Computer Science Journal of Moldova, vol.**13**, no.1(37), 2005

Nash equilibria sets in mixed extension of $2 \times 2 \times 2$ games

Valeriu Ungureanu, Ana Botnari

Abstract

We describe the Nash equilibria set as an intersection of graphs of players' best responses. The problem of Nash equilibria set construction for three-person extended $2 \times 2 \times 2$ games is studied.

Mathematics Subject Classification 2000: 90C05, 90C10, 90C29, 90C31.

Keywords and phrases: Noncooperative game; Nash equilibrium, Nash equilibria set, graph of best responses.

1 Introduction and preliminary results

The problem of the Nash equilibria set construction is rarely encountered in literature. There are diverse explanations of this fact. The main reason is the complexity of this problem [1].

We consider a noncooperative game:

$$\Gamma = \langle N, \{X_i\}_{i \in N}, \{f_i(x)\}_{i \in N} \rangle,$$

where $N = \{1, 2, ..., n\}$ is a set of players, X_i is a set of strategies of player $i \in N$ and $f_i : X \to R$ is a player's $i \in N$ payoff function defined on the Cartesian product $X = \times_{i \in N} X_i$. Elements of X are named outcomes of the game (situations or strategy profiles).

The outcome $x^* \in X$ of the game is the Nash equilibrium [3] (shortly NE) of Γ if

$$f_i(x_i, x^*_{-i}) \le f_i(x^*_i, x^*_{-i}), \forall x_i \in X_i, \ \forall i \in N,$$

where

$$x^*_{-i} = (x^*_1, x^*_2, ..., x^*_{i-1}, x^*_{i+1}, ..., x^*_n),$$

$$x^*_{-i} \in X_{-i} = X_1 \times X_2 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n,$$

$$(x_i, x^*_{-i}) = (x^*_1, x^*_2, ..., x^*_{i-1}, x_i, x^*_{i+1}, ..., x^*_n) \in X.$$

There are diverse alternative formulations of a Nash equilibrium [1] as:

- a fixed point of the best response correspondence;
- a fixed point of a function;
- a solution of a non-linear complementarity problem;
- a solution of a stationary point problem;
- a minimum of a function on a polytope;
- a semi-algebraic set.

We study the Nash equilibria set as an intersection of graphs of players' best responses [4], i.e. intersection of the sets:

$$Gr_i = \{(x_i, x_{-i}) \in X : x_{-i} \in X_{-i}, x_i \in \operatorname{Arg} \max_{x_i \in X_i} f_i(x_i, x_{-i})\},\ i \in N.$$

Theorem 1. *The outcome $x^* \in X$ is a Nash equilibrium if and only if $x^* \in \bigcap\limits_{i \in N} Gr_i$.*

The proof follows from the definition of the Nash equilibrium.

Corollary. $NE(\Gamma) = \bigcap\limits_{i \in N} Gr_i$.

If all strategy sets $X_i, i \in N$, are finite, then a mixed extension of Γ is

$$\Gamma_m = \langle M_i, f^*_i(\mu), i \in N \rangle,$$

where

$$f_i^*(\mu) = \sum_{x \in X} f_i(x)\mu_1(x_1)\mu_2(x_2)\dots\mu_n(x_n),$$

$$\mu = (\mu_1, \mu_2, ..., \mu_n) \in M = \times_{i \in N} M_i,$$

M_i is a set of mixed strategies of the player $i \in N$.

Theorem 2. *If X is a finite set, then the set $NE(\Gamma_m)$ is a nonempty compact subset of the M. Moreover, it contains the set $NE(\Gamma)$:*

$$NE(\Gamma) \subset NE(\Gamma_m) \neq \emptyset.$$

One of the simplest solvable problems of the NE set determination is the similar problem in the mixed extension of two-person 2×2 game [1, 2, 4]. In this paper the class partition of all three-person $2 \times 2 \times 2$ games is considered and the NE set is determined for mixed extension of the games of each class.

2 Main results

Consider a three-person matrix game Γ with matrices:

$$A = (a_{ijk}),\ B = (b_{ijk}),\ C = (c_{ijk}),\ \ i = \overline{1,2},\ j = \overline{1,2},\ k = \overline{1,2}.$$

The game $\Gamma_m = \langle \{1,2,3\}; X, Y, Z;\ f_1, f_2, f_3 \rangle$ is the mixed extension of Γ, where

$$X = \{\mathbf{x} = (x_1, x_2) \in R^2 : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\},$$

$$Y = \{\mathbf{y} = (y_1, y_2) \in R^2 : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\},$$

$$Z = \{\mathbf{z} = (z_1, z_2) \in R^2 : z_1 + z_2 = 1, z_1 \geq 0, z_2 \geq 0\},$$

$$f_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \textstyle\sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 a_{ijk} x_i y_j z_k,$$

$$f_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \textstyle\sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 b_{ijk} x_i y_j z_k,$$

$$f_3(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \textstyle\sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 c_{ijk} x_i y_j z_k.$$

By substitutions:

$$x_1 = x,\ x_2 = 1 - x,\ x \in [0,1];$$

$$y_1 = y,\ y_2 = 1 - y,\ y \in [0,1];$$

$$z_1 = z,\ z_2 = 1 - z,\ z \in [0,1],$$

the game Γ_m is reduced to the equivalent normal form game:

$$\Gamma'_m = \langle \{1,2,3\}; [0,1], [0,1], [0,1]; \varphi_1, \varphi_2, \varphi_3 \rangle.$$

where

$$\begin{aligned}
&\varphi_1(x,y,z) = \\
&((a_{111} - a_{211})yz + (a_{112} - a_{212})y(1-z) + (a_{121} - a_{221})(1-y)z + \\
&(a_{122} - a_{222})(1-y)(1-z))x + \\
&((a_{211} - a_{221})z + (a_{212} - a_{222})(1-z))y + (a_{221} - a_{222})z + a_{222};
\end{aligned}$$

$$\begin{aligned}
&\varphi_2(x,y,z) = \\
&((b_{111} - b_{121})xz + (b_{112} - b_{122})x(1-z) + (b_{211} - b_{221})(1-x)z + \\
&(b_{212} - b_{222})(1-x)(1-z))y + \\
&((b_{121} - b_{221})z + (b_{122} - b_{222})(1-z))x + (b_{221} - b_{222})z + b_{222};
\end{aligned}$$

$$\begin{aligned}
&\varphi_3(x,y,z) = \\
&((c_{111} - c_{112})xy + (c_{121} - c_{122})x(1-y) + (c_{211} - c_{212})(1-x)y + \\
&(c_{221} - c_{222})(1-x)(1-y))z + \\
&((c_{112} - c_{212})y + (c_{122} - c_{222})(1-y))x + (c_{212} - c_{222})y + c_{222}.
\end{aligned}$$

Thus, Γ_m is reduced to the game Γ'_m on the unit cube.

If $NE(\Gamma'_m)$ is known, then it is easy to construct the set $NE(\Gamma_m)$.

Basing on properties of strategies of each player of the initial pure strategies game Γ, diverse classes of games are considered and for every class the set of $NE(\Gamma'_m)$ is determined.

Proposition 1. *If all players have equivalent strategies, then*

$$NE(\Gamma'_m) = [0,1]^3.$$

Remark. *In the case, considered in proposition 1, players have the following linear payoff functions:*

$\varphi_1(x,y,z) = ((a_{211} - a_{221})z + (a_{212} - a_{222})(1-z))y + (a_{221} - a_{222})z + a_{222},$

$\varphi_2(x,y,z) = ((b_{121} - b_{221})z + (b_{122} - b_{222})(1-z))x + (b_{221} - b_{222})z + b_{222},$

$\varphi_3(x,y,z) = ((c_{112} - c_{212})y + (c_{122} - c_{222})(1-y))x + (c_{212} - c_{222})y + c_{222}.$

Every player doesn't influence on his payoff function, but his strategy is essential for payoff values of the rest of the players.

Proposition 2. *If all the players have dominant strategies in* Γ*, then* $NE(\Gamma'_m)$ *contains only one point:*

$$NE(\Gamma'_m) = \begin{cases} (0,0,0) & \textit{if strategies (2,2,2) are dominant;} \\ (0,0,1) & \textit{if strategies (2,2,1) are dominant;} \\ (0,1,0) & \textit{if strategies (2,1,2) are dominant;} \\ (0,1,1) & \textit{if strategies (2,1,1) are dominant;} \\ (1,0,0) & \textit{if strategies (1,2,2) are dominant;} \\ (1,0,1) & \textit{if strategies (1,2,1) are dominant;} \\ (1,1,0) & \textit{if strategies (1,1,2) are dominant;} \\ (1,1,1) & \textit{if strategies (1,1,1) are dominant.} \end{cases}$$

Proof. It is easy to observe that graphs coincide with facets of unite cube.

For first player:

$$\operatorname{Arg}\max_{x \in [0,1]} \varphi_1(x,y,z) = \begin{cases} \{1\} & \text{if the 1-st strategy is dominant in } \Gamma, \\ \{0\} & \text{if the 2-nd strategy is dominant in } \Gamma, \end{cases}$$

$\forall (y,z) \in [0,1]^2$. Hence,

$$Gr_1 = \begin{cases} 1 \times [0,1] \times [0,1] & \text{if the 1-st strategy is dominant,} \\ 0 \times [0,1] \times [0,1] & \text{if the 2-nd strategy is dominant.} \end{cases}$$

For second player:

$$\operatorname{Arg}\max_{y\in[0,1]} \varphi_2(x,y,z) = \begin{cases} \{1\} & \text{if the 1-st strategy is dominant in } \Gamma, \\ \{0\} & \text{if the 2-nd strategy is dominant in } \Gamma, \end{cases}$$

$\forall (x,z) \in [0,1]^2$. So,

$$Gr_2 = \begin{cases} [0,1]\times 1 \times [0,1] & \text{if the 1-st strategy is dominant,} \\ [0,1]\times 0 \times [0,1] & \text{if the 2-nd strategy is dominant.} \end{cases}$$

For third player:

$$\operatorname{Arg}\max_{z\in[0,1]} \varphi_3(x,y,z) = \begin{cases} \{1\} & \text{if the 1-st strategy is dominant in } \Gamma, \\ \{0\} & \text{if the 2-nd strategy is dominant in } \Gamma, \end{cases}$$

$\forall (x,y) \in [0,1]^2$. Hence,

$$Gr_3 = \begin{cases} [0,1]\times[0,1] \times 1 & \text{if the 1-st strategy is dominant,} \\ [0,1]\times[0,1] \times 0 & \text{if the 2-nd strategy is dominant.} \end{cases}$$

Consequently, the NE set contains only one vertex of unit cube. □

Proposition 3. *If the first and the second players have dominant strategies and the third player has incomparable strategies, then:*

$$NE(\Gamma'_m) = \begin{cases} (1,1,0) & \textit{if (1,1,}\cdot\textit{) are dominant and } c_{111} < c_{112}, \\ (1,1,1) & \textit{if (1,1,}\cdot\textit{) are dominant and } c_{111} > c_{112}, \\ 1\times 1\times[0,1] & \textit{if (1,1,}\cdot\textit{) are dominant and } c_{111} = c_{112}, \\ (0,0,0) & \textit{if (2,2,}\cdot\textit{) are dominant and } c_{221} < c_{222}, \\ (0,0,1) & \textit{if (2,2,}\cdot\textit{) are dominant and } c_{221} > c_{222}, \\ 0\times 0\times[0,1] & \textit{if (2,2,}\cdot\textit{) are dominant and } c_{221} = c_{222}, \\ (1,0,0) & \textit{if (1,2,}\cdot\textit{) are dominant and } c_{121} < c_{122}, \\ (1,0,1) & \textit{if (1,2,}\cdot\textit{) are dominant and } c_{121} > c_{122}, \\ 1\times 0\times[0,1] & \textit{if (1,2,}\cdot\textit{) are dominant and } c_{121} = c_{122}, \\ (0,1,0) & \textit{if (2,1,}\cdot\textit{) are dominant and } c_{211} < c_{212}, \\ (0,1,1) & \textit{if (2,1,}\cdot\textit{) are dominant and } c_{211} > c_{212}, \\ 0\times 1\times[0,1] & \textit{if (2,1,}\cdot\textit{) are dominant and } c_{211} = c_{212}. \end{cases}$$

Similarly the NE set can be constructed in two other possible cases:

- *players 1 and 3 have dominant strategies, and player 2 has incomparable strategies;*

- *players 2 and 3 have dominant strategies, and player 1 has incomparable strategies.*

So, the NE set is either one vertex of a unit cube or one edge of this cube.

Proposition 4. *If the first and the second players have dominant strategies and the third one has equivalent strategies, then*

$$NE(\Gamma'_m) = \begin{cases} 1 \times 1 \times [0,1] & \textit{if (1,1,}\cdot\textit{) are dominant,} \\ 0 \times 0 \times [0,1] & \textit{if (2,2,}\cdot\textit{) are dominant,} \\ 1 \times 0 \times [0,1] & \textit{if (1,2,}\cdot\textit{) are dominant,} \\ 0 \times 1 \times [0,1] & \textit{if (2,1,}\cdot\textit{) are dominant.} \end{cases}$$

Similarly the NE set can be constructed in the following cases:

- *players 1 and 3 have dominant strategies, and player 2 has equivalent strategies;*

- *players 2 and 3 have dominant strategies, and player 1 has equivalent strategies.*

Thus, the NE set is an edge of unit cube.

Proposition 5. *If the first and the second players have equivalent strategies, and the third player has dominant strategy, then*

$$NE(\Gamma'_m) = \begin{cases} [0,1] \times [0,1] \times 1 & \textit{if the 1-st strategy is dominant,} \\ [0,1] \times [0,1] \times 0 & \textit{if the 2-nd strategy is dominant.} \end{cases}$$

Similarly the NE set can be constructed in the following cases:

- *players 1 and 3 have equivalent strategies, and player 2 has dominant strategy;*

- *players 2 and 3 have equivalent strategies, and player 1 has dominant strategy.*

In such a way, the NE set is a facet of a unit cube.

Proposition 6. *If the first player has equivalent strategies, the second player has dominant strategy and the third player has incomparable strategies, then*

$$NE(\Gamma'_m) = Gr_3,$$

where

$$Gr_3 = [0,1]^3 \cap \begin{cases} \begin{cases} [0; -\frac{\gamma_2}{\gamma_1}) \times 1 \times 0 \cup \\ -\frac{\gamma_2}{\gamma_1} \times 1 \times [0,1] \cup \\ (-\frac{\gamma_2}{\gamma_1}; 1] \times 1 \times 1 \end{cases} & \textit{if } \gamma_1 > 0, \\ \begin{cases} [0; -\frac{\gamma_2}{\gamma_1}) \times 1 \times 1 \cup \\ -\frac{\gamma_2}{\gamma_1} \times 1 \times [0,1] \cup \\ (-\frac{\gamma_2}{\gamma_1}; 1] \times 1 \times 0 \end{cases} & \textit{if } \gamma_1 < 0, \\ [0,1] \times 1 \times 0 & \textit{if } \gamma_1 = 0,\ \gamma_2 < 0, \\ [0,1] \times 1 \times 1 & \textit{if } \gamma_1 = 0,\ \gamma_2 > 0, \\ [0,1] \times 1 \times [0,1] & \textit{if } \gamma_1 = \gamma_2 = 0, \end{cases}$$

$\gamma_1 = c_{111} - c_{112} - c_{211} + c_{212}$, $\gamma_2 = c_{211} - c_{212}$, $\gamma_3 = c_{112} - c_{212}$, $\gamma_4 = c_{212}$

if the 1-st strategy of the second player is dominant,
and

$$Gr_3 = [0,1]^3 \cap \begin{cases} \begin{cases} [0; -\frac{\gamma_6}{\gamma_5}) \times 0 \times 0 \cup \\ -\frac{\gamma_6}{\gamma_5} \times 0 \times [0,1] \cup \\ (-\frac{\gamma_6}{\gamma_5}; 1] \times 0 \times 1 \end{cases} & \textit{if } \gamma_5 > 0, \\ \begin{cases} [0; -\frac{\gamma_6}{\gamma_5}) \times 0 \times 1 \cup \\ -\frac{\gamma_6}{\gamma_5} \times 0 \times [0,1] \cup \\ (-\frac{\gamma_6}{\gamma_5}; 1] \times 0 \times 0 \end{cases} & \textit{if } \gamma_5 < 0, \\ [0,1] \times 0 \times 0 & \textit{if } \gamma_5 = 0,\ \gamma_6 < 0, \\ [0,1] \times 0 \times 1 & \textit{if } \gamma_5 = 0,\ \gamma_6 > 0, \\ [0,1] \times 0 \times [0,1] & \textit{if } \gamma_5 = \gamma_6 = 0, \end{cases}$$

$\gamma_5 = c_{121} - c_{122} - c_{221} + c_{222}$, $\gamma_6 = c_{221} - c_{222}$, $\gamma_7 = c_{122} - c_{222}$, $\gamma_8 = c_{222}$

if the 2-nd strategy of the second player is dominant.

Proof. If the 1-st strategy of the second player is dominant, then

$$\varphi_3(x,y,z) = (x(c_{111}-c_{112})+(1-x)(c_{211}-c_{212}))z+(c_{112}-c_{212})x+c_{212} =$$

$$= (\gamma_1 x + \gamma_2)z + \gamma_3 x + \gamma_4.$$

From this the truth of proposition follows evidently.

If the 2-nd strategy of the second player is dominant, then

$$\varphi_3(x,y,z) = (x(c_{121}-c_{122})+(1-x)(c_{221}-c_{222}))z+(c_{122}-c_{222})x+c_{222} =$$

$$= (\gamma_5 x + \gamma_6)z + \gamma_7 x + \gamma_8.$$

From this the truth of the second part of the proposition results. □

Similarly the NE set can be constructed in the following cases:

- *player 1 has equivalent strategies, player 3 has dominant strategy, and player 2 has incomparable strategies;*

- *player 2 has equivalent strategies, player 1 has dominant strategy, and player 3 has incomparable strategies;*

- *player 2 has equivalent strategies, player 3 has dominant strategy, and player 1 has incomparable strategies;*

- *player 3 has equivalent strategies, player 1 has dominant strategy, and player 2 has incomparable strategies;*

- *player 3 has equivalent strategies, player 2 has dominant strategy, and player 1 has incomparable strategies.*

Proposition 7. *If the first and the second players have incomparable strategies and the third player has dominant strategy, then*

$$NE(\Gamma'_m) = Gr_1 \cap Gr_2,$$

where

$$Gr_1 = [0,1]^3 \cap \begin{cases} \begin{cases} 0\times[0; -\frac{\alpha_2}{\alpha_1}) \times 1 \cup \\ [0,1]\times -\frac{\alpha_2}{\alpha_1} \times 1 \cup \\ 1 \times (-\frac{\alpha_2}{\alpha_1}; 1] \times 1 \end{cases} & \text{if } \alpha_1 > 0, \\ \begin{cases} 1 \times [0; -\frac{\alpha_2}{\alpha_1}) \times 1 \cup \\ [0,1]\times -\frac{\alpha_2}{\alpha_1} \times 1 \cup \\ 0 \times (-\frac{\alpha_2}{\alpha_1}; 1] \times 1 \end{cases} & \text{if } \alpha_1 < 0, \\ 0 \times [0,1] \times 1 & \text{if } \alpha_1 = 0,\ \alpha_2 < 0, \\ 1 \times [0,1] \times 1 & \text{if } \alpha_1 = 0,\ \alpha_2 > 0, \\ [0,1]\times[0,1] \times 1 & \text{if } \alpha_1 = \alpha_2 = 0, \end{cases}$$

$$Gr_2 = [0,1]^3 \cap \begin{cases} \begin{cases} [0; -\frac{\beta_2}{\beta_1}) \times 0 \times 1 \cup \\ -\frac{\beta_2}{\beta_1} \times [0,1] \times 1 \cup \\ (-\frac{\beta_2}{\beta_1}; 1] \times 1 \times 1 \end{cases} & \text{if } \beta_1 > 0, \\ \begin{cases} [0; -\frac{\beta_2}{\beta_1}) \times 1 \times 1 \cup \\ -\frac{\beta_2}{\beta_1} \times [0,1] \times 1 \cup \\ (-\frac{\beta_2}{\beta_1}; 1] \times 0 \times 1 \end{cases} & \text{if } \beta_1 < 0, \\ [0,1]\times 0 \times 1 & \text{if } \beta_1 = 0,\ \beta_2 < 0, \\ [0,1]\times 1 \times 1 & \text{if } \beta_1 = 0,\ \beta_2 > 0, \\ [0,1]\times[0,1] \times 1 & \text{if } \beta_1 = \beta_2 = 0, \end{cases}$$

$\alpha_1 = a_{111} - a_{211} - a_{121} + a_{221}, \alpha_2 = a_{121} - a_{221}, \alpha_3 = a_{211} - a_{221}, \alpha_4 = a_{221}$,

$\beta_1 = b_{111} - b_{121} - b_{211} + b_{221}, \beta_2 = b_{211} - b_{221}, \beta_3 = b_{121} - b_{221}, \beta_4 = b_{221}$

if the 1-st strategy of the third player is dominant,
and

$$Gr_1 = [0,1]^3 \cap \begin{cases} \begin{cases} 0 \times [0; -\frac{\alpha_6}{\alpha_5}) \times 0 \cup \\ [0,1]\times -\frac{\alpha_6}{\alpha_5} \times 0 \cup \\ 1 \times (-\frac{\alpha_6}{\alpha_5}; 1] \times 0 \end{cases} & \text{if } \alpha_5 > 0, \\ \begin{cases} 1 \times [0; -\frac{\alpha_6}{\alpha_5}) \times 0 \cup \\ [0,1]\times -\frac{\alpha_6}{\alpha_5} \times 0 \cup \\ 0 \times (-\frac{\alpha_6}{\alpha_5}; 1] \times 0 \end{cases} & \text{if } \alpha_5 < 0, \\ 0 \times [0,1] \times 1 & \text{if } \alpha_5 = 0,\ \alpha_6 < 0, \\ 1 \times [0,1] \times 0 & \text{if } \alpha_5 = 0,\ \alpha_6 > 0, \\ [0,1]\times[0,1] \times 0 & \text{if } \alpha_5 = \alpha_6 = 0, \end{cases}$$

$$Gr_2 = [0,1]^3 \cap \begin{cases} \begin{cases} [0; -\frac{\beta_6}{\beta_5}) \times 0 \times 0 \cup \\ -\frac{\beta_6}{\beta_5} \times [0,1] \times 0 \cup \\ (-\frac{\beta_6}{\beta_5}; 1] \times 1 \times 0 \end{cases} & \textit{if } \beta_5 > 0, \\ \begin{cases} [0; -\frac{\beta_6}{\beta_5}) \times 1 \times 0 \cup \\ -\frac{\beta_6}{\beta_5} \times [0,1] \times 0 \cup \\ (-\frac{\beta_6}{\beta_5}; 1] \times 0 \times 0 \end{cases} & \textit{if } \beta_5 < 0, \\ [0{,}1] \times 0 \times 0 & \textit{if } \beta_5 = 0,\ \beta_6 < 0, \\ [0{,}1] \times 1 \times 0 & \textit{if } \beta_5 = 0,\ \beta_6 > 0, \\ [0{,}1] \times [0,1] \times 0 & \textit{if } \beta_5 = \beta_6 = 0, \end{cases}$$

$\alpha_5 = a_{112} - a_{212} - a_{122} + a_{222}, \alpha_6 = a_{122} - a_{222}, \alpha_7 = a_{212} - a_{222}, \alpha_8 = a_{222},$

$\beta_5 = b_{112} - b_{122} - b_{212} + b_{222}, \beta_6 = b_{212} - b_{222}, \beta_7 = b_{122} - b_{222}, \beta_8 = b_{222}$

if the 2-nd strategy of the third player is dominant.

Proof. If the 1-st strategy of the third player is dominant, then

$$\varphi_1(x,y,z) = (y(a_{111} - a_{211}) + (1-y)(a_{121} - a_{221}))x + (a_{211} - a_{221})y + a_{221} =$$

$$= (\alpha_1 y + \alpha_2)x + \alpha_3 y + \alpha_4,$$

$$\varphi_2(x,y,z) = (x(b_{111} - b_{121}) + (1-x)(b_{211} - b_{221}))y + (b_{121} - b_{221})x + b_{221} =$$

$$= (\beta_1 x + \beta_2)y + \beta_3 x + \beta_4.$$

From the above the truth of the proposition follows.

If the 2-nd strategy of the third player is dominant, then

$$\varphi_1(x,y,z) = (y(a_{112} - a_{212}) + (1-y)(a_{122} - a_{222}))x + (a_{212} - a_{222})y + a_{222} =$$

$$(\alpha_5 y + \alpha_6)x + \alpha_7 y + \alpha_8,$$

$$\varphi_2(x,y,z) = (x(b_{112} - b_{122}) + (1-x)(b_{212} - b_{222}))y + (b_{122} - b_{222})x + b_{222} =$$

$$(\beta_5 x + \beta_6)y + \beta_7 x + \beta_8.$$

From this the truth of the second part of the proposition results. □

Similarly the NE set can be constructed in the following cases:

- *players 1 and 3 have incomparable strategies, player 2 has dominant strategy;*

- *players 2 and 3 have incomparable strategies, player 1 has dominant strategy.*

Proposition 8. *If the first and the second players have equivalent strategies and the third player has incomparable strategies, then*

$$NE(\Gamma'_m) = Gr_3,$$

where

$$Gr_3 = [0,1]^3 \cap \{X_< \times Y_< \times 0 \cup X_= \times Y_= \times [0,1] \cup X_> \times Y_> \times 1\},$$

$$X_< \times Y_< = \{(x,y) : x \in [0,1],\, y \in [0,1],\, \gamma_1 xy + \gamma_2 x + \gamma_3 y + \gamma_4 < 0\},$$

$$X_= \times Y_= = \{(x,y) : x \in [0,1],\, y \in [0,1],\, \gamma_1 xy + \gamma_2 x + \gamma_3 y + \gamma_4 = 0\},$$

$$X_> \times Y_> = \{(x,y) : x \in [0,1],\, y \in [0,1],\, \gamma_1 xy + \gamma_2 x + \gamma_3 y + \gamma_4 > 0\}.$$

$$\gamma_1 = c_{111} - c_{112} - c_{121} + c_{122} - c_{211} + c_{212} + c_{221} - c_{222},$$

$$\gamma_2 = c_{121} - c_{122} - c_{221} + c_{222}, \gamma_3 = c_{211} - c_{212} - c_{221} + c_{222}, \gamma_4 = c_{221} - c_{222}.$$

Proof. The truth of the proposition results from the following representation of the cost function:

$$\varphi_3(x,y,z) = (xy(c_{111} - c_{112}) + x(1-y)(c_{121} - c_{122}) + (1-x)y(c_{211} - c_{212}) +$$
$$+(1-x)(1-y)(c_{221} - c_{222}))z + (y(c_{112} - c_{212}) + (1-y)(c_{122} - c_{222}))x +$$
$$+(c_{212} - c_{222})y + c_{222} =$$
$$= (\gamma_1 xy + \gamma_2 x + \gamma_3 y + \gamma_4)z + \gamma_5 xy + \gamma_6 x + \gamma_7 y + \gamma_8,$$

where

$$\gamma_5 = c_{112} - c_{212} - c_{122} + c_{222}, \gamma_6 = c_{122} - c_{222}, \gamma_7 = c_{212} - c_{222}, \gamma_8 = c_{222}.$$

□

Similarly the NE set can be constructed in the following cases:

- *players 1 and 3 have equivalent strategies, player 2 has incomparable strategies;*

- *players 2 and 3 have equivalent strategies, player 1 has incomparable strategies.*

Proposition 9. *If the first and the second players have incomparable strategies and the third player has equivalent strategies, then*

$$NE(\Gamma'_m) = Gr_1 \cap Gr_2,$$

where

$$Gr_1 = [0,1]^3 \cap \{0 \times Y_< \times Z_< \cup [0,1] \times Y_= \times Z_= \cup 1 \times Y_> \times Z_>\},$$

$$Gr_2 = [0,1]^3 \cap \{X_< \times 0 \times Z_< \cup X_= \times [0,1] \times Z_= \cup X_> \times 1 \times Z_>\},$$

$$Y_< \times Z_< = \{(y,z) \,:\, y \in [0,1],\, z \in [0,1],\, \alpha_1 yz + \alpha_2 y + \alpha_3 z + \alpha_4 < 0\},$$

$$Y_= \times Z_= = \{(y,z) \,:\, y \in [0,1],\, z \in [0,1],\, \alpha_1 yz + \alpha_2 y + \alpha_3 z + \alpha_4 = 0\},$$

$$Y_> \times Z_> = \{(y,z) \,:\, y \in [0,1],\, z \in [0,1],\, \alpha_1 yz + \alpha_2 y + \alpha_3 z + \alpha_4 > 0\},$$

$$X_< \times Z_< = \{(x,z) \,:\, x \in [0,1],\, z \in [0,1],\, \beta_1 xz + \beta_2 x + \beta_3 z + \beta_4 < 0\},$$

$$X_= \times Z_= = \{(x,z) \,:\, x \in [0,1],\, z \in [0,1],\, \beta_1 xz + \beta_2 x + \beta_3 z + \beta_4 = 0\},$$

$$X_> \times Z_> = \{(x,z) \,:\, x \in [0,1],\, z \in [0,1],\, \beta_1 xz + \beta_2 x + \beta_3 z + \beta_4 > 0\},$$

$$\alpha_1 = a_{111} - a_{211} - a_{112} + a_{212} - a_{121} + a_{221} + a_{122} - a_{222},$$

$$\alpha_2 = a_{112} - a_{212} - a_{122} + a_{222}, \alpha_3 = a_{121} - a_{221} - a_{122} + a_{222}, \alpha_4 = a_{122} - a_{222},$$

$$\beta_1 = b_{111} - b_{121} - b_{112} + b_{122} - b_{211} + b_{221} + b_{212} - b_{222},$$

$$\beta_2 = b_{112} - b_{122} - b_{212} + b_{222}, \beta_3 = b_{211} - b_{221} - b_{212} + b_{222}, \beta_4 = b_{212} - b_{222}.$$

Proof. The truth of the proposition results from the following representation of the payoff functions:

$$\varphi_1(x,y,z) = (yz(a_{111} - a_{211}) + y(1-z)(a_{112} - a_{212}) + (1-y)z(a_{121} - a_{221}) +$$

$$+ (1-y)(1-z)(a_{122} - a_{222}))x +$$

$$+ (z(a_{211} - a_{221}) + (1-z)(a_{212} - a_{222}))y + (a_{221} - a_{222})z + a_{222} =$$

$$= (\alpha_1 yz + \alpha_2 y + \alpha_3 z + \alpha_4)x + \alpha_5 yz + \alpha_6 y + \alpha_7 z + \alpha_8,$$

$$\varphi_2(x,y,z) = (xz(b_{111}-b_{121})+x(1-z)(b_{112}-b_{122})+(1-x)z(b_{211}-b_{221})+$$
$$+(1-x)(1-z)(b_{212}-b_{222}))y+$$
$$+(z(b_{121}-b_{221})+(1-z)(b_{122}-b_{222}))x+(b_{221}-b_{222})z+b_{222}=$$
$$= (\beta_1 xz + \beta_2 x + \beta_3 z + \beta_4)y + \beta_5 xz + \beta_6 x + \beta_7 z + \beta_8.$$

□

Similarly the NE set can be constructed in the following cases:

- *players 1 and 3 have incomparable strategies, player 2 has equivalent strategies;*

- *players 2 and 3 have incomparable strategies, player 1 has equivalent strategies.*

Proposition 10. *If all players have incomparable strategies, then*

$$NE(\Gamma'_m) = Gr_1 \cap Gr_2 \cap Gr_3,$$

where

$$Gr_1 = [0,1]^3 \cap \{0 \times Y_< \times Z_< \cup [0,1] \times Y_= \times Z_= \cup 1 \times Y_> \times Z_>\},$$
$$Gr_2 = [0,1]^3 \cap \{X_< \times 0 \times Z_< \cup X_= \times [0,1] \times Z_= \cup X_> \times 1 \times Z_>\},$$
$$Gr_3 = [0,1]^3 \cap \{X_< \times Y_< \times 0 \cup X_= \times Y_= \times [0,1] \cup X_> \times Y_> \times 1\},$$

the components of the Gr_1, Gr_2, Gr_3 are defined as above.

Proof. The truth of proposition results from the following representation of the payoff functions:

$$\varphi_1(x,y,z) = (yz(a_{111}-a_{211}) + y(1-z)(a_{112}-a_{212})+$$
$$+(1-y)z(a_{121}-a_{221}) + (1-y)(1-z)(a_{122}-a_{222}))x+$$
$$+(z(a_{211}-a_{221}) + (1-z)(a_{212}-a_{222}))y + (a_{221}-a_{222})z + a_{222} =$$

$$= (\alpha_1 yz + \alpha_2 y + \alpha_3 z + \alpha_4)x + \alpha_5 yz + \alpha_6 y + \alpha_7 z + \alpha_8,$$

$$\varphi_2(x,y,z) = (xz(b_{111} - b_{121}) + x(1-z)(b_{112} - b_{122}) +$$

$$+(1-x)z(b_{211} - b_{221}) + (1-x)(1-z)(b_{212} - b_{222}))y +$$

$$+(z(b_{121} - b_{221}) + (1-z)(b_{122} - b_{222}))x + (b_{221} - b_{222})z + b_{222} =$$

$$= (\beta_1 xz + \beta_2 x + \beta_3 z + \beta_4)y + \beta_5 xz + \beta_6 x + \beta_7 z + \beta_8,$$

$$\varphi_3(x,y,z) = (xy(c_{111} - c_{112}) + x(1-y)(c_{121} - c_{122}) +$$

$$+(1-x)y(c_{211} - c_{212}) + (1-x)(1-y)(c_{221} - c_{222}))z +$$

$$+(y(c_{112} - c_{212}) + (1-y)(c_{122} - c_{222}))x + (c_{212} - c_{222})y + c_{222} =$$

$$= (\gamma_1 xy + \gamma_2 x + \gamma_3 y + \gamma_4)z + \gamma_5 xy + \gamma_6 x + \gamma_7 y + \gamma_8.$$

□

3 Conclusions

The NE set can be described as an intersection of graphs of players' best responses.

The solution of the problem of NE set construction in the mixed extension of the $2 \times 2 \times 2$ game illustrates that the NE set is not necessarily convex even in convex game. Moreover, the NE set is frequently disconnected. Thus, new conceptual methods "which derive from the theory of semi-algebraic sets are required for finding all equilibria" [1]. In this article we make an attempt to give an idea of such a method.

References

[1] McKelvey R.D., McLenan A., *Computation of equilibria in finite games,* Handbook of Computational Economics, 1, Amsterdam, Elsevier, 1996, pp. 87-142.

[2] Moulen H., *Théorie des jeux pour l'éqonomie et la politique,* Paris, 1981 (in Russian: Games Theory, Moscow, Mir, 1985, 200 p.).

[3] Nash J.F., *Noncooperative game*, Annals of Mathematics, 54, 1951, pp. 280-295.

[4] Sagaidac M., Ungureanu V., *Operational research*, Chişinău, CEP USM, 2004, 296 p. (in Romanian).

V. Ungureanu, A. Botnari,
Received March 18, 2005

State University of Moldova,
60, A. Mateevici str.,
Chişinău, MD−2009, Moldova.
E–mail: *valungureanu@usm.md*