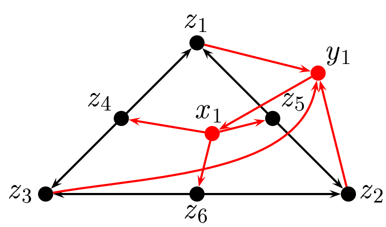

Fig. 1. The graph $\vec{J_1}$.

By this way, we obtain that all undirected d-convex simple graphs G from [2, 3], has at least one direction graph $\vec{G}$, that is a directed, d-convex simple graph. This means that the set of directed d-convex simple graphs contains, in this sense, the set of undirected d-convex simple graphs.

References

[1] S. Cataranciuc, N. Vicol. *Convexity in directed graphs*, Balance of Science Activity of MSU in 2000-2002, 2003, pp. 104–106 (in Romanian).

[2] N. Sur. *About Division of d-Convex Simple Graphs in M-Prime Graphs*, Buletinul Acad. de Ştiinţe, Matematica, nr 3(58), 2008, Chişinău, will appear.

[3] N. Sur. *The closed and complete sets of d-convex simple graphs and theirs generators*, Mathematics modelling, optimisation and informatics technology, The International Conference, ATIC, Chişinău, Evrica, 2008, pp. 199–204 (in Romanian).

[4] N. Sur. *About a new class of d-convex simple graphs*, Scientific Annals of MSU, Section "Phys-Math Science", Chişinău, 2006, pp. 123–126 (in Romanian).

[5] N. Sur. *An algebraic operation in the set of d-convex simple graphs*, International scientific conference dedicated to 60 years of MSU, Chişinău, 2006, pp. 55–56 (in Romanian).

[6] N. Sur. *A method of description for d-convex simple and non-oriented graphs*, Annals of Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, vol. 3, 2005, pp. 35–38.

[7] S. Cataranciuc *About simple convexity of graphs*, The PhD thesis, Chişinău, 1990, p. 178 (in Russian).

[8] V. Soltan. *Introduction to axiomatic convex theory*, Chişinău, 1984, p. 222 (in Russian).

[9] C. Berge. *Theorie des graphes et ses applications*, 277, DUNOD, Paris, 1958, p. 277.

Nadejda Sur, Sergiu Cataranciuc,

Received May 26, 2008

Faculty of Mathematics and Informatics,
Moldova State University,
A. Mateevici street, 60, MD-2009
Republic of Moldova
E–mail: *nadejda_sur@rambler.ru*, *caseg@usm.md*