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On a k-clique-join of a class of partitionable graphs

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Abstract

We call a graph G *O-graph* if there is an optimal coloring of the set of vertices and an optimal (disjoint) covering with cliques such that any class of colors intersects any clique. In this paper, it has been established the relation to $[p, q, r]$-*partite graphs* and the fact that the O-graphs admit a k-*clique-join*.

Key Words: perfect graphs, (α,ω)-partitionable graphs, [p,q,r]-partite graphs, k-clique-join.

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1 Introduction.

Throughout this paper $G = (V, E)$ is a simple (i.e. finite, undirected, without loops and multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$, with $\alpha = \alpha(G) \geq 2$ and $\omega = \omega(G) \geq 2$. $\overline{G}$ designates the complement of G. If $e = xy \in E$, we shall also write $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. If $A \subseteq V$, then $G[A]$ (or $[A]$, or $[A]_G$) is the subgraph of G induced by $A \subseteq V$. By $G - W$ we mean the graph $(V, E - W)$, whenever $W \subseteq E$. For $A, B \subset V$, $A \cap B = \emptyset$, the set $\{ab|a \in A, b \in B, ab \in E\}$ will be denoted by (A, B), and we write $A \sim B$ whenever $ab \in E$ holds for any $a \in A$ and $b \in B$.

By P_n, C_n and K_n we mean a chordless path on $n \geq 3$ vertices, the chordless cycle on $n \geq 3$ vertices, and the complete graph on $n \geq 1$ vertices. A *hole* is a chordless cycle of length at least four; an *antihole* is the complement of such a cycle. A *Berge* graph is a graph which contains no odd hole and no odd antihole.

A *stable* set in G is a set of mutually non-adjacent vertices, and the *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set.

By $S(G)$ we shall denote the family of all maximal stable sets of G, and $S_\alpha(G) = \{S|S \in S(G), |S| = \alpha(G)\}$. A *clique* in G is a subset A of $V(G)$ that induces a complete subgraph in G, and $C(G) = S(\overline{G})$, $\omega(G) = \alpha(\overline{G})$, while $C_\omega(G) = S_\alpha(\overline{G})$. Clearly, $S_\alpha(G) \subseteq S(G)$ and $C_\omega(G) \subseteq C(G)$ are true for any graph G.

The *chromatic number* and the *clique covering number* of G (i.e. the chromatic number of $\overline{G}$) will be denoted respectively, by $\chi(G)$ and $\theta(G)$. The *density* of G is the size of a largest clique in G, i.e., $\omega(G)=\alpha(\overline{G})$.

A graph G is *perfect* if $\alpha(H) = \theta(H)$ (or, equivalently, $\chi(H) = \omega(H)$) holds for any induced subgraph H of G.

Definition. *A graph G is called (α,ω)-partitionable (see Golumbic, [6], Olaru, [8]), if for any $v \in V(G)$, G-v admits a partition of α ω-cliques and a partition of ω α-stable sets.*

Properties referring to the (α,ω)-partitionable graphs can be found in (Chvatal, Graham, Perold, Whitesides, [4], also see Golumbic, [6], Olaru, [8]) and are given by:

Theorem. *Let G be a graph with n vertices, and $\alpha = \alpha(G)$ and $\omega = \omega(G)$. If G is (α,ω)-partitionable then the following statements hold:*

(i) n= $\alpha\omega + 1$;

(ii) G has exactly n ω-cliques and n α-stable sets;

(iii) Each vertex of G belongs to exactly α ω-cliques and to exactly ω α-stable sets;

(iv) Each ω-clique intersects exactly n-1 α-stable sets and is disjoint from exactly one and each α-stable set intersects exactly n-1 ω-cliques and is disjoint from exactly one.

From the previous Definition and Theorem a question is asked, what properties do the graphs that admit an optimal coloring and covering with cliques have, such that any clique intersects any class of colors. We call this type of graphs, O-graphs (to be seen [10]). We hope that this class of graphs makes a step towards a well-characterization of the graphs that admit a k-clique-join ([12]).

On the web page from [12] there are open problems concerning Perfect Graphs.

Perfect graphs have proved to be one of the most stimulating and fruitful concepts of modern graph theory: there are three books ([6], [2], [9]) and nearly six hundred papers ([5]) on the subject. The origin of this development was the Strong Perfect Graph Conjecture ([1]):

a graph is perfect if and only if neither it nor its complement contains a chordless cycle whose length is odd and at least five.

There are theorems that elucidate the structure of objects in some class C by showing that every object in C has either a prescribed and relatively transparent structure or one of prescribed structural faults, along which it can be decomposed. M. Conforti, G. Cornuejols and K. Vuskovic proved that

every square-free Berge
(meaning Berge graph containing no hole of length four)
either belongs to one of two basic classes
(bipartite graphs and line-graphs of bipartite graphs),
or else it has one of two structural faults (star-cutset or 2-join).

Therefore every square-free Berge graph is perfect.

In 2002, M. Chudnovsky and P. Seymour, as well as, N. Robertson and R. Thomas announced that they had completed the proof of the Strong Perfect Graphs Conjecture. Their structural theorem asserts that

every Berge graph either belongs to one of five basic classes
(namely: bipartite graphs, their complements, line-graphs of
bipartite graphs, and their complements, double split graphs)
or else it has one of four structural faults
(namely: 2-join, 2-join in the complement, M-join,
a balanced skew partition).

Therefore every Berge graph is perfect (namely the Strong Perfect Graph Conjecture became, in May 2002, the Strong Perfect Graph

Theorem by Maria Chudnovsky, Neil Robertson, Paul Seymour, Robin Thomas ([3])).

2 The Results.

In the beginning we show that any O-graph admits a partition of the set of vertices in α ω-cliques and one in ω α-stable sets.

Definition 1. *A graph G is called O-graph if there is an optimal coloring $(S_1, ..., S_p)$ of vertices and optimal covering with cliques $(Q_1, ..., Q_r)$ such that any class of colors intersects any clique (i.e. $S_i \cap Q_j \neq \emptyset$, $(1 \leq i \leq p,\ 1 \leq j \leq r)$).*

We specify that in an optimal covering $(Q_1, ..., Q_r)$ with cliques of an O-graph, $S_i \cap Q_j \neq \emptyset$ $(1 \leq i < j \leq n)$.

If G is O-graph, we denote with $Q(G)$ and, respectively, $I(G)$ the covering set with cliques, respectively colorings of G with the property that any covering from $Q(G)$ and any coloring from $I(G)$ satisfies the condition from Definition 1.

We remark that G is O-graph if and only if $\overline{G}$ is O-graph and *any even cycle is O-graph* and *any even chain is O-graph.*

Lemma 1. *If G=(V,E) is an O-graph with n vertices, then for any p-coloration from I(G) and any r-covering from Q(G) the following statements hold:*

1) $p = \omega(G)(= \omega)$; $r = \alpha(G)(= \alpha)$ and

2) $n = \alpha\omega$.

Proof. We denote with $S = (S_1, ..., S_p)$ and $C = (Q_1, ..., Q_r)$ a p-coloration from $I(G)$ and respectively a r-covering from $Q(G)$. We prove that $|S_i| = \alpha$ and $|Q_j| = \omega$, $\forall i = 1, ..., p, \forall j = 1, ..., r$. Let S_i be fixed. Because $|S_i \cap Q_j| = 1$, $\forall j = 1, .., r$, we have $|S_i| \geq r$, that means that $\alpha \geq |S_i| \geq r$. Because $(Q_1, ..., Q_r)$ is a covering with cliques, it results that $|S_i| \leq r$. So we have $|S_i| = r$. We have, for any stable set S, $|S| \leq r$. If $|S| = \alpha$, then, in particularly $\alpha \leq r$. Because $\alpha \geq r$, we obtain $r = \alpha$. As conclusion, we have $|S_i| = \alpha$, $\forall i = 1, ..., p$ and we prove $|Q_j| = \omega$, $\forall j = 1, ..., r$ the same way. Because S is a partition of $V(G)$, we obtain $n = |V(G)| = \sum_{i=1}^{\omega} |S_i| = \alpha\omega$.

Corollary 1. *A graph G is O-graph if and only if there is a partition of the set of vertices in ω stable sets with α elements and a partition in α cliques with ω elements.*

Proof. The direct statement results from Lemma 1, and a graph with the property from the Corollary is obviously an O-graph.

Corollary 2. *If G is O-graph then:*

for any clique Q from an optimal covering with cliques of G:
$\alpha(G-Q) = \alpha(G) - 1$;

for any stable set S from an optimal coloring of G:
$\omega(G-S) = \omega(G) - 1$.

Corollary 3. *For any O-graph G, any class of colors from any optimal coloring intersects any clique from any optimal covering with cliques.*

Proof. We suppose that there exists a clique Q_j and a class of colors S_i, disjoint. Then $|Q_j| < \omega$ or $|S_i| < \alpha$, but $|V(G)| = \sum_{j=1}^{\alpha} |Q_j| = \sum_{i=1}^{\omega} |S_i|$, so $|V(G)| < \alpha\omega$, contradicting Lemma 1.

Remark 1. *If G is a (α,ω)-partitionable graph of order n then for any vertex v, G-v is O-graph.*

Proof. G being (α,ω)-partitionable, results that, for any $v \in V(G)$, G-v admits a partition of α ω-cliques and a partition of ω α-stable sets. If there is a clique Q_j from an optimal covering with cliques of G-v disjoint of a class S_j of colors from an optimal coloring of G-v then $|Q_j| < \omega$ or $|S_i| < \alpha$, but $|V(G)| - 1 = \sum_{j=1}^{\alpha} |Q_j| = \sum_{i=1}^{\omega} |S_i|$, so $|V(G)| - 1 < \alpha\omega$, contradicting that G is (α,ω)-partitionable.

Next, it is established a theorem of characterization of O-graphs, it is given an example of non-perfection of an O-graph and is shown in which condition an O-graph is perfect. For this it is given the definition of $[p, q, r]$-partite graphs.

Definition 2. *An [p,q,r]-partite graph ([11]) is a graph whose set of vertices, V, is partitioned in p independent sets $S_1, ..., S_p$, each containing exactly q vertices, and $S_i \cup S_j$ contains exactly r independent edges, for $1 \le i < j \le p$.*

Theorem 1. *Let G=(V,E) be a graph with $\alpha = \alpha(G)$ and $\omega = \omega(G)$. G is O-graph if and only if G is $[\omega, \alpha, \alpha]$- partite and $\overline{G}$ is $[\alpha, \omega, \omega]$-partite.*

Proof. Let G be an O-graph and $(S_1, ..., S_\omega)$ a partition of G in ω α-stable sets, and $(Q_1, ..., Q_\alpha)$ a partition in α ω-cliques with $S_i \cap Q_k \neq \emptyset, (1 \leq i \leq \omega, 1 \leq k \leq \alpha)$. We must show that $\forall i, j, i = 1, ..., \omega, j = 1, ..., \omega$ with $i \neq j$ $S_i \cup S_j$ admits a maximum matching with α elements. We denote with $\{x_k^i\} = S_i \cap Q_k$, $(1 \leq i \leq \omega, 1 \leq k \leq \alpha)$. For $1 \leq k, l \leq \alpha$, $k \neq l$ we have $x_k^i \neq x_l^i$ $(1 \leq i \leq \omega)$ because $Q_k \cap Q_l = \emptyset$. Therefore $S_i = \{x_1^i, ..., x_\alpha^i\}$ $(1 \leq i \leq \omega)$. For $1 \leq k \leq \alpha$, we have $x_k^i x_k^j \in E(G)$, because $\{x_k^i, x_k^j\} \subseteq Q_k$, $\forall i, j, i = 1, ..., \omega, j = 1, ..., \omega$ with $i \neq j$; so the set of edges $\{x_k^i x_k^j | k = 1, ..., \alpha\}$ is a matching in $[S_i \cup S_j]$ $\forall i, j, i = 1, ..., \omega, j = 1, ..., \omega$ with $i \neq j$. Because $\overline{G}$ is O-graph with $\alpha(\overline{G}) = \omega$, $\omega(\overline{G}) = \alpha$, it results that $\overline{G}$ is $[\alpha, \omega, \omega]$-partite graph. Let G be $[\omega, \alpha, \alpha]$ -partite graph and $\overline{G}$ $[\alpha, \omega, \omega]$-partite graph. It results that there is a partition of V in $S = (S_1, ..., S_\omega)$ α -stable sets and in $C = (Q_1, ..., Q_\alpha)$ with Q_i cliques and $|Q_i| = \omega$, that means G is O-graph.

An example of O-graph which is not perfect is the reunion of a four disjoint four-cliques, adding four edges such that form an induced C_5.

Corollary 4. *A graph G is perfect O-graph if and only if G is* $[\omega, \alpha, \alpha]$*-partite,* $\{C_{2l+1}, \overline{C}_{2l+1}\}$*-free* $(l \geq 2)$ *and* $\overline{G}$ *is* $[\alpha, \omega, \omega]$*-partite graph.*

Proof. We suppose that G is $[\omega, \alpha, \alpha]$-partite and $\overline{G}$ is $[\alpha, \omega, \omega]$-partite graph. From Theorem 1 it results that G is O-graph. We suppose, on the contrary, that a minimal contraexample is minimal imperfect. Then G is unbreakable (Chvatal, to be seen [7]). So any vertex x of G is in a disk ([7]) (i.e. $x \in C_k$ or $x \in \overline{C}_k$, $k \geq 5$). Because G is $\{C_{2l+1}, \overline{C}_{2l+1}\}$-free $(l \geq 2)$, it results that $G \cong C_{2p}$ or $G \cong \overline{C}_{2p} (p \geq 3)$, a contradiction. So G is perfect O-graph. If G is O-graph it results that G is $[\omega, \alpha, \alpha]$-partite and $\overline{G}$ is $[\alpha, \omega, \omega]$-partite. If G contains $\{C_{2p+1}, (p \geq 2)$ as an induced subgraph then G is not perfect.

Next, it is shown that an O-graph admits a (k, α)-clique-join and the way in which an O-graph is obtained (built).

In [12] it is asked to be found a characterization for a k-clique-join.

Definition 3. *Let k be positive integer. A k-clique-join ([12]) of a graph G=(V,E) is a set of pairs* $\{(A_0, B_0), (A_1, B_1), ..., (A_k, B_k)\}$, *where* $\{A_0, B_0\}$ *is a partition of V, both* A_0 *and* B_0 *contain at least one* ω*-clique, and* $A_i \subseteq A_0, B_i \subseteq B_0 (i = 1, ..., k)$ *(not necessarily disjoint), moreover*

(i) If $x \in A_i$ *and* $y \in B_i$*, then* $xy \in E$

(ii) If K is an ω*-clique of G that intersects both* A_0 *and* B_0 *then there exists i so that* $K \subseteq A_i \cup B_i$

Definition 4. *Let k, s be positive integers. A k-clique-join* $\{(A_0, B_0), (A_1, B_1), ..., (A_k, B_k)\}$ *of a graph G=(V,E) is called (k,s)-clique-join if* $A_{i_1} \cup B_{i_1}$,...,$A_{i_{s-2}} \cup B_{i_{s-2}}$,$A_0 - \cup_{l=1}^{s-2} A_{i_l}$, $B_0 - \cup_{l=1}^{s-2} B_{i_l}$ *are s disjoint* ω*-cliques, for all* $i_1, ..., i_{s-2}$ *of the set* $\{1, ..., k\}$.

Theorem 2. *Let G be a graph with* $\alpha = \alpha(G)$, $\omega = \omega(G)$ *and* $\alpha \geq 2$. *G is O-graph if and only if there are k positive integers such that G admits a* (k, α)*-clique-join.*

Proof. Let $G = (V, E)$ be an O-graph and $(S_1, ..., S_\omega)$ a partition in ω α-stable sets of G and $(Q_1, ..., Q_\alpha)$ a disjoint covering with α ω-cliques of G with $S_i \cap Q_j \neq \phi$ $(1 \leq i \leq \omega,\ 1 \leq j \leq \alpha)$. Next, we define a (k, α)-clique-join $\{(A_0, B_0), (A_1, B_1), ..., (A_k, B_k)\}$. Because $\alpha \geq 2$, there are at least two disjoint ω-cliques. Without restricting the generality, we take $Q_1 \subseteq A_0, Q_2 \subseteq B_0, Q_1, Q_2$, the two ω-cliques. More, we consider $A_0 = Q_1 \cup \cup_{i=3}^{\alpha} M_i$, $B_0 = Q_2 \cup \cup_{i=3}^{\alpha} (Q_i - M_i)$, where $M_i \subset Q_i (M_i \neq Q_i)(3 \leq i \leq \alpha)$. We denoted $A_l = M_{l+2}, B_l = Q_{l+2} - M_{l+2} (1 \leq l \leq \alpha - 2)$ and we obtained:

$\{A_0, B_0\}$ is a partition of V, A_0 (B_0) contains at least a ω-clique Q_1 (Q_2);

If $x \in A_i$ and $y \in B_i (1 \leq i \leq \alpha - 2)$, then $xy \in E$ (because A_i is totally adjacent to B_i $(A_i \sim B_i)$);

If $K = Q_i$ for some i $(1 \leq i \leq \alpha)$ then (ii) is hold from Definition 3. Because a disjoint reunion of α ω-cliques is O-graph, it results that $k \geq \alpha - 2$. $\{(A_0, B_0), (A_1, B_1), ..., (A_k, B_k)\}$ is a k-clique-join for G with $k = \alpha - 2$. If K' is an ω-clique of G that intersects both A_0 and B_0, but $K' = Q_i$, $\forall i = 1, ..., \alpha$, then we denote $A_s = K' \cap A_0$, $B_s = K' \cap B_0$ for some s. We have $K' = A_s \cup B_s$ ((ii) holds from Definition 3), and if $x \in A_s$ and $y \in B_s$ then $xy \in E$ ((i) holds from Definition 3). For each such ω-clique K', we add the pair (A_s, B_s) to the previous k-clique-join and add one to k.

So $\{(A_0, B_0), (A_1, B_1), ..., (A_k, B_k)\}$ is a (k, α)-clique -join of G.

Reverse, we suppose that $\{(A_0, B_0), (A_1, B_1), ..., (A_k, B_k)\}$ is a (k, α)-clique -join of G and we show that G is a $[\omega, \alpha, \alpha]$-partite graph and $\overline{G}$ is a $[\alpha, \omega, \omega]$-partite graph. Without restricting the generality, we consider $A_1 \cup B_1, ..., A_{\alpha-2} \cup B_{\alpha-2}, A_0 - \cup_{i=1}^{\alpha-2} A_i, B_0 - \cup_{i=1}^{\alpha-2} B_i$ the α disjoint ω-clique, and let $Q_{l+2} = A_l \cup B_l$ be for $1 \leq l \leq \alpha - 2$, $Q_1 = A_0 - \cup_{i=1}^{\alpha-2} A_i$, $Q_2 = B_0 - \cup_{i=1}^{\alpha-2} B_i$. We denote $Q_j = \{x_j^1, ..., x_j^\omega\}(1 \leq j \leq \alpha)$. Because $Q_j(1 \leq j \leq \alpha)$ is a ω-clique, we have: $\forall x \in V - Q_j, \exists y \in Q_j$ so that $xy \notin E$. For $1 \leq j, t \leq \alpha, j \neq t$, we suppose $x_j^i x_t^i \notin E$. We denote $S_i = \{x_1^i, ..., x_\alpha^i\}(1 \leq i \leq \omega)$. Because $x_j^i \in S_i \cap Q_j, x_t^i \in S_i \cap Q_t(1 \leq j, t \leq \alpha, j \neq t)$, it results that S_i is α-stable. Because $E([S_i \cup S_s]) = \{x_j^i x_j^s | 1 \leq j \leq \alpha\}$, it results that $[S_i \cup S_s]$ contains α independent edges $(1 \leq i, s \leq \omega, i \neq s)$. $\overline{G}$ is a $[\alpha, \omega, \omega]$-partite graph, because Q_j $(1 \leq j \leq \alpha)$ is a stable set in $\overline{G}$ with ω elements, S_i $(1 \leq i \leq \omega)$ is a clique in $\overline{G}$ with α elements and $E([Q_j \cup Q_t]_{\overline{G}}) = \{x_j^i x_t^i | 1 \leq i \leq \omega\}$ $(1 \leq j, t \leq \alpha, j \neq t)$.

Proposition 1. *Let G=(V,E) be a simple graph with $\alpha = \alpha(G)$ and $\omega = \omega(G)$. G is O-graph if and only if it can be obtained from a disjoint reunion of α ω-cliques adding an edge between each two vertices non contained in any class of a partition of V in ω α-stable sets and in any class of a partition of V in α ω-cliques.*

Proof. Let $G = (V, E)$ be a simple graph and $S = \{S_1, ..., S_\omega\}$ $(C = \{Q_1, ..., Q_\alpha\})$ a partition of V in ω α-stable sets (α ω-cliques). We denote $S_i = \{x_1^i, ..., x_\alpha^i\}(1 \leq i \leq \omega)$ and $Q_j = \{x_j^1, ..., x_j^\omega\}(1 \leq j \leq \alpha)$. Clearly, the graph $H = [\cup_{i=1}^{\alpha} Q_j]_G$ is O-graph with $\alpha(H) = \alpha$ and $\omega(H) = \omega$. Let H' be the graph obtained from H to which an edge $e = xy$ $(x = x_k^i,\ y = x_q^p)$ of G is added. Because x and y belong to two distinct α-stable sets of S, it results that $i \neq p$. If $k = q$ then the edge e would be added to a ω-clique Q_k, contradicting the fact that

G is a simple graph. The graph H' has $\alpha(H') = \alpha$, $\omega(H') = \omega$ and the same S partition (respectively C) in ω α-stable sets (respectively α ω-cliques) as H. So H' is O-graph. Repeating the above procedure of adding edges from G, we obtain the graph G and the fact that G is an O-graph.

Reverse, if we suppose that $G = (V, E)$ is O-graph with $\alpha = \alpha(G)$, $\omega = \omega(G)$ and we consider the disjoint covering with α ω -cliques $C = \{Q_1, ..., Q_\alpha\}$ and the covering with ω α-stable sets $S = \{S_1, ..., S_\omega\}$ and we apply the procedure:

begin

H:=G;
while ($\exists e = xy \in E$ with $\{x, y\} \not\subset Q_j$ $(1 \leq j \leq \alpha)$) *do*
H:=H-e;
end;

we obtain that $[\cup_{j=1}^{\alpha} Q_j]_G$ is an O-graph.

Indeed, the partition $C = \{Q_1, ..., Q_\alpha\}$ with α ω -cliques (and the covering with ω α-stable sets $S = \{S_1, ..., S_\omega\}$) of G, by deleting edges e=xy with $\{x, y\} \not\subset Q_j$ $(1 \leq j \leq \alpha)$ (according to the above procedure) remains the same also for $H = [\cup_{j=1}^{\alpha} Q_j]_G$, that means that H is an O-graph.

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