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An algebraic approach to a study of two-dimensional affine differential system

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Abstract

In a present paper a problem of classification of $Aff(2,\mathbb{R})$-orbits' dimensions is considered on example of an autonomous two-dimensional affine differential system of first order. Methods of Lie algebras are used in the work, as well as methods of group analysis. Computer algebra systems "Bergman" and "Mathematica 5.0" are widely used.

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1 Introduction

In a present work we consider autonomous polynomial differential system, written in general form as follows

$$\frac{dx^j}{dt} = \sum_{m_i \in \Gamma} P^j_{m_i}(x), \ (j = \overline{1,2}), \tag{1}$$

where $\Gamma = \{m_i\}_{i=1}^{l}$ is some finite set of different non-negative integers, and

$$P^j_{m_i}(x) = \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}^j_k (x^1)^{m_i-k}(x^2)^k, \ (j = \overline{1,2}; i = \overline{1,l})$$

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are homogeneous polynomials with order m_i in $(x^1, x^2, ..., x^n)$. Coefficients and variables of system (1) are defined over the field of real numbers $\mathbb{R}$. Further we will denote system (1) by $s^2(\Gamma)$ for special Γ. The variable t is independent one, and x^1, x^2 are dependent functions (variables) on t.

System (1) will be considered with group of affine transformations $Aff(2,\mathbb{R})$ given by equalities

$$\overline{x}^1 = \alpha x^1 + \beta x^2 + h_1,\ \overline{x}^2 = \gamma x^1 + \delta x^2 + h_2,\quad \left(\Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0\right), \quad (2)$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ h_1,\ h_2$ are real parameters, ever varying in $\mathbb{R}$. Further we will consider transformations (2) given by matrix $q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and when we say "q belongs to group $Aff(2,\mathbb{R})$", we write this as $q \in Aff(2,\mathbb{R})$.

Note that the application of group $Aff(2,\mathbb{R})$ to qualitative investigation of systems (1) is remarkable as the system keeps its form after affine transformation. And coefficients of the system are varying in according to law of tensors, being basic geometrical objects of Invariant Theory. Thus, we can conclude that to perform complete qualitative investigation of system (1) it is necessary to apply the method of algebraic invariants. Remark, that this method was founded in works by K.Sibirsky [1].

Adaptation of Lie algebras of operators and techniques of group analysis in study of systems (1) has appeared as a certain step in development of this method. Results of such researches are quoted in works by M.Popa [2] and his disciples. These works are devoted to investigation of algebraic objects (finite-dimensional Lie algebras and corresponding algebras of invariants), obtained due to representation of linear groups of transformations in space of coefficients of systems (1). Besides, the classification's tasks are considered in these works, concerned with dimensions of orbits, as well as with problems of existence of invariant integrals.

As appeared, an answer to the question about existence of such integrals is thoroughly connected with classification of orbits' dimensions and of invariant varieties of considering groups, particularly, group $Aff(2,\mathbb{R})$. Therefore it became necessary to construct such classifications for further investigation of systems (1).

Remark, that solution of classifications' questions for systems (1) with more than one homogeneity in right-hand sides requires implication of computer algebra systems and was impossible until nowadays due to intricate calculations.

2 Basic notions and definitions

Throughout the work we will need some notions.

Definition 2.1. *Call the linear space L_r over the field $\mathbb{R}$ a Lie algebra, if for any two of its elements X, Y the operation of commutation $[X,Y]$ is defined, which returns the element from L_r (commutator of elements X, Y) and satisfies the following axioms:*

1) bilinearity: for any $X,Y,Z \in L$ and $\alpha,\beta \in \mathbb{R}$

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z],$$

$$[X, \alpha Y + \beta Z] = \alpha[X,Y] + \beta[X,Z];$$

2) anti-symmetry: for any $X,Y \in L$

$$[X,Y] = -[Y,X];$$

3) identity of Jacobi: for any $X,Y,Z \in L$

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

It is shown in [2] that Lie algebra, corresponding to linear representation of group $Aff(2,\mathbb{R})$ in the space of coefficients and variables of system (1), is six-dimensional Lie algebra $L_6 = \{X_1, X_2, X_3, X_4, X_5, X_6\}$. This algebra can be given by Lie operators [2]:

$$X_1 = x^1\frac{\partial}{\partial x^1} - D_1,\ X_2 = x^2\frac{\partial}{\partial x^1} - D_2,\ X_3 = x^1\frac{\partial}{\partial x^2} - D_3,$$
$$X_4 = x^2\frac{\partial}{\partial x^2} - D_4,\ X_5 = \frac{\partial}{\partial x^1} - D_5,\ X_6 = \frac{\partial}{\partial x^2} - D_6, \qquad (3)$$

where

$$D_1 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[(m_i-k-1)\overset{i}{a}{}^1_k\frac{\partial}{\partial \overset{i}{a}{}^1_k} + (m_i-k)\overset{i}{a}{}^2_k\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right],$$

$$D_2 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[k\left(\overset{i}{a}{}^1_{k-1}\frac{\partial}{\partial \overset{i}{a}{}^1_k} + \overset{i}{a}{}^2_{k-1}\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right) - \overset{i}{a}{}^2_k\frac{\partial}{\partial \overset{i}{a}{}^1_k}\right],$$

$$D_3 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[(m_i-k)\left(\overset{i}{a}{}^1_{k+1}\frac{\partial}{\partial \overset{i}{a}{}^1_k} + \overset{i}{a}{}^2_{k+1}\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right) - \overset{i}{a}{}^1_k\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right],$$

$$D_4 = \sum_{i=1}^{l}\sum_{k=0}^{m_i}\left[k\overset{i}{a}{}^1_k\frac{\partial}{\partial \overset{i}{a}{}^1_k} + (k-1)\overset{i}{a}{}^2_k\frac{\partial}{\partial \overset{i}{a}{}^2_k}\right],$$

$$D_5 = \sum_{i=1}^{l}\sum_{k=0}^{i-1} i\left(\overset{i}{a}{}^1_k\frac{\partial}{\partial \overset{i-1}{a}{}^1_k} + \overset{i}{a}{}^2_k\frac{\partial}{\partial \overset{i-1}{a}{}^2_k}\right),$$

$$D_6 = \sum_{i=1}^{l}\sum_{k=0}^{i-1}\left(\overset{i}{a}{}^1_{k+1}\frac{\partial}{\partial \overset{i-1}{a}{}^1_k} + \overset{i}{a}{}^2_{k+1}\frac{\partial}{\partial \overset{i-1}{a}{}^2_k}\right). \qquad (4)$$

According to [2], in order to solve the problem of classification of orbits' dimensions, we will consider only operators D_1 - D_6, since they form six-dimensional Lie algebra L_6, corresponding to linear representation of group $Aff(2,\mathbb{R})$ in the space of coefficients of system (1).

Let $a = \left(\overset{1}{a}{}^1_0, \overset{1}{a}{}^1_1, \ldots, \overset{l}{a}{}^2_{m_l}\right) \in E(a)$, where $E(a)$ is Euclidean space of coefficients of right-hand sides of system (1).

Denote by $a(q)$ a point from $E(a)$ corresponding to a system, obtained from system (1) with coefficients a after transformation $q \in Aff(2,\mathbb{R})$.

Definition 2.2. *The set $O(a) = \{a(q); q \in Aff(2,\mathbb{R})\}$ is called an $Aff(2,\mathbb{R})$-orbit of a point* **a** *for system* (1).

Definition 2.3 *The set $M \subseteq E(a)$ is called an $Aff(2,\mathbb{R})$-invariant set if for any point $a \in M$ its orbits $O(a) \subseteq M$.*

It is known from [3] - [4] that space $\mathfrak{g}(a)$, constructed on coordinate vectors of operators (4), is the tangent space to $Aff(2,\mathbb{R})$ - orbit $O(a)$ in point $a \in E(a)$, such that

$$dim_{\mathbb{R}} O(a) = dim_{\mathbb{R}} \mathfrak{g}(a). \tag{5}$$

On the other hand,

$$dim_{\mathbb{R}} \mathfrak{g}(a) = rank M_1, \tag{6}$$

where M_1 is a matrix, constructed on coordinate vectors of operators (4).

From (5) - (6) it is evident

$$dim_{\mathbb{R}} O(a) = rank M_1. \tag{7}$$

Denote by

$$M = \begin{pmatrix} x^1 & 0 \\ x^2 & 0 \\ 0 & x^1 \\ 0 & x^2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will denote the matrix (M, M_1) by $(\xi(x), \eta(a))$ when it represents a reflection in space of coefficients and variables $E(x,a)$ of system (1).

Further we will consider varieties Ψ given implicitly in finite-dimensional space $E(x,a)$ [4].

This means that an open set $U \subset E(x,a)$ is given together with reflection $\psi : U \to \mathbb{R}$ of class $C_\infty(U)$, and $\psi(x_0,a_0) = 0$ for some point $(x_0,a_0) \in U$ and the set $\psi(U_0)$ is open in $\mathbb{R}$ for any vicinity $U_0 \subset U$ of the point (x_0,a_0). Variety Ψ can be defined in these conditions as locus of $(x,a) \in U$, for which holds

$$\psi(x,a) = 0. \tag{8}$$

Equality (8) is called the equation of variety Ψ.

Definition 2.4. *Call the variety* Ψ *an invariant if for any point* $a \in \Psi$ *its orbit* $O(a) \subseteq \Psi$.

Definition 2.5. *Call the number*

$$r_* = r_*(\xi, \eta) = \max_{(x,a)\in U} rank(\xi(x), \eta(a))$$

a general rang of the reflection (ξ, η) *onto open set* $U \subset E(x, a)$.

Definition 2.6. *Call the point* $(x, a) \in E(x, a)$ *a singular point (of group* $Aff(2, \mathbb{R})$ *or its Lie algebra* L_6*), if*

$$rank(\xi(x), \eta(a)) < r_*,$$

and non-singular point (of group $Aff(2, \mathbb{R})$ *or its Lie algebra* L_6*) if*

$$rank(\xi(x), \eta(a)) = r_*.$$

Definition 2.7. *Call the variety* $\Psi \subset U$ *a singular variety of group* $Aff(2, \mathbb{R})$ *(or its Lie algebra* $L_6(\xi, \eta)$*) if all its points are singular and if the reflection* (ξ, η) *has the rang on* Ψ*, i.e. for any point* $(x, a) \in \Psi$ *we obtain*

$$rank(\xi(x), \eta(a)) = r_*(M|\Psi) < r_*.$$

Definition 2.8. *Call the variety* $\Psi \subset U$ *a non-singular variety of group* $Aff(2, \mathbb{R})$ *(or its Lie algebra* $L_6(\xi, \eta)$*) if all its points are non-singular, i.e. if the following equality holds*

$$r_*(M|\Psi) = r_*.$$

According to last definitions, all invariant varieties of group $Aff(2, \mathbb{R})$ can be divided into singular and non-singular $Aff(2, \mathbb{R})$-invariant varieties.

From this viewpoint, the classification of dimensions of $Aff(2, \mathbb{R})$-orbits of differential equations' system can be represented as a classification of invariant varieties of group $Aff(2, \mathbb{R})$. Remark, that $Aff(2, \mathbb{R})$-orbits of maximal dimension correspond to non-singular invariant varieties of group $Aff(2, \mathbb{R})$.

From Theorem of representation [4] follows

Theorem 2.1. *If non-singular variety of Lie algebra $L_6(\xi,\eta)$ is given regularly by equation* (8)*, then such invariant $F : E(x,a) \to \mathbb{R}$ of this algebra exists, that this variety can be given by equality $F(x,a) = 0$.*

Definition 2.9. *Call the integer rational function $K(x,a)$, in variables x and coefficients a of system* (1) *an affine comitant if it meets the condition*

$$K(\bar{x},\bar{a}) = \Delta^{-g} K(x,a)$$

for any values of x and a and any transformations of group $Aff(2,\mathbb{R})$. Number g is called a weight of affine comitant.

Definition 2.10. *If an affine comitant $K(x,a)$ does not depend on variables x, it is called an affine invariant of system* (1)*.*

From [2] and [4] it is known

Theorem 2.2. *The integer rational function $K(x,A)$ $(I(A))$ in variables x and coefficients a of system* (1) *is an affine comitant (invariant) of this system with weight g if and only if it meets conditions*

$$X_1(K) = X_4(K) = -gK,\ X_2(K) = X_3(K) = X_5(K) = X_6(K) = 0;$$
$$D_1(I) = D_4(I) = -gI,\ D_2(I) = D_3(I) = D_5(I) = D_6(I) = 0,$$

where X_1 - X_6 and D_1 - D_6 are defined in (3) *and* (4).

3 Classification of dimensions of $Aff(2,\mathbb{R})$ - orbits for system $s^2(0,1)$.

Let us apply above stated theory to investigation of affine differential system $s^2(0,1)$.

Consider system (1) for $\Gamma = \{0,1\}$. According to [1] we will write it in tensor form as follows

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha, \quad (j,\alpha = 1,2). \tag{9}$$

System (9) will be considered with group $Aff(2,\mathbb{R})$, defined in (2).

Further we will use affine comitants and invariants known from works [1], [5], [6]:

$$K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq},\ K_{21} = a^p x^q \varepsilon_{pq},\ K_{22} = a^\alpha a^p_\alpha x^q \varepsilon_{pq},$$
$$I_1 = a^\alpha_\alpha,\ I_2 = a^\alpha_\beta a^\beta_\alpha,\ I_{21} = a^\alpha a^q a^p_\alpha \varepsilon_{pq},$$
$$Q = I_{21} + I_1 K_{22} - I_2 K_{21} + \frac{1}{2}(I_1^2 - I_2)K_2, \qquad (10)$$

where ε^{pq} and ε_{pq} are unit bi-vectors with coordinates $\varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$.

Remark [6], that invariants I_1, I_2 and comitant Q form minimal polynomial basis of affine comitants for system (9).

In order to simplify further expressions we will use the following notations

$$x^1 = x,\ x^2 = y,\ a^1 = a,\ a^2 = b,\ a^1_1 = c,\ a^1_2 = d,\ a^2_1 = e,\ a^2_2 = f. \quad (11)$$

According to (3) - (4) and (11), we will write Lie operators for system (9):

$$X_1 = x\frac{\partial}{\partial x} - D_1,\ X_2 = y\frac{\partial}{\partial x} - D_2,\ X_3 = x\frac{\partial}{\partial y} - D_3,$$
$$X_4 = y\frac{\partial}{\partial y} - D_4,\ X_5 = \frac{\partial}{\partial x} - D_5,\ X_6 = \frac{\partial}{\partial y} - D_6,$$

where

$$D_1 = -a\frac{\partial}{\partial a} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e},\ D_2 = -b\frac{\partial}{\partial a} - e\frac{\partial}{\partial c} + (c-f)\frac{\partial}{\partial d} + e\frac{\partial}{\partial f},$$
$$D_3 = -a\frac{\partial}{\partial b} + d\frac{\partial}{\partial c} - (c-f)\frac{\partial}{\partial e} - d\frac{\partial}{\partial f},\ D_4 = -b\frac{\partial}{\partial b} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e},$$
$$D_5 = c\frac{\partial}{\partial a} + e\frac{\partial}{\partial b},\ D_6 = d\frac{\partial}{\partial a} + f\frac{\partial}{\partial b}. (12)$$

Matrix M_1, constructed on coordinate vectors of operators (12), takes the form

$$M_1(0,1) = \begin{pmatrix} -a & 0 & 0 & -d & e & 0 \\ -b & 0 & -e & c-f & 0 & e \\ 0 & -a & d & 0 & f-c & -d \\ 0 & -b & 0 & d & -e & 0 \\ c & e & 0 & 0 & 0 & 0 \\ d & f & 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Remark 3.1. *One can verify that rank of matrix* (13) *is less than 5. Therefore, according to* (7)*, the dimension of* $Aff(2,\mathbb{R})$*-orbit for system* (9) *is less than 5.*

Remark 3.2. *Using* (10)*, one can verify that* $K_2 \equiv 0$ *yields* $Q \equiv 0$*.*

To define a rank of matrix $M_1(0,1)$ it is necessary to construct all its minors of all possible orders. It is done using computer algebra system "Mathematica 5.0". In order to find affine-invariant conditions for rank of matrix $M_1(0,1)$ its minors of each order are considered separately along with invariants and semi-invariants (corresponding coefficients of affine comitants with each degree of variable x) of system (9). As these objects are polynomials depending on coefficients of system (9) and forming an ideal, the corresponding Gröbner bases [7] can be used to obtain linear dependency among them. Namely, the set of minors of each order is divided in subsets with respect to their types. All possible combinations of invariants, semi-invariants and their products of each type are composed. The corresponding Gröbner bases then has been constructed for them with the help of computer algebra system "Bergman" [8]. Analyzing such a bases one can figure out its element representing linear dependency between minors of matrix (13) and affine invariants and semi-invariants, as this element should contain only names of minors, invariants and semi-invariants, not the coefficients of system (9). According to this algorithm all types of minors of matrix (13) have been treated and corresponding Gröbner bases are constructed, therefore, desired dependencies are obtained. This technique is used throughout the proofs of Lemmas 3.1 - 3.4.

Lemma 3.1. *Rank of matrix* $M_1(0,1)$ *is equal to 4 if and only if holds*

$$K_2 Q \not\equiv 0, \tag{14}$$

where K_2 *and* Q *are defined in* (10).

Proof. Let us prove the necessity. Assume the contradiction. Namely, assume that for

$$K_2 Q \equiv 0 \tag{15}$$

even one non-zero minor of 4th order of matrix (13) exists. Equality (15) holds at least for $K_2 \equiv 0$ or $Q \equiv 0$.

Examine $K_2 \equiv 0$. Than, taking into consideration (10) and (11), we obtain the following values for coefficients of system (9)

$$e = d = 0,\ c = f. \tag{16}$$

After substitution of values (16) to matrix (13) one can verify that all 4th order's minors of this matrix are equal to zero. Thus, the assumption is not true in this case.

Examine $Q \equiv 0$. Than, taking into consideration (10) and (11), we obtain the following series of values for coefficients of system (9):

$$e = d = 0,\ c = f, \tag{17}$$

$$a = c = d = 0, \tag{18}$$

$$b = e = f = 0, \tag{19}$$

$$d = f = 0,\ e = \frac{bc}{a},\ a \neq 0, \tag{20}$$

$$a = b = 0,\ d = \frac{fc}{e},\ e \neq 0, \tag{21}$$

$$c = -f,\ d = \frac{af}{b},\ e = -\frac{bf}{a},\ ab \neq 0, \tag{22}$$

$$c = e = 0,\ d = \frac{af}{b},\ b \neq 0, \tag{23}$$

$$c = f,\ d = \frac{af}{b},\ e = \frac{bf}{a},\ ab \neq 0. \tag{24}$$

Case (17) coincides with case (16), obtained for $K_2 \equiv 0$, and will not be considered.

After substitution of each of series (18) - (24) to matrix (13) we obtain that all its 4th order minors are equal to zero. So, the above stated assumption is not true in this case too. Therefore we conclude the necessity of conditions (14).

Sufficiency of conditions (14) is ensured by equality

$$K_2Q = \Delta^{1256}_{1235}x^4 + 2\Delta^{1256}_{1236}x^3y + (2\Delta^{1256}_{1234} - \Delta^{2356}_{1236})x^2y^2 + 2\Delta^{1356}_{1236}xy^3 + \\ +\Delta^{1356}_{1234}y^4 + (\Delta^{1245}_{1236} + 2\Delta^{2345}_{1235})x^3 + (\Delta^{2345}_{1236} - 2\Delta^{1236}_{1235})x^2y + (2\Delta^{2345}_{1234} - \\ -\Delta^{1236}_{1236})xy^2 + (\Delta^{1346}_{1236} - 2\Delta^{1345}_{1234})y^3 + \Delta^{1234}_{1235}x^2 - \Delta^{1234}_{1236}xy - \Delta^{1234}_{1234}y^2,$$

where Δ^{ijhk}_{lmnp} is 4th order minor of matrix (13), constructed on lines i, j, h, k ($1 \le i, j, h, k \le 6$) and columns l, m, n, p ($1 \le l, m, n, p \le 6$). Lemma 3.1 is proved.

Lemma 3.2. *Rank of matrix $M_1(0,1)$ is equal to 3 if and only if hold*

$$Q \equiv 0, \;\; K_2 \not\equiv 0, \tag{25}$$

where K_2 and Q are defined in (10).

Proof. Necessity of conditions (25) follows from Lemma 3.1. Let us prove sufficiency. We will consider each of cases (18) - (24) separately. Note, that case (17) contradicts to conditions of Lemma 3.2.

Denote by Δ^{ijk}_{lmn} a 3rd order minor of matrix (13) constructed on lines i, j, h, ($1 \le i, j, k \le 6$) and columns l, m, n ($1 \le l, m, n \le 6$).

As conditions (18) hold, comitant K_2 takes the form $K_2 = -ex^2 - fxy$. For $K_2 \not\equiv 0$ non-zero 3rd order's minors of matrix (13) will be at least $\Delta^{125}_{145} = -e^3$ or $\Delta^{236}_{245} = f^3$.

As conditions (19) hold, comitant K_2 takes the form $K_2 = cxy + dy^2$. For $K_2 \not\equiv 0$ non-zero 3rd order's minors of matrix (13) will be at least $\Delta^{136}_{134} = -d^3$ or $\Delta^{235}_{145} = c^3$.

As conditions (20) hold, comitant K_2 takes the form $K_2 = c(-\frac{b}{a}x^2 + xy)$. For $K_2 \not\equiv 0$ non-zero 3rd order's minor of matrix (13) will be at least $\Delta^{235}_{145} = c^3$.

As conditions (21) hold, comitant K_2 takes the form $K_2 = -ex^2 + (c-f)xy + \frac{cf}{e}y^2$. Remark, that $e \ne 0$. So, $K_2 \not\equiv 0$ and non-zero 3rd order's minor of matrix (13) will be at least $\Delta^{125}_{145} = -e^3$.

As conditions (22) or (24) hold, comitant K_2 takes the form $K_2 = f(-\frac{b}{a}x^2 - 2xy + \frac{a}{b}y^2)$ or $K_2 = f(-\frac{b}{a}x^2 + \frac{a}{b}y^2)$, correspondingly. In both cases for $K_2 \not\equiv 0$ non-zero 3rd order's minor of matrix (13) will be at least $\Delta_{134}^{125} = f^3$.

As conditions (23) hold, comitant K_2 takes the form $K_2 = f(-xy+\frac{a}{b}y^2)$. For $K_2 \not\equiv 0$ non-zero 3rd order's minor of matrix (13) will be at least $\Delta_{245}^{236} = f^3$.

Sufficiency of conditions (25) is proved completely. Lemma 3.2 is proved.

Lemma 3.3. *Rank of matrix $M_1(0,1)$ is equal to 2 if and only if hold*

$$K_2 \equiv 0, \quad K_{21}^2 + I_1^2 \not\equiv 0, \tag{26}$$

where K_2, K_{21}, I_1 are defined in (10).

Proof. Denote by Δ_{hk}^{ij} a 2nd order minor of matrix (13) constructed on lines i, j $(1 \leq i, j \leq 6)$ and columns h, k $(1 \leq h, k \leq 6)$.

Necessity of equality from (26) follows from Lemmas 3.1 - 3.2 and Remark 3.2. Let us prove necessity of inequality from (26). Assume the contradiction. Namely, assume that for

$$K_{21}^2 + I_1^2 \equiv 0 \tag{27}$$

at least one non-zero 2nd order's minor of matrix (13) exists. For $K_2 \equiv 0$, taking into consideration (10) and (16), invariant I_1 takes the form

$$I_1 = 2f. \tag{28}$$

According to (10) and (11), comitant K_{21} can be written as follows

$$K_{21} = -bx + ay. \tag{29}$$

As $K_2 \equiv 0$ holds, all non-zero 2nd order's minors of matrix (13) will coincide to sign with one of the following

$$\Delta_{12}^{13} = a^2,\ \Delta_{12}^{14} = ab,\ \Delta_{12}^{24} = b^2,\ \Delta_{12}^{16} = af,\ \Delta_{12}^{26} = bf,\ \Delta_{12}^{56} = f^2. \tag{30}$$

As (27) holds, from (28) and (29) follows that $a = b = f = 0$ and all minors (30) are equal to zero. This contradiction confute our assumption and confirms the necessity of inequality from (26).

Sufficiency of conditions (26) is ensured by equality

$$K_{21}^2 + I_1^2 = \Delta_{12}^{24}x^2 - 2\Delta_{12}^{14}xy + \Delta_{12}^{13}y^2 + 4\Delta_{12}^{56}.$$

Lemma 3.3 is proved.

From Lemmas 3.1 - 3.3 evidently follows

Lemma 3.4. *Rank of matrix* $M_1(0,1)$ *is equal to 0 if and only if hold*

$$K_2 \equiv 0, \quad K_{21}^2 + I_1^2 \equiv 0, \tag{31}$$

where K_2, K_{21}, I_1 *are defined in* (10).

From Lemmas 3.1 - 3.4, Remark 3.1 and equality (7) follows

Theorem 3.1. $Aff(2,\mathbb{R})$ *- orbit of system* (9) *has the dimension*

$$\begin{array}{lll} 4 & \textit{for} & QK_2 \not\equiv 0; \qquad (32)\\ 3 & \textit{for} & Q \equiv 0,\ K_2 \not\equiv 0; \qquad (33)\\ 2 & \textit{for} & K_2 \equiv 0,\ K_{21}^2 + I_1^2 \not\equiv 0; \qquad (34)\\ 0 & \textit{for} & K_2 \equiv 0,\ K_{21}^2 + I_1^2 \equiv 0, \qquad (35) \end{array}$$

where K_2, K_{21}, Q, I_1 *are defined in* (10).

According to Definition 2.3 from Theorem 3.1 follows

Theorem 3.2. *Sets* M_1, M_2, M_3, M_4, *defined by expressions* (32), (33), (34) *and* (35) *correspondingly, form* $Aff(2,\mathbb{R})$*-invariant partition of space* $E(a)$ *of coefficients of system* (9), *i.e.*

$$\bigcup_{i=1}^{4} M_i = E(a), \quad M_i \bigcap M_j = \emptyset$$

and each set M_1 $(i = \overline{1,4})$ *is* $Aff(2,\mathbb{R})$*-invariant.*

Remark 3.3. *Set* M_1 *with conditions* (32) *represents non-singular invariant variety of group* $Aff(2,\mathbb{R})$.

Remark 3.4. *Sets* M_2*-*M_4 *with conditions* (33) - (35) *correspondingly represent singular invariant varieties of group* $Aff(2,\mathbb{R})$.

Some results of this paper were announced in a common report with V.Orlov at the Conference "Algebraic systems and their applications in differential equations and other domains of mathematics", see [9].

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