Computer Science Journal of Moldova, vol.**13**, no.2(38), 2005

Nash equilibria sets in mixed extended 2×3 games

Valeriu Ungureanu, Ana Botnari

Abstract

We describe the Nash equilibria set as an intersection of best response graphs. The problem of Nash equilibria set construction for two-person mixed extended 2×3 games is studied.

Mathematics Subject Classification 2000: 91A05, 91A06, 91A10, 91A43, 91A44.

Keywords and phrases: Noncooperative game; Nash equilibrium, Nash equilibria set, best response graph.

1 Introduction

We construct the Nash equilibria set as an intersection of best response graphs [4, 5]. This paper may be considered a continuation of [5] and it has to illustrate the practical opportunity of a mentioned characteristic.

Consider a noncooperative game:

$$\Gamma = \langle N, \{X_i\}_{i \in N}, \{f_i(x)\}_{i \in N} \rangle,$$

where $N = \{1, 2, ..., n\}$ is a set of players, X_i is a set of strategies of player $i \in N$ and $f_i : X \to R$ is a player's $i \in N$ payoff function defined on the Cartesian product $X = \times_{i \in N} X_i$. Elements of X are named outcomes of the game (situations or strategy profiles).

The outcome $x^* \in X$ of the game is the Nash equilibrium [3] (shortly NE) of Γ if

$$f_i(x_i, x^*_{-i}) \leq f_i(x^*_i, x^*_{-i}), \forall x_i \in X_i, \ \forall i \in N,$$

where

$$x^*_{-i} = (x^*_1, x^*_2, ..., x^*_{i-1}, x^*_{i+1}, ..., x^*_n),$$

$$x^*_{-i} \in X_{-i} = X_1 \times X_2 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n,$$

$$(x_i, x^*_{-i}) = (x^*_1, x^*_2, ..., x^*_{i-1}, x_i, x^*_{i+1}, ..., x^*_n) \in X.$$

There are diverse alternative formulations of a Nash equilibrium [1]: as a fixed point of the best response correspondence, as a fixed point of a function, as a solution of a non-linear complementarity problem, as a solution of a stationary point problem, as a minimum of a function on a polytope, as a semi-algebraic set. We study the Nash equilibria set as an intersection of best response graphs [4, 5], i.e. intersection of the sets:

$$Gr_i = \{(x_i, x_{-i}) \in X : x_{-i} \in X_{-i}, x_i \in \text{Arg} \max_{x_i \in X_i} f_i(x_i, x_{-i})\},\ i \in N.$$

From the players views not all Nash equilibria are equally attractive. They may be Pareto ranked. Therefore Nash equilibrium may dominate or it may be dominated. There are also different other criteria for Nash equilibria distinguishing such as perfect equilibria, proper equilibria, sequential equilibria, stable sets etc. Thus the methods that found only a sample of Nash equilibrium don't guarantee that determined Nash equilibrium complies all the players demands and refinement conditions. Evidently, a method for all Nash equilibria determination is useful and required. Other theoretical and practical factors that argue for NE set determination exist [1].

This paper as the continuation of [5] investigates the problems of NE set construction in the games that permit simple graphic illustrations and that elucidate the usefulness of the interpretation of NE as an intersection of best response graphs [4, 5].

2 Main results

Consider a two-person matrix game Γ with matrices:

$$A = (a_{ij}),\ B = (b_{ij}),\ i = \overline{1,2},\ j = \overline{1,3}.$$

The game $\Gamma_m = \langle \{1,2\}; X, Y; f_1, f_2\rangle$ is the mixed extension of Γ, where

$X = \{\mathbf{x} = (x_1, x_2) \in R^2 : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$,

$Y = \{\mathbf{y} = (y_1, y_2, y_3) \in R^3 : y_1 + y_2 + y_3 = 1, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}$,

$f_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} x_i y_j$,

$f_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{2} \sum_{j=1}^{3} b_{ij} x_i y_j$.

This game is reduced to the game on the unit prism. For the reduced game the class partition of the strategy sets is considered and the NE set is determined for each possible "subgame" (see the following propositions).

2.1 Reduction to game on a prism

By substitutions:

$x_1 = x$, $x_2 = 1 - x$, $x \in [0, 1]$,

$y_3 = 1 - y_1 - y_2, \quad y_3 \in [0, 1]$,

the game Γ_m is reduced to the equivalent game:

$$\Gamma'_m = \langle \{1,2\}; [0,1], \triangle; \varphi_1, \varphi_2\rangle,$$

where

$\triangle = \{\mathbf{y} = (y_1, y_2) \in R^2 : y_1 + y_2 \leq 1, y_1 \geq 0, y_2 \geq 0\}$,

$$\begin{aligned}
&\varphi_1(x, \mathbf{y}) = (a_{11}y_1 + a_{12}y_2 + a_{13}(1 - y_1 - y_2))x + \\
&\qquad (a_{21}y_1 + a_{22}y_2 + a_{23}(1 - y_1 - y_2))(1 - x) = \\
&((a_{11} - a_{21} + a_{23} - a_{13})y_1 + (a_{12} - a_{22} + a_{23} - a_{13})y_2 + a_{13} - a_{23})x + \\
&\qquad (a_{21} - a_{23})y_1 + (a_{22} - a_{23})y_2 + a_{23} = \\
&((a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 + (a_{13} - a_{23})(1 - y_1 - y_2))\, x + \\
&\qquad nbb(a_{21} - a_{23})y_1 + (a_{22} - a_{23})y_2 + a_{23};
\end{aligned}$$

$$\begin{aligned}\varphi_2(x,\mathbf{y}) = &(b_{11}y_1 + b_{12}y_2 + b_{13}(1-y_1-y_2))x + \\ &(b_{21}y_1 + b_{22}y_2 + b_{23}(1-y_1-y_2))(1-x) = \\ &((b_{11}-b_{13}+b_{23}-b_{21})x + b_{21} - b_{23})y_1 + \\ &((b_{12}-b_{13}+b_{23}-b_{22})x + b_{22} - b_{23})y_2 + \\ &(b_{13}-b_{23})x + b_{23} = \\ &((b_{11}-b_{13})x + (b_{21}-b_{23})(1-x))y_1 + \\ &((b_{12}-b_{13})x + (b_{22}-b_{23})(1-x))y_2 + \\ &(b_{13}-b_{23})x + b_{23}.\end{aligned}$$

Thus, Γ_m is reduced to the game Γ'_m on the prism $\Pi = [0,1] \times \triangle$. If $NE(\Gamma'_m)$ is known, then it is easy to construct the set $NE(\Gamma_m)$.

Basing on properties of strategies of each player of the initial pure strategies game Γ, diverse classes of games are considered and for every class the sets $NE(\Gamma'_m)$ are determined.

For commodity, we use notation:

$$\triangle_= = \{\mathbf{y} = (y_1, y_2) \in R^2 : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}.$$

2.2 Both players have either equivalent strategies or dominant strategies

Proposition 1. *If all the players have equivalent strategies, then* $NE(\Gamma'_m) = \Pi$.

Proof. From the equivalence of strategies

$$\varphi_1(x,\mathbf{y}) = (a_{21} - a_{23})y_1 + (a_{22} - a_{23})y_2 + a_{23},$$

$$\varphi_2(x,\mathbf{y}) = (b_{13} - b_{23})x + b_{23}.$$

From this the truth of the proposition results. $\square$

Proposition 2. *If all the players have dominant strategies in* Γ, *then:*

$$NE(\Gamma'_m) = \begin{cases} (0,0,0) & \textit{if strategies (2,3) are dominant,} \\ (0,0,1) & \textit{if strategies (2,2) are dominant,} \\ (0,1,0) & \textit{if strategies (2,1) are dominant,} \\ 0 \times \Delta_{=} & \textit{if strategies (2,1 } \sim \textit{ 2) are dominant,} \\ 0 \times [0,1] \times 0 & \textit{if strategies (2,1 } \sim \textit{ 3) are dominant,} \\ 0 \times 0 \times [0,1] & \textit{if strategies (2,2 } \sim \textit{ 3) are dominant,} \\ (1,0,0) & \textit{if strategies (1,3) are dominant,} \\ (1,0,1) & \textit{if strategies (1,2) are dominant,} \\ (1,1,0) & \textit{if strategies (1,1) are dominant,} \\ 1 \times \Delta_{=} & \textit{if strategies (1,1 } \sim \textit{ 2) are dominant,} \\ 1 \times [0,1] \times 0 & \textit{if strategies (1,1 } \sim \textit{ 3) are dominant,} \\ 1 \times 0 \times [0,1] & \textit{if strategies (1,2 } \sim \textit{ 3) are dominant.} \end{cases}$$

Proof. It is easy to observe that

$$\operatorname{Arg} \max_{x \in [0,1]} \varphi_1(x, \mathbf{y}) = \begin{cases} 1 & \text{if the 1-st strategy is dominant in } \Gamma, \\ 0 & \text{if the 2-nd strategy is dominant in } \Gamma, \end{cases}$$

$\forall \mathbf{y} \in \triangle$. Hence,

$$Gr_1 = \begin{cases} 1 \times \triangle & \text{if the 1-st strategy is dominant,} \\ 0 \times \triangle & \text{if the 2-nd strategy is dominant.} \end{cases}$$

For the second player:

$$\operatorname{Arg} \max_{\mathbf{y} \in \triangle} \varphi_2(x, \mathbf{y}) = \begin{cases} (1,0) & \text{if the 1-st strategy is dominant in } \Gamma, \\ (0,1) & \text{if the 2-nd strategy is dominant in } \Gamma, \\ (0,0) & \text{if the 3-rd strategy is dominant in } \Gamma, \\ \Delta_{=} & \text{if strategies } 1 \sim 2 \text{ dominate } 3, \\ [0,1] \times 0 & \text{if strategies } 1 \sim 3 \text{ dominate } 2, \\ 0 \times [0,1] & \text{if strategies } 2 \sim 3 \text{ dominate } 1, \end{cases}$$

$\forall \mathbf{x} \in [0,1]$. Hence,

$$Gr_2 = \begin{cases} [0,1]\times(1,0) & \text{if the 1-st strategy is dominant,} \\ [0,1]\times(0,1) & \text{if the 2-nd strategy is dominant,} \\ [0,1]\times(0,0) & \text{if the 3-rd strategy is dominant.} \\ [0,1]\times\Delta_{=} & \text{if strategies } 1 \sim 2 \text{ dominate 3,} \\ [0,1]\times[0,1]\times 0 & \text{if strategies } 1 \sim 3 \text{ dominate 2,} \\ [0,1]\times 0\times[0,1] & \text{if strategies } 2 \sim 3 \text{ dominate 1.} \end{cases}$$

Thus, the NE set contains either only one vertex of a unit prism $\triangle$ as an intersection of one facet Gr_1 with one edge Gr_2 or only one edge of a unit prism $\triangle$ as an intersection of one facet Gr_1 with one edge Gr_2. □

2.3 One player has dominant strategy

Proposition 3A. *If the 1-st strategy of the first player is dominant, then*

$$NE(\Gamma'_m) = \begin{cases} (1,1,0) & \textit{if } b_{11} > \max\{b_{12}, b_{13}\}, \\ (1,0,1) & \textit{if } b_{12} > \max\{b_{11}, b_{13}\}, \\ (1,0,0) & \textit{if } b_{12} > \max\{b_{11}, b_{13}\}, \\ 1 \times \triangle_{=} & \textit{if } b_{11} = b_{12} > b_{13}, \\ 1 \times [0,1] \times 0 & \textit{if } b_{11} = b_{13} > b_{12}, \\ 1 \times 0 \times [0,1] & \textit{if } b_{12} = b_{13} > b_{11}, \\ 1 \times \triangle & \textit{if } b_{12} = b_{13} = b_{11}, \end{cases}$$

if the 2-nd strategy of the first player is dominant, then

$$NE(\Gamma'_m) = \begin{cases} (0,1,0) & \textit{if } b_{11} > \max\{b_{12}, b_{13}\}, \\ (0,0,1) & \textit{if } b_{12} > \max\{b_{11}, b_{13}\}, \\ (0,0,0) & \textit{if } b_{12} > \max\{b_{11}, b_{13}\}, \\ 0 \times \triangle_{=} & \textit{if } b_{11} = b_{12} > b_{13}, \\ 0 \times [0,1] \times 0 & \textit{if } b_{11} = b_{13} > b_{12}, \\ 0 \times 0 \times [0,1] & \textit{if } b_{12} = b_{13} > b_{11}, \\ 0 \times \triangle & \textit{if } b_{12} = b_{13} = b_{11}. \end{cases}$$

Proof. If the first player has dominant strategy, then

$$Gr_1 = \begin{cases} 1 \times \triangle & \text{if the 1-st strategy is dominant,} \\ 0 \times \triangle & \text{if the 2-nd strategy is dominant} \end{cases}$$

is one triangle facet of the prism.

If the 1-st strategy of the first player is dominant, then

$$\varphi_2(1, \mathbf{y}) = (b_{11} - b_{13})y_1 + (b_{12} - b_{13})y_2 + b_{13} =$$

$$= b_{11}y_1 + b_{12}y_2 + b_{13}(1 - y_1 - y_2).$$

From this we obtain that

$$\operatorname{Arg}\max_{\mathbf{y}\in\triangle} \varphi_2(1, \mathbf{y}) = \begin{cases} (1,0) & \text{if } b_{11} > \max\{b_{12}, b_{13}\}, \\ (0,1) & \text{if } b_{12} > \max\{b_{11}, b_{13}\}, \\ (0,0) & \text{if } b_{12} > \max\{b_{11}, b_{13}\}, \\ \triangle_{=} & \text{if } b_{11} = b_{12} > b_{13}, \\ [0,1]\times 0 & \text{if } b_{11} = b_{13} > b_{12}, \\ 0 \times [0,1] & \text{if } b_{12} = b_{13} > b_{11}, \\ \triangle & \text{if } b_{12} = b_{13} = b_{11} \end{cases}$$

and

$$Gr_2 = 1 \times \operatorname{Arg}\max_{\mathbf{y}\in\triangle} \varphi_2(1, \mathbf{y})$$

is a vertex, edge or triangle facet of the prism Π. Hence, the truth of the first part of proposition follows.

Analogically the proposition can be proved when the second strategy is dominant. $\square$

Proposition 3B. *If the second player has only one dominant strat-*

egy, then

$$NE(\Gamma'_m) = \begin{cases} (0,1,0) & \textit{if } (\cdot,1) \textit{ is dominant and } a_{11} < a_{21}, \\ (1,1,0) & \textit{if } (\cdot,1) \textit{ is dominant and } a_{11} > a_{21}, \\ [0,1]\times 1 \times 0 & \textit{if } (\cdot,1) \textit{ is dominant and } a_{11} = a_{21}, \\ (0,0,1) & \textit{if } (\cdot,2) \textit{ is dominant and } a_{12} < a_{22}, \\ (1,0,1) & \textit{if } (\cdot,2) \textit{ is dominant and } a_{12} > a_{22}, \\ [0,1]\times 0 \times 1 & \textit{if } (\cdot,2) \textit{ is dominant and } a_{12} = a_{22}, \\ (0,0,0) & \textit{if } (\cdot,3) \textit{ is dominant and } a_{13} < a_{23}, \\ (1,0,0) & \textit{if } (\cdot,3) \textit{ is dominant and } a_{13} > a_{23}, \\ [0,1]\times 0 \times 0 & \textit{if } (\cdot,3) \textit{ is dominant and } a_{13} = a_{23}. \end{cases}$$

Proof. If the 3-rd strategy of the second player is dominant, then

$$\operatorname{Arg}\max_{\mathbf{y}\in\triangle} \varphi_2(x,\mathbf{y}) = \operatorname{Arg}\max_{\mathbf{y}\in\triangle}((b_{11}-b_{13})x + (b_{21}-b_{23})(1-x))y_1 +$$

$$((b_{12}-b_{13})x + (b_{22}-b_{23})(1-x))y_2 + (b_{13}-b_{23})x + b_{23} = (0,0),$$

and

$$Gr_2 = [0,1] \times (0,0)$$

is an edge of a prism Π.

For the first player, we obtain $\varphi_1(x,\mathbf{0}) = a_{13}x + a_{23}(1-x)$ and

$$Gr_1 = \begin{cases} (1,0,0) & \text{if } a_{13} > a_{13}, \\ (0,0,0) & \text{if } a_{13} < a_{13}, \\ [0,1]\times 0 \times 0 & \text{if } a_{13} = a_{13}. \end{cases}$$

Consequently, NE set is a vertex or edge of the prism Π.

Similarly, the remained part of the proposition can be proved in the other two subcases. $\square$

Proposition 3C. *If the second player has two dominant strategies, then*

$$NE(\Gamma'_m) = Gr_1 \cap Gr_2,$$

where:

$$Gr_1 = 0 \times Y_{12}^{<} \cup 1 \times Y_{12}^{>} \cup [0,1] \times Y_{12}^{=}$$

$$Gr_2 = [0,1] \times \Delta_=$$

if the 1-st and 2-nd strategies are equivalent and they dominate the 3-rd strategy;

$$Gr_1 = 0 \times Y_1^< \cup 1 \times Y_1^> \cup [0,1] \times Y_1^=$$

$$Gr_2 = [0,1] \times [0,1] \times 0$$

if the 1-st and 3-rd strategies are equivalent and they dominate the 2-nd strategy;

$$Gr_1 = 0 \times Y_2^< \cup 1 \times Y_2^> \cup [0,1] \times Y_2^=$$

$$Gr_2 = [0,1] \times 0 \times [0,1]$$

if the 2-nd and 3-rd strategies are equivalent and they dominate the 1-st strategy;

$$\begin{aligned}
Y_{12}^< &= \{\mathbf{y} \in \triangle_= : (a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 < 0\},\\
Y_{12}^> &= \{\mathbf{y} \in \triangle_= : (a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 > 0\},\\
Y_{12}^= &= \{\mathbf{y} \in \triangle_= : (a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 = 0\},\\
Y_1^< &= \{\mathbf{y} \in R^2 : (a_{11} - a_{21} + a_{23} - a_{13})y_1 + a_{13} - a_{23} < 0,\\
&\qquad y_1 \in [0,1], y_2 = 0\},\\
Y_1^> &= \{\mathbf{y} \in R^2 : (a_{11} - a_{21} + a_{23} - a_{13})y_1 + a_{13} - a_{23} > 0,\\
&\qquad y_1 \in [0,1], y_2 = 0\},\\
Y_1^= &= \{\mathbf{y} \in R^2 : (a_{11} - a_{21} + a_{23} - a_{13})y_1 + a_{13} - a_{23} = 0,\\
&\qquad y_1 \in [0,1], y_2 = 0\},\\
Y_2^< &= \{\mathbf{y} \in R^2 : (a_{12} - a_{22} + a_{23} - a_{13})y_2 + a_{13} - a_{23} < 0,\\
&\qquad y_2 \in [0,1], y_1 = 0\},\\
Y_2^> &= \{\mathbf{y} \in R^2 : (a_{12} - a_{22} + a_{23} - a_{13})y_2 + a_{13} - a_{23} > 0,\\
&\qquad y_2 \in [0,1], y_1 = 0\},\\
Y_2^= &= \{\mathbf{y} \in R^2 : (a_{12} - a_{22} + a_{23} - a_{13})y_2 + a_{13} - a_{23} = 0,\\
&\qquad y_2 \in [0,1], y_1 = 0\}.
\end{aligned}$$

Proof. If the 1-st and 2-nd strategies of the second player are equivalent and they dominate the third strategy, then

$$\operatorname{Arg}\max_{\mathbf{y}\in\triangle} \varphi_2(x,\mathbf{y}) = \operatorname{Arg}\max_{\mathbf{y}\in\triangle}((b_{11} - b_{13})x + (b_{21} - b_{23})(1-x))(y_1 + y_2) +$$

$$+(b_{13} - b_{23})x + b_{23} = \Delta_{=}$$

and

$$Gr_2 = [0, 1] \times \Delta_{=}$$

is a facet of a prism Π.

For the first player, we obtain

$$\begin{aligned}\varphi_1(x, \mathbf{y}) = &((a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2)\, x + \\ &+ (a_{21} - a_{23})y_1 + (a_{22} - a_{23})y_2 + a_{23};\end{aligned}$$

and

$$Gr_1 = 0 \times Y_{12}^{<} \cup 1 \times Y_{12}^{>} \cup [0, 1] \times Y_{12}^{=}$$

where:

$$\begin{aligned}Y_{12}^{<} &= \{\mathbf{y} \in \triangle_{=} : (a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 < 0\},\\ Y_{12}^{>} &= \{\mathbf{y} \in \triangle_{=} : (a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 > 0\},\\ Y_{12}^{=} &= \{\mathbf{y} \in \triangle_{=} : (a_{11} - a_{21})y_1 + (a_{12} - a_{22})y_2 = 0\}.\end{aligned}$$

Similarly, the remained part of the proposition can be proved in the other two subcases. □

Evidently, propositions 3A, 3B and 3C elucidate the case when one player has dominant strategy (strategies) and the other player has equivalent strategies.

2.4 One player has equivalent strategies

Proposition 4A. *If the first player has equivalent strategies, then*

$$NE(\Gamma'_m) = Gr_2,$$

$$\begin{aligned}Gr_2 = &X_1 \times (1, 0) \cup X_2 \times (0, 1) \cup X_3 \times (0, 0) \cup \\ &X_{12} \times \triangle_{=} \cup X_{13} \times [0, 1] \times 0 \cup X_{23} \times 0 \times [0, 1] \cup \\ &X_{123} \times \triangle,\end{aligned}$$

where:

$$X_1 = \left\{ x \in [0, 1] : \begin{array}{l} (b_{11} - b_{21})x + b_{21} > (b_{12} - b_{22})x + b_{22} \\ (b_{11} - b_{21})x + b_{21} > (b_{13} - b_{23})x + b_{23} \end{array} \right\},$$

$$X_2 = \left\{ x \in [0,1] : \begin{array}{l} (b_{12} - b_{22})x + b_{22} > (b_{11} - b_{21})x + b_{21} \\ (b_{12} - b_{22})x + b_{22} > (b_{13} - b_{23})x + b_{23} \end{array} \right\},$$

$$X_3 = \left\{ x \in [0,1] : \begin{array}{l} (b_{13} - b_{23})x + b_{23} > (b_{11} - b_{21})x + b_{21} \\ (b_{13} - b_{23})x + b_{23} > (b_{12} - b_{22})x + b_{22} \end{array} \right\},$$

$$X_{12} = \left\{ x \in [0,1] : \begin{array}{l} (b_{11} - b_{21})x + b_{21} = (b_{12} - b_{22})x + b_{22} \\ (b_{11} - b_{21})x + b_{21} > (b_{13} - b_{23})x + b_{23} \end{array} \right\},$$

$$X_{13} = \left\{ x \in [0,1] : \begin{array}{l} (b_{11} - b_{21})x + b_{21} > (b_{12} - b_{22})x + b_{22} \\ (b_{11} - b_{21})x + b_{21} = (b_{13} - b_{23})x + b_{23} \end{array} \right\},$$

$$X_{23} = \left\{ x \in [0,1] : \begin{array}{l} (b_{12} - b_{22})x + b_{22} > (b_{11} - b_{21})x + b_{21} \\ (b_{12} - b_{22})x + b_{22} = (b_{13} - b_{23})x + b_{23} \end{array} \right\},$$

$$X_{123} = \left\{ x \in [0,1] : \begin{array}{l} (b_{11} - b_{21})x + b_{21} = (b_{12} - b_{22})x + b_{22} \\ (b_{11} - b_{21})x + b_{21} = (b_{13} - b_{23})x + b_{23} \end{array} \right\}.$$

Proof. If the strategies of the first player are equivalent, then $Gr_1 = \Pi$.

Suppose that $x \in [0,1]$ is fixed. The payoff function of the second player can be represented in the form

$$\begin{aligned} \varphi_2(x, \mathbf{y}) = & ((b_{11} - b_{21})x + b_{21})y_1 + ((b_{12} - b_{22})x + b_{22})y_2 + \\ & ((b_{13} - b_{23})x + b_{23})(1 - y_1 - y_2). \end{aligned}$$

It's evident that for:

$x \in X_1$ the minimum of the cost function is realized on $(1,0) \in \triangle$,

$x \in X_2$ the minimum is realized on $(0,1) \in \triangle$,

$x \in X_3$ the minimum is realized on $(0,0) \in \triangle$,

$x \in X_{12}$ the minimum is realized on $\triangle_=$,

$x \in X_{13}$ the minimum is realized on $[0,1] \times 0 \in \triangle$,

$x \in X_{23}$ the minimum is realized on $0 \times [0,1] \in \triangle$,

$x \in X_{123}$ the minimum is realized on $\triangle$.

From the above the truth of the proposition follows. □

Proposition 4B. *If all three strategies of the second player are equivalent, then*

$$NE(\Gamma'_m) = Gr_1 = 1 \times Y_1 \cup 0 \times Y_2 \cup [0,1] \times Y_{12},$$

where

$$\begin{aligned}
&Y_1 = \{\mathbf{y} \in \triangle : \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 > 0\},\\
&Y_2 = \{\mathbf{y} \in \triangle : \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 < 0\},\\
&Y_{12} = \{\mathbf{y} \in \triangle : \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 = 0\},\\
&\alpha_1 = a_{11} - a_{13} + a_{23} - a_{21},\\
&\alpha_2 = a_{12} - a_{13} + a_{23} - a_{22},\\
&\alpha_3 = a_{13} - a_{23}.
\end{aligned}$$

Proof. If all three strategies of the second player are equivalent, then $b_{11} = b_{12} = b_{13}$, $b_{21} = b_{22} = b_{23}$ and $Gr_2 = \Pi$.

The cost function of the first player can be represented in the following form

$$\begin{aligned}
\varphi_1(x, \mathbf{y}) = {} & ((a_{11} - a_{13})y_1 + (a_{12} - a_{13})y_2 + a_{13})x + \\
& ((a_{21} - a_{23})y_1 + (a_{22} - a_{23})y_2 + a_{23})(1 - x).
\end{aligned}$$

It's evident that for

$$\begin{aligned}
\mathbf{y} \in Y_1 = \{\mathbf{y} \in \triangle : {} & a_{11}y_1 + a_{12}y_2 + a_{13}(1 - y_1 - y_2) > \\
& > a_{21}y_1 + a_{22}y_2 + a_{23}(1 - y_1 - y_2)\ \} = \\
= \{\mathbf{y} \in \triangle : {} & (a_{11} - a_{13} + a_{23} - a_{21})y_1 + \\
& + (a_{12} - a_{13} + a_{23} - a_{22})y_2 + a_{13} - a_{23} > 0\}
\end{aligned}$$

the 1-st strategy of the first player is optimal, for

$$\begin{aligned}
\mathbf{y} \in Y_2 = \{\mathbf{y} \in \triangle : {} & a_{11}y_1 + a_{12}y_2 + a_{13}(1 - y_1 - y_2) < \\
& < a_{21}y_1 + a_{22}y_2 + a_{23}(1 - y_1 - y_2)\ \} = \\
= \{\mathbf{y} \in \triangle : {} & (a_{11} - a_{13} + a_{23} - a_{21})y_1 + \\
& + (a_{12} - a_{13} + a_{23} - a_{22})y_2 + a_{13} - a_{23} < 0\}
\end{aligned}$$

the 2-nd strategy of the first player is optimal, and for

$$\begin{aligned}\mathbf{y} \in Y_{12} = \{\mathbf{y} \in \triangle : a_{11}y_1 + a_{12}y_2 + a_{13}(1 - y_1 - y_2) = \\ = a_{21}y_1 + a_{22}y_2 + a_{23}(1 - y_1 - y_2)\ \} = \\ = \{\mathbf{y} \in \triangle : (a_{11} - a_{13} + a_{23} - a_{21})y_1 + \\ + (a_{12} - a_{13} + a_{23} - a_{22})y_2 + a_{13} - a_{23} = 0\}\end{aligned}$$

every strategy $x \in [0, 1]$ of the first player is optimal. From this, the truth of the proposition follows. □

2.5 The players don't have dominant strategies

Proposition 5. *If the both players don't have dominant strategies, then*

$$NE(\Gamma'_m) = Gr_1 \cap Gr_2,$$

where Gr_1, Gr_2 are defined as in propositions 4A, 4B.

The truth of the proposition follows from the above.

2.6 Algorithm

From the above a simple solving procedure follows. In this procedure only one step from **1°** to **5°** is executed.

0° The game Γ'_m is considered (see subsection 2.1);

1° If the both players have equivalent strategies in Γ, then the NE set in Γ_m is $X \times Y$ (see the proposition 1);

2° If the both players have dominant strategies in Γ, then the NE set in Γ_m is constructed in compliance with proposition 2 and substitutions of subsection 2.1;

3A° If only the first player has dominant strategy in Γ, then the NE set in Γ_m is constructed in conformity with proposition 3A and substitutions of subsection 2.1;

3B° If only the second player has only one dominant strategy in Γ, then the NE set in Γ_m is constructed in conformity with proposition 3B and substitutions of subsection 2.1;

3C° If the second player has two dominant strategies that dominate the other strategy in Γ, then the NE set in Γ_m is constructed in conformity with proposition 3B and substitutions of subsection 2.1;

4A° If only the first player has equivalent strategies in Γ, then the NE set in Γ_m is constructed in accordance with proposition 4A and substitutions of subsection 2.1;

4B° If only the second player has equivalent strategies in Γ, then the NE set in Γ_m is constructed in accordance with proposition 4B and substitutions of subsection 2.1;

5° If the both players don't have dominant strategies in Γ, then the NE set in Γ_m is constructed in compliance with proposition 5 and substitutions of subsection 2.1.

References

[1] McKelvey R.D., McLenan A., *Computation of equilibria in finite games,* Handbook of Computational Economics, 1, Amsterdam, Elsevier, 1996, pp. 87-142.

[2] Moulen H., *Théorie des jeux pour l'éqonomie et la politique,* Paris, 1981 (in Russian: Games Theory, Moscow, Mir, 1985, 200 p.).

[3] Nash J.F., *Noncooperative game,* Annals of Mathematics, 54, 1951, pp. 280-295.

[4] Sagaidac M., Ungureanu V., *Operational research,* Chişinău, CEP USM, 2004, 296 p. (in Romanian).

[5] Ungureanu V., Botnari A., *Nash equilibria sets in mixed extension of* $2 \times 2 \times 2$ *games*, Computer Science Journal of Moldova, 1, 2005, pp. 1-15.

V. Ungureanu, A. Botnari,

Received June 23, 2005

State University of Moldova,
60, A. Mateevici str.,
Chişinău, MD−2009, Moldova.
E–mail: *valungureanu@usm.md*