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The multiobjective transportation fractional programming model

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Abstract

In this paper the multiobjective transportation problem of linear-fractional type with one nonlinear time constraint criterion is investigated. Particularly, the case of the identical denominators is studied. The concrete procedure to find the set of all basic efficient solutions for this model is proposed. This algorithm is tested on an annexed example.

Keywords and phrases: multiobjective problem, optimization, linear-fractional criterion, basic efficient solution, bottleneck restriction

The transportation problem dealing with the total cost minimizing criterion, considered as a classical one, is well-known and sufficiently analyzed in the respective sources.

The transportation model of a "bottleneck" type is a specific problem within the transportation classical issue, the objective function of which is a non-linear one. Special cases of these types of problems are investigated in many paper-works like [1], [2], [5], [6], where concrete algorithms in order to solve them are carried out. The transportation model of the "bottleneck" type with 2 criteria, where the first one is providing the total transportation cost minimization and the second one, that is non-linear, is strangling in time, is studied in article [7], where the authors propose the concrete algorithm to solve it. The special algorithm for solving transportation model of the "bottleneck" type with 3 criteria is presented in paper [8], where it is tested on a concrete example.

One should mention that in our daily life the multiobjective fractional programming models are of great interest. We are often concerned about the optimization of the ratios like the summary cost of the total transportation expenditures to the maximal necessary time to satisfy the demands, the total benefits or production values into time unit, the total depreciation into time unit and many other important similar criteria, which may appear in order to evaluate the economical activities and make the correct managerial decisions. These problems led to the "bottleneck" transportation model with multiple fractional criteria, where the "bottleneck" criteria appear as a "minmax" time constraining. The common characteristic of these objective ratios is the identical denominators. Concrete algorithms for solving of special models of transportation type with one criterion, where the objective function is a fractional one, are proposed in papers [3],[4].

The multicriterial transportation model of "bottleneck" type with two fractional criteria is defined as follows:

$$\min z_1 = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}}{\max\limits_{i,j}\{t_{ij}|x_{ij}>0\}} \tag{1}$$

$$\min z_2 = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij}}{\max\limits_{i,j}\{t_{ij}|x_{ij}>0\}} \tag{2}$$

$$\min z_3 = \max_{i,j}\{t_{ij}|x_{ij}>0\} \tag{3}$$

in conditions (4)-(7)

$$\sum_{j=1}^{n} x_{ij} = a_i, \forall i = \overline{1,m} \tag{4}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \forall j = \overline{1,n} \tag{5}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{6}$$

$$x_{ij} \geq 0, i = \overline{1,m}, j = \overline{1,n} \tag{7}$$

where c_{ij} - cost of transporting a unit from source i to destination j, d_{ij} - deterioration of a unit while transporting from source i to destination j, a_i - availability at source i, b_j - requirement at destination j, x_{ij} - amount transported from source i to destination j, t_{ij} - time of transporting a unit from source i to destination j.

A non traditional algorithm of building numerous efficient solutions of the models is carried out here. There is no sense to look for an optimal solution to settle the multicriterial mathematical models. As it often occurs, there are no solutions at all.

That is why, one should better determine the multitude of non-dominant solutions, which are known as efficient solutions or optimal in the terms of Pareto.

In order to solve the multi criteria model the notion of an efficient solution has been introduced.

DEF: The feasible solution for the multiple criteria model is considered to be efficient if and only if another feasible solution, for which we obtain a better value at least for one criterion while the values of the rest criteria remain unmodified, doesn't exist.

In order to solve the problem (1)- (7) by finding the set of the efficient basic solutions, we reduce it to the following model:

$$\min z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{8}$$

$$\min z_2 = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} \tag{9}$$

$$\min z_3 = \max_{i,j} \{ t_{ij} | x_{ij} > 0 \} \tag{10}$$

in conditions (4)-(7)

The authors Y.P. Aneja and K.P.K. Nair in their paper: "Bicriteria transportation problem" [1] propose an algorithm to solve the model (8)-(10) in conditions (4)-(7). The algorithm determines the multitude of extreme non-dominant solutions within the admissible space of solutions. The algorithm is theoretically and scientifically tested and proved in a concrete case.

The algorithm of solving the model (1)-(7) develops a procedure of building a multitude of all efficient, basic solutions. This set coincides with the set of the efficient basic solutions for the model (8)-(10) in conditions (4)-(7). That is why we reduce the multicriterial fractional transportation model of "bottleneck" type (1)-(7) to the problem (8)-(10) in the restrictions (4)-(7) in order to find the set of its basic efficient solutions.

Theorem 1. *The set of the efficient basic solutions of the model (1)-(7) and the model (8)-(10) coincide.*

Proof. Let X^1 be an efficient basic solution for the model (1)-(7), and $T^1 = \max_{i,j}\{t_{ij}/x^1_{ij} > 0\}$. We state that for each available solution X^2 of this model and corresponding T^2, where $T^2 = \max_{i,j}\{t_{ij}/x^2_{ij} > 0\}$, based on the definition of the efficient solution, the following inequalities are true:

$$\begin{array}{c} Z_1(X^1) < Z_1(X^2) \text{ and } Z_2(X^1) \le Z_2(X^2) \\ \text{or} \\ Z_1(X^1) \le Z_1(X^2) \text{ and } Z_2(X^1) < Z_2(X^2) \end{array} \qquad (11)$$

where $T^2 \le T^1$, $T^1 \ge 0$, $T^2 \ge 0$.

We suppose that the solution X^1 is not efficient for the model (8)-(10) in the conditions (4)-(7). Analogously to the antecedent reasoning, using the definition of the efficient solution, it follows that there exists the available solution X^2 of this model and corresponding T^2, for which the following inequalities are true:

$$\begin{array}{c} \frac{Z_1(X^2)}{T^2} < \frac{Z_1(X^1)}{T^1} \text{ and } \frac{Z_2(X^2)}{T^2} \le \frac{Z_2(X^1)}{T^1} \\ \text{or} \\ \frac{Z_1(X^2)}{T^2} \le \frac{Z_1(X^1)}{T^1} \text{ and } \frac{Z_2(X^2)}{T^2} < \frac{Z_2(X^1)}{T^1} \end{array} \qquad (12)$$

where $T^2 \le T^1$, $T^1 \ge 0$, $T^2 \ge 0$.

Multiplying inequalities (12) by T^1 and supposing $k = \frac{T^1}{T^2}$, we obtain that the following inequalities are true:

$$\begin{array}{c} kZ_1(X^2) < Z_1(X^1) \text{ and } kZ_2(X^2) \leq Z_2(X^1) \\ \text{or} \\ kZ_1(X^2) \leq Z_1(X^1) \text{ and } kZ_2(X^2) < Z_2(X^1) \end{array} \tag{13}$$

where $T^2 \leq T^1$, $T^1 \geq 0$, $T^2 \geq 0$.

As it is obvious that $k \geq 1$, from (13) we conclude that for the solution X^2 the following inequalities are true:

$$\begin{array}{c} Z_1(X^2) < Z_1(X^1) \text{ and } Z_2(X^2) \leq Z_2(X^1) \\ \text{or} \\ Z_1(X^2) \leq Z_1(X^1) \text{ and } Z_2(X^2) < Z_2(X^1) \end{array} \tag{14}$$

where $T^2 \leq T^1$, $T^1 \geq 0$, $T^2 \geq 0$,
that contradicts (11).

It can be proved analogously that each efficient solution of the model (8)-(10) is also an efficient solution for the model (1)-(7).

The theorem is proved.

Generalizing this idea for the model with multiple number of fractional criteria with the "bottleneck" constraining criterion, we conclude that it may be reduced to the model (15) in order to find the set of its efficient basic solutions, that is defined as follows:

$$\begin{array}{c} \min z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^1 x_{ij} \\ \min z_2 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^2 x_{ij} \\ \ldots\ldots \\ \min z_r = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^r x_{ij} \\ \min z_{r+1} = \max_{i,j}\{t_{ij} | x_{ij} > 0\} \\ \text{in conditions (4)-(7)} \end{array} \tag{15}$$

Values C_{ij}^k, $k = 1, ..., r$, $i = 1, ..., m$, $j = 1, ..., n$ correspond to the concrete interpretation of the respective criteria.

If there are some criteria of "max" type among the set of criteria from the model (15), it is not difficult to reduce this case to the initial one. It is obvious that the model (8)-(10) is a particular case of the model (15), and so the algorithm to solve the model (15) in conditions (4)-(7) can be used to solve the model (8)-(10).

The truthfulness of the above theorem for the model (15) is proved similarly.

The algorithm of finding the set of the efficient basic solutions for the model (15) is an interactive one. Initially we consider at least $(m+n-1)$ cells from the tables C^k, $k = 1, 2, \ldots, r$, in order to find the first efficient basic solution of model (15). The indexes' order is maintained the same as in the table T, where the cells are numbered according to the respective time values well arranged in the increasing order. Each iteration supposes a deep levels' exploration and a completion of the multitude of efficient basic solutions for a new unblocked stochastic time-variable.

The exploration procedure of each time instant chain is finite in depth and ends on every branch, in the case when the same solutions have been found at upper level of other branch or when all possibilities of improvement have been spent at this level.

At the time when the solution of a certain configuration detains the form recorded in another link, which has been investigated earlier, its depth exploration has no justification, that is why it is eventually stopped.

We propose the logic scheme to construct the algorithm for solving the multiple criteria transportation models of "bottleneck" type with a finite number of criteria, which is presented at Fig.1, where $\Delta_{ij} = (u_i + v_j) - c_{ij}$, $n_i < p$ (p is defined by the dimension of the problem, n_i is an index of ordering the cells by data from the table T).

The logical blocks are to be verified before every logical ramification according to Fig.1.

ALGORITHM

1. Table T with the increasing order of time values which uses the k index is being well arranged. The index order is maintained for the

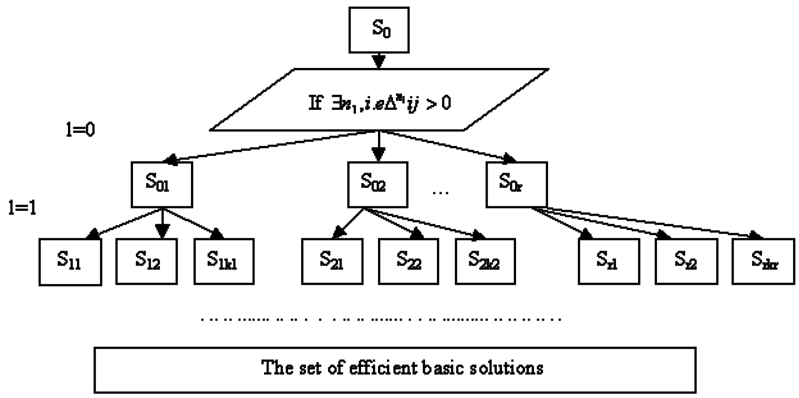

Figure 1.

respective cells from the tables C^k, $k = 1, ..., r$.

2. The adoption of an initial, basic solution in the first $p = (m+n-1)$ cells from the table C is performed. The other cells are considered to be blocked.

3. All configurations of basic solutions can be recorded at the level $l = 0$, using only the non-blocked cells and providing the dozing in all those cells with $(i, j | x_{ij} > 0)$, for which the relation $\Delta_{ij} \geq 0$ to be true at least for one criterion.

Each configuration of the solution is iteratively investigated, obtaining in such a way the following records at the next level: $l = l + 1$.

If a certain level of a basic solution, which was previously obtained, is found, the latter won't be further studied. Since the problem covers a finite dimension, the multitude, consequently, of all basic solutions for the unblocked cells will be obtained by exploring a finite number of levels in depth.

4. If $p < m*n$, the following $p = p+1$ cell is unblocked, and for this purpose the exploration of the basic solutions is revived, then we start with the level 0. The 4th step will be repeated until we get $p = m*n$.

The basic efficient solutions set is selected out of the multitude of the basic solutions.

Theorem 2. *The set of all efficient basic solutions for the multiple criteria transportation problem of "bottleneck" type is found by applying the above algorithm.*

Proof. Let L be a list of efficient basic solutions of model (15) being found by applying the above algorithm. We suppose, that the efficient basic solution S_1, that was not found using the above algorithm exists and $S_1 \notin L$. Let S_1 corresponds to T_1. We will fix it on the branch that corresponds to the T_1 beginning with the level 0, when corresponding cells from table T are cleared. Wide exploration of the fixed branch leads to the registration of all basic solutions of branch T_1. So, all the basic solutions that correspond to time T_1 are contained in this set. We will separate from the set L_{T_1} the efficient basic solutions corresponding to time T_1. It is obvious that $L_{T_1} \subset L$. As a result, if $S_1 \in L_{T_1}$, then S_1 is a basic efficient solution found by applying the above algorithm or if $S_1 \notin L_{T_1}$, then S_1 is not a basic solution and moreover it is not a basic efficient solution. So, iether S_1 is not a basic solution or it is contained in list L. The theorem is proved.

Example

Consider the following 3-criteria problem.

Time, Supply, Demand=

10	95	73	52	8
68	66	30	21	19
37	63	19	23	17
11	3	14	16	$b_j \| a_i$

Cost 1,2=

1 4	2 4	7 3	7 4
1 5	9 8	3 9	4 10
8 6	9 2	4 5	6 1

Using the above proposed algorithm we have found the following 11 efficient basic solutions:

S1=(176,207,68); S2=(164,276,68); S3=(178,203,68);
S4=(172,213,68); S5=(158,283,68); S6=(208,167,73);
S7=(202,173,73); S8=(156,200,95); S9=(176,175,95);
S10=(143,265,95); S11=(186,171,95).

The authors of the article [1], using their own algorithm for this example, have obtained 9 efficient extreme solutions.

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