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The minimum cost multicommodity flow problem in dynamic networks and an algorithm for its solving

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Abstract

The dynamic version of the minimum cost multicommodity flow problem that generalizes the static minimum cost multicommodity flow problem is formulated and studied. This dynamic problem is considered on directed networks with a set of commodities, time-varying capacities, fixed transit times on arcs, and a given time horizon. We assume that cost functions, defined on edges, are nonlinear and depend on time and flow and the demand function also depends on time. The corresponding algorithm, based on reducing the dynamic problem to a static problem on a time-expanded network, to solve the minimum cost dynamic multicommodity flow problem is proposed and some details concerning its complexity are discussed.

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1 Introduction

Multicommodity flows are among the most important and challenging problems in network optimization, due to the large size of these models in real world applications. Many product distribution, scheduling planning, telecommunication, transportation, communication, and management problems can be formulated and solved as multicommodity flow problems (see, for example, [1]). The multicommodity flow problem consists of shipping several different commodities from their respective sources to their sinks through a given network so that the total flow going through each edge does not exceed its capacity. No commodity ever transforms into another commodity, so that each one has its own flow conservation constraints, but they compete for the resources of the common network. Considered multicommodity network flow problem requires to find the minimum cost flow of a set of commodities through a network, where the arcs have an individual capacity for each commodity, and a mutual capacity for all the commodities.

While there is substantial literature on the static multicommodity flow problem, hardly any results on multicommodity dynamic flows are known, although the dynamic multicommodity flows are much more closer to reality than the static ones. In considered dynamic models the flow requires a certain amount of time to travel through each arc, it can be delayed at nodes, flow values on arcs and the network parameters can change with time. Dynamic flows are widely used to model different network-structured, decision-making problems over time (see, for example, [2, 3]), but because of their complexity, dynamic flow models have not been investigated as well as classical flow models.

In this paper we study the dynamic version of the minimum cost multicommodity flow problem on networks with time-varying capacities of edges. We assume that cost functions, defined on edges, are nonlinear and depend on time and flow and the demand function also depends on time. The minimum cost multicommodity dynamic flow problem asks for a feasible flow over time with given time horizon, satisfying all supplies and demands with minimum cost. We propose an algorithm for solving this problem, which is based on reducing the dynamic problem to the classical static problem on a time-expanded network.

2 Problem formulation

We consider a directed network $N = (V, E, K, w, u, \tau, d, \varphi)$ with set of vertices V, set of edges E and set of commodities K that must be routed through the same network. Each edge $e \in E$ has a nonnegative time-varying capacity $w_e^k(t)$ which bounds the amount of flow of each commodity $k \in K$ allowed on each arc $e \in E$ in every moment of time $t \in \mathbb{T}$. We also consider that every arc $e \in E$ has a nonnegative time-varying capacity for all commodities, which is known as the mutual capacity $u_e(t)$. Moreover, each edge $e \in E$ has an associated positive transit time τ_e which determines the amount of time it takes for flow to travel from the tail to the head of that edge. The underlying network also consists of demand function d: $V \times K \times \mathbb{T} \to R$ and cost function φ: $E \times R_+ \times K \times \mathbb{T} \to R_+$, where $\mathbb{T} = \{0, 1, 2, \ldots, T\}$.

The demand function $d_v^k(t)$ satisfies the following conditions:

a) there exists $v \in V$ for every $k \in K$ with $d_v^k(0) < 0$;

b) if $d_v^k(t) < 0$ for a node $v \in V$ for commodity $k \in K$ then $d_v^k(t) = 0$, $t = 1, 2, \ldots, T$;

In order for the flow to exist we require that $\sum\limits_{t \in \mathbb{T}} \sum\limits_{v \in V} d_v^k(t) = 0, \forall k \in$ $\in K$. Nodes $v \in V$ with $\sum\limits_{t \in \mathbb{T}} d_v^k(t) < 0,\ k \in K$ are called sources for commodity k, nodes $v \in V$ with $\sum\limits_{t \in \mathbb{T}} d_v^k(t) > 0,\ k \in K$ are called sinks for commodity k and nodes $v \in V$ with $\sum\limits_{t \in \mathbb{T}} d_v^k(t) = 0,\ k \in K$ are called intermediate for commodity k. We denote by V_-^k, V_+^k and V_0^k the set of sources, sinks and intermediate nodes for commodity k, respectively. The sources are nodes through which flow enters the network and the sinks are nodes through which flow leaves the network. The sources and sinks are sometimes called terminal nodes, while the intermediate nodes are called non-terminals.

To model transit costs, which may change over time, we define the cost function $\varphi_e^k(x_e^k(t), t)$ with the meaning that flow of commodity k of value $\xi = x_e^k(t)$ entering edge e at time t will incur a transit cost of $\varphi_e^k(\xi, t)$. We consider the discrete time model, in which all times are integral and bounded by horizon T. Time is measured in discrete steps, so that if one unit of flow leaves node u at time t on arc $e = (u, v)$, then one unit of flow arrives at node v at time $t + \tau_e$, where τ_e is the transit time of arc e. The time horizon (finite or infinite) is the time until which the flow can travel in the network and defines the makespan $\mathbb{T} = \{0, 1, \ldots, T\}$ of time moments we consider.

We start with the definition of static multicommodity flows. A static multicommodity flow x on $N = (V, E, K, w, u, d, \varphi)$ assigns to every arc $e \in E$ for each commodity $k \in K$ a non-negative flow value x_e^k such that the following flow conservation constraints are obeyed:

$$\sum_{e \in E^+(v)} x_e^k - \sum_{e \in E^-(v)} x_e^k = d_v^k, \ \forall\, v \in V, \ \forall\, k \in K,$$

where $E^+(v) = \{(u, v) \,|\, (u, v) \in E\}, \quad E^-(v) = \{(v, u) \,|\, (v, u) \in E\}$.

The multicommodity flow x satisfies the demands if one-commodity flow x^k, $\forall k \in K$ satisfies the demands d_v^k for all $v \in V$.

Multicommodity flow x is called feasible if it obeys the mutual capacity constraints:

$$\sum_{k \in K} x_e^k \le u_e, \ \forall\, e \in E \tag{1}$$

and individual capacities of every arc for each commodity:

$$0 \le x_e^k \le w_e^k, \ \forall\, e \in E, \ \forall k \in K. \tag{2}$$

Constraints (1) and (2) are called weak and strong forcing constraints, respectively.

The total cost of the static multicommodity flow x is defined as follows:

$$c(x) = \sum_{k \in K} \sum_{e \in E} \varphi_e^k(x_e^k).$$

A feasible dynamic flow on $N = (V, E, K, w, u, \tau, d, \varphi)$ is a function $x\colon E \times K \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e \in E^+(v) \\ t - \tau_e \ge 0}} x_e^k(t - \tau_e) - \sum_{e \in E^-(v)} x_e^k(t) = d_v^k(t), \ \forall\, t \in \mathbb{T}, \ \forall\, v \in V, \ \forall k \in K; \tag{3}$$

$$\sum_{k \in K} x_e^k(t) \le u_e(t), \ \forall\, t \in \mathbb{T}, \ \forall e \in E; \tag{4}$$

$$0 \le x_e^k(t) \le w_e^k(t), \ \ \forall\, t \in \mathbb{T}, \ \forall\, e \in E, \ \forall k \in K; \tag{5}$$

$$x_e^k(t) = 0, \ \forall\, e \in E, \ t = \overline{T - \tau_e + 1, T}, \ \forall k \in K. \tag{6}$$

Here the function x defines the value $x_e^k(t)$ of flow of commodity k entering edge e at time t. It is easy to observe that the flow does not enter edge e at time t if it will have to leave the edge after time T; this is ensured by condition (6). Capacity constraints (5) mean that in a feasible dynamic flow, at most $w_e^k(t)$ units of flow of commodity k can enter the arc e at time t. Mutual capacity constraints (4) mean that in a feasible dynamic flow, at most $u_e(t)$ units of flow can enter the arc e at time t. Conditions (3) represent flow conservation constraints.

The total cost of the dynamic multicommodity flow x is defined as follows:

$$c(x) = \sum_{t=0}^{T} \sum_{k \in K} \sum_{e \in E} \varphi_e^k(x_e^k(t), t). \tag{7}$$

The minimum-cost multicommodity dynamic flow problem is to find a feasible flow that minimizes the objective function (7).

It is easy to observe that if $\tau_e = 0, \forall\, e \in E$ and $T = 0$ then the formulated problem becomes the static minimum cost multicommodity flow problem.

3 The main results

In this paper we propose an approach for solving the formulated problem, which is based on its reduction to a static flow problem. We show that the minimum cost multicommodity flow problem on dynamic network N can be reduced to the minimum cost static flow problem on auxiliary static network N^T; we name it the time-expanded network. In such a way, a dynamic flow problem in a given network with transit times on the arcs can be transformed into an equivalent static flow problem in the corresponding time-expanded network. A discrete dynamic flow in the given network can be interpreted as a static flow in the corresponding time-expanded network. The advantage of this approach is that it turns the problem of determining an optimal flow over time into a classical static network flow problem in the time-expanded network.

The time-expanded network is a static representation of the dynamic network. Such a time-expanded network contains copies of the node set of the underlying network for each discrete interval of time, building a time layer. Copies of an arc of the considered network join copies of its end-nodes in time layers whose distances equal the transit time of that arc. We define this network as follows:

1. $V^T \colon = \{v(t) \,|\, v \in V,\ t \in \mathbb{T}\}$;
2. $E^T \colon = \{e(t) = (v(t), w(t+\tau_e)) \,|\, e = (v,w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $u^T_{e(t)} \colon = u_e(t)$ for $e(t) \in E^T$;
4. ${w^k_{e(t)}}^T \colon = w^k_e(t)$ for $e(t) \in E^T,\ k \in K$.
5. ${\varphi^k_{e(t)}}^T({x^k_{e(t)}}^T) \colon = \varphi^k_e(x_e(t), t)$ for $e(t) \in E^T,\ k \in K$;
6. ${d^k_{v(t)}}^T \colon = d^k_v(t)$ for $v(t) \in V^T,\ k \in K$.

The essence of the time-expanded network is that it contains a copy of the vertices of the dynamic network for each time $t \in \mathbb{T}$, and the transit times and flows are implicit in the edges linking those copies.

Let $e(t) = (v(t), w(t + \tau_e)) \in E^T$ and let $x^k_e(t)$ be a flow of commodity $k \in K$ on the dynamic network N. The corresponding function on the time-expanded network N^T is defined as follows:

$${x^k_{e(t)}}^T = x^k_e(t),\ \forall k \in K.$$

Using the method from [4, 5] it can be proved that the set of feasible flows on the dynamic network N corresponds to the set of feasible flows on the time-expanded network N^T and that any dynamic flow corresponds to a static flow in the time-expanded network of equal cost, and vice versa. In such a way, for each minimum-cost flow in the dynamic network there is a corresponding minimum-cost flow in the static network and vice-versa.

Therefore, the minimum cost multicommodity flow problem on dynamic networks can be solved by static flow computations in the corresponding time-expanded network. If the cost function of dynamic network is linear with regard to flow, then the cost function of the time-expanded network will be linear. In this case we can apply well-known methods for minimum cost flow problems, including linear programming algorithms, combinatorial algorithms, as well as other developments, like [6]. If there is exactly one source and the cost function of the dynamic network is concave with regard to flow, then the cost function of the time-expanded network will be concave. If the cost function of dynamic network is convex with regard to flow, then the cost function of the time-expanded network will be convex. In this case we can apply methods from convex programming and the specialization of such methods for minimum cost flow problems.

4 The algorithm

Let the dynamic network N be given. Our object is to solve the minimum cost multicommodity flow problem on N. Proceedings are following:

1. Building the time-expanded network N^T for the given dynamic network N.
2. Solving the classical minimum cost multicommodity flow problem on the static network N^T, using one of the known algorithms (see, for example, [7, 8, 9, 10, 11]).
3. Reconstructing the solution of the static problem on N^T to the dynamic problem on N. $\square$

The complexity of this algorithm depends on the complexity of the algorithm used for the minimum cost multicommodity flow problem in static networks.

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