The Maple Computer Algebra System

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Maple is a comprehensive general purpose computer algebra system. It is used primarily in education and scientific research in the sciences, in mathematics, and in engineering. Maple can do both symbolic and numerical calculations and has facilities for 2 and 3-dimensional graphical output. The newest version of Maple, Maple V Release 2, sports a new user interface that integrates these three facilities as well as text into a document called a *worksheet*. Worksheets are a nice medium for presenting and communicating results and teaching material. A new feature of Release 2 that makes worksheets especially appealing is that results computed by Maple are displayed in a high quality font with full support for math symbols, Greek characters, subscripts, etc.

Maple is also a programming language. In fact almost all of the mathematical and graphical facilities are written in Maple and not in a systems implementation language like other computer algebra systems. These Maple programs reside on disk in the Maple library and are loaded on demand. The programming language supports procedural and functional programming. We will not give specific details of the programming language beyond this, even though this is perhaps a key part of the Maple system. Details and examples can be found in the two manuals [1] and [2] cited in the references. We wish to mention instead a bit of Maple history.

The main design goal of Maple in 1980 was to design a language for implementing mathematical algorithms which was powerful and efficient enough so that almost all of the mathematical algorithms could be written in it, rather than in a systems implementation language like C or Lisp. Another main design objective was to minimize overall memory requirements so that larger problems could be solved, and Maple could run on smaller computers, and support many simultaneous users on time-shared systems. The outcome is a system consisting

of three components, the user interface, the kernel, and the Maple library. The kernel, programmed in C, is about 500Kbytes in size. The Maple library, programmed in the Maple programming language, consists of about 2,500 user level functions which totals some 150,000 lines of Maple code. Because code from the Maple library is loaded on demand, Maple runs on small computers e.g. Amigas, PC's, Macintoshes with as little as 2 megabytes of RAM, as well as on most workstations, mainframes and supercomputers. This is attractive for educational laboratories and for students. The attention to space efficiency helps in solving large problems and contributes to the overall time efficiency of the system. Also, because the clean separation of the user interface from the kernel and library, the Maple kernel and library are used as the symbolic component in other commercial and research systems. For example, the system MathCAD, a numerical system popular amongst engineers, makes some of Maples symbolic functionality accessible to MathCAD users.

There is also a growing number of functions and packages in the Maple *share* library which have been contributed by users from around the world. The share library also contains additional documentation, other software tools, etc. It is updated quarterly as new applications codes are contributed and is now distributed electronically by an electronic mail server and anonymous `ftp`. Other sources of information include

- the Maple newsgroup, an electronic forum for discussion on Maple - maple@daisy.waterloo.edu

- the Maple ROOTS newsletter, published by the Maple company, contains information about new releases, Maple events, etc.,

- the Maple Technical Newsletter, published by Birkhäuser, contains application and technical articles on Maple, and

- the Geddes textbook [3] which contains detailed information and references for many the algorithms used in Maple.

Further information about Maple can be obtained from

In Europe	North America and elsewhere
Waterloo Maple Software	Waterloo Maple Software,
Tiergartenstrasse 17	160 Columbia Street West,
W-6900 Heidelberg 1	Waterloo, Ontario, N2L 3L3
Germany	Canada
Phone: +49-6221-487 180	Phone: (519) 747-2373
FAX: +49-6221-487 184	FAX: (519) 747-5284
	Email: info@maplesoft.on.ca

We summarize below the capabilities of Maple under the headings Numeric, Symbolic and Graphics and follow with some examples illustrating some of the new features of Release 2.

Numeric facilities

- Arbitrary precision integer, floating point, and complex numeric arithmetic,

- Numerical evaluation of the elementary functions,

- A library of special functions and constants including the error function, the Gamma and related functions, the exponential integral, the Riemann zeta function, Bessel and related functions, Hypergeometric functions, statistical distribution functions,

- Numerical linear algebra: linear systems, eigenvalues and eigenvectors, SVD,

- Numerical approximation: interpolating splines, B-splines, least squares, Pade, continued fractions, Chebyshev series, Remez algorithm,

- Numerical calculus: numerical integration, numerical evaluation of infinite sums and products, solution of ODE's, root finding, FFT.

Symbolic facilities

- Indefinite and definite integration of rational functions, elementary functions, algebraic functions, and special functions,

- Laplace, Fourier, Mellin, and Z transforms and inverse transforms,

- Definite and indefinite summation, infinite products, solution of recurrence equations, Taylor series, asymptotic expansions, symbolic limits,

- Polynomial arithmetic over finite and algebraic number fields, including GCDs, resultants, decomposition, factorization, root finding, Galois groups,

- Symbolic linear algebra: matrix determinant and inverse, eigenvalues and eigenvectors, Row echelon, Smith, Hermite, and Jordan canonical forms,

- Solving equations: linear systems, polynomial systems, Gröbner bases, ODE's,

- Specialized packages: finite groups, number theory, tensors, generating functions, geometry, orthogonal polynomials, differential forms, symmetric functions, graph theory, statistics, etc.

Graphic facilities

Basic facilities support the display of an arbitrary number of curves, surfaces, polygons, and text. Surfaces may be displayed in various styles, e.g. with hidden line removal, gray scale shading, color shading. Shaded surfaces can include contours, grids, lighting, etc. Also support for 2 and 3D frame sequence animation. Hardcopy output is available for a variety of printers and may be included in a Maple worksheet. Specific plots (all programmed in Maple) include

- Curves in cartesian, and polar co-ordinates,

- Surfaces in cartesian, cylindrical or spherical co-ordinates,

- Curves and surfaces defined parametrically or implicitly as the solution of a polynomial equation,

- Contour plots, log plots, density plots, conformal plots, vector plots,

- Phase portraits and direction field plots, plotting of systems of first order ODE's.

A Pot Pouri of integration and summation examples

New users of computer algebra systems are intrigued by their ability to compute formulae for integrals, and also sums. Here are some examples from Maple V Release 2. In this first example, we ask Maple to compute $\int e^x x^3 \mathrm{d}x$. The Maple command to compute this integral is

```
> int( exp(x)*x^3, x );
```

The > character is the Maple prompt. The result computed by Maple is then displayed on the users terminal. It looks like

$$e^x x^3 - 3\,x^2 e^x + 6\,x e^x - 6\,e^x$$

Maple computed this integral by repeated application of *integration by parts*. This is a traditional method for computing integrals of this kind. This following rational function integral is computed using the method of *partial fractions*.

```
> int( 3*x/(x^3-1), x );
```

$$\ln(x-1) - \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\arctan\left(\frac{1}{3}(2\,x+1)\sqrt{3}\right)$$

A more powerful method for rational function integration where the denominator of the rational function does not factor nicely into linear and quadratic factors is looked at later. The next example shows that Maple knows something about special functions, in this case the Sine integral and the Cosine integrals.

```
> int(sin(x)/(x-1), x);
```

$$\text{Si}(x-1)\cos(1) + \text{Ci}(x-1)\sin(1)$$

Consider the non-trival function f

```
> f := ((2*x-1)*exp(4*x)+(4*x^2+2*x-1)*exp(2*x)) /
>       (exp(2*x)*x^2+x^2);
```

$$f := \frac{(2\,x-1)\,e^{4\,x} + (4\,x^2 + 2\,x - 1)\,e^{2\,x}}{e^{2\,x}x^2 + x^2}$$

The integral of f is computed by the Risch integration algorithm [8], an algorithm for integrating elementary functions.

```
> int(f,x);
```

$$\frac{e^{2\,x}}{x} + 2\,\ln\left(e^{2\,x} + 1\right)$$

Here is a definite integral where we are using the `Int` command to produce the *unevaluated* integral. In this way, we can display what we are computing, and the result that Maple obtains.

```
> f := exp(-t^2)*ln(t)*t^2:
> Int(f,t=0..infinity) = int(f,t=0..infinity);
```

$$\int_0^\infty e^{-t^2}\ln(t)t^2dt = \frac{1}{4}\sqrt{\pi} - \frac{1}{8}\sqrt{\pi}\gamma - \frac{1}{4}\sqrt{\pi}\ln(2)$$

This example illustrates the difference between numerical computation and symbolic computation. Instead of a numerical value for the definite integral, we get an exact formula. The formula involves the constants π, Euler's constant γ, and $\ln(2)$.

Here are some summations computed by Maple. First a formula for the sum of the first n natural numbers

54

```
> sum( i, i=0..n );
```

$$\frac{1}{2}(n+1)^2 - \frac{1}{2}n - \frac{1}{2}$$

Probably you will not recognize that in this form, let's factor it

```
> factor( sum(i,i=0..n) );
```

$$\frac{1}{2}n(n+1)$$

Here is an indefinite sum of a hypergeometric function computed by Gosper's algorithm [4]

```
> sum((-3*n^2+28*n+57)*13^n/(n+1)!/(3*n+8)/(3*n+5), n);
```

$$13\frac{13^{n-1}}{(3\,n+5)\,n!}$$

Finally, an infinite sum of a rational function.

```
> f := (i^4+2*i^2+2*i+2)/(i^6+2*i^5+3*i^4+2*i^3+2*i^2):
> Sum(f,i=1..infinity) = sum(f,i=1..infinity);
```

$$\sum_{i=1}^{\infty} \frac{i^4 + 2\,i^2 + 2\,i + 2}{i^6 + 2\,i^5 + 3\,i^4 + 2\,i^3 + 2\,i^2} = -\frac{1}{2} + \frac{1}{6}\pi^2$$

This sum was computed by Moecnk's algorithm [7] which was used to split the indefinite sum as follows

$$\sum_i f \;=\; \left(i^2+1\right)^{-1} + \sum_i i^{-2}.$$

The examples shown already include the use of some modern algorithms for integration and summation. We will show some examples of current research results later in this paper. First, however, we'd like to show that Maple can also compute with numbers. This includes integers, rational numbers, decimal numbers, algebraic numbers, and complex numbers.

```
> 41! + 1;
```

$$815915283247897734345611269596115894272000000000$$

Integers can have as many digits as you need. Because you can work with such integers, it is feasible to do number theoretic calculations on large numbers. For example, it happens that this integer is prime

```
> isprime( 41! + 1 );
```

$$true$$

The integer primality test used in Maple note is a probabilistic primality test. The complex unit $i = \sqrt{-1}$ is input as `I` or `sqrt(-1)` in Maple.

```
> (6+2*I)/(3-I);
```

$$\frac{8}{5} + \frac{6}{5}I$$

Decimal or floating point numbers are input using a decimal point or in scientific notation.

```
> 2.0 / 3.0;
```

$$.6666666667$$

Arithmetic is done in decimal, not binary, and by default, 10 digits of precision are used. The precision is adjustable. An important property of Maple and other computer algebra systems is that when you try to compute with exact numbers, i.e. non-decimal numbers, then Maple always computes the "exact answer". If you compute with decimal numbers, then Maple will compute a decimal number. For example, compare these two calculations

```
> arcsin(sqrt(2)/2), arcsin(sqrt(2.0)/2.0);
```

$$\frac{1}{4}\pi, \ .7853981631$$

Examples from polynomial factorization

Maple V can factor univariate polynomials over the rationals and more generally, algebraic number fields, and also finite fields. For example, the polynomial $x^8 - 8$, which is irreducible over $\mathbf{Q}$, factors over $\mathbf{Q}(\sqrt[4]{2})$ as follows

```
> factor(x^8-8,2^(1/4));
```

$$\left(x^4 + 2\sqrt{2}\right)\left(x^2 - 2^{3/4}\right)\left(x^2 + 2^{3/4}\right)$$

Release 2 includes factorization of multivariate polynomials over any algebraic number or algebraic function field. See [9] for details. In the example below, we factor a bivariate polynomial over $\mathbf{Q}(\sqrt{2})$

```
> factor( y^4-2*x*y^2-x^2, sqrt(2) );
```

$$-\left(x + y^2 - y^2\sqrt{2}\right)\left(x + y^2 + y^2\sqrt{2}\right)$$

These examples have shown how one can compute over an algebraic number field defined by a simple radical extension, the $\sqrt{2}$. It is possible to compute over a field defined by the roots of an irreducible polynomial using Maple's general `RootOf` notation. For example, consider the polynomial $x^6 + x^2y + y^2$. It is irreducible over the rationals.

```
> f := factor( x^6+x^2*y+y^2 );
```

$$x^6 + x^2y + y^2$$

But if we let α be a root of this polynomial, then $x - \alpha$ will divide this polynomial. Perhaps it has other non-trivial factors? We do this in Maple as follows.

```
> alias(alpha=RootOf(f,x)):
> factor(f,alpha);
```

$$\left(x^4 + \alpha^2 x^2 + y + \alpha^4\right)(x + \alpha)(x - \alpha)$$

Other facilities of Maple that exploit these new facilities for algebraic functions include integration, symbolic eigenvalues and eigenvectors, and the solution of systems of polynomial equations.

Examples from linear algebra

Maple can compute exact formulae for eigenvalues and eigenvectors as well as numerical approximations. E.g. given the matrix

```
> A := matrix(3,3,[sqrt(2),x,alpha,
>                   x,sqrt(2),x,
>                   alpha,x,sqrt(2)]);
```

$$\begin{bmatrix} \sqrt{2} & x & \alpha \\ x & \sqrt{2} & x \\ \alpha & x & \sqrt{2} \end{bmatrix}$$

The eigenvalues are the roots of the characteristic polynomial

```
> factor(charpoly(A,lambda));
```

$$\left(\lambda^2 - 2\sqrt{2}\lambda - \alpha\lambda + 2 + \alpha\sqrt{2} - 2x^2 \right) \left(\lambda - \sqrt{2} + \alpha \right)$$

Note that Maple has factored the characteristic polynomial over $Q(\sqrt{2})$. Solving for λ we obtain

```
> e := eigenvals(A);
```

$$\frac{\alpha}{2} + \sqrt{2} + \frac{\sqrt{\alpha^2 + 8x^2}}{2}, \frac{\alpha}{2} + \sqrt{2} - \frac{\sqrt{\alpha^2 + 8x^2}}{2}, -\alpha + \sqrt{2}$$

For each eigenvalue λ_i, a basis for its corresponding eigenvectors is given by the nullspace of characteristic matrix $\lambda_i I - A$, i.e.

```
> seq( nullspace(charmat(A,lambda)), lambda = e );
```

$$\left\{ [1, -\frac{\alpha - \sqrt{\alpha^2 + 8x^2}}{2x}, 1] \right\}, \quad \left\{ [1, -\frac{\alpha + \sqrt{\alpha^2 + 8x^2}}{2x}, 1] \right\}, \quad \{[-1, 0, 1]\}$$

Examples from rational function integration

Consider integrating the following rational function

```
> f := (6*x^5+6*x^2-4*x+8*x^3-4)/(x^6+2*x^3-2*x^2+1);
```

$$\frac{6\,x^5 + 8\,x^3 + 6\,x^2 - 4\,x - 4}{x^6 + 2\,x^3 - 2\,x^2 + 1}$$

The difficulty in this example is that the denominator does not factor over $\mathbf{Q}$, hence we cannot use the traditional method of splitting the rational function into partial fractions. The Trager-Rothstein algorithm [11] implemented in Maple V yields the nice result

$$\int f\mathrm{d}x \;=\; \sum_{\alpha^2-2=0} (1+\alpha)\ln(x^3 - \alpha\,x + 1)$$

That is, there are two logs corresponding to two roots $\pm\sqrt{2}$ of the polynomial $\alpha^2 - 2 = 0$. Hence, Maple obtains

```
> int(f,x);
```

$$\left(1 - \sqrt{2}\right)\ln\left(x^3 + \sqrt{2}x + 1\right) + \left(\sqrt{2} + 1\right)\ln\left(x^3 - \sqrt{2}x + 1\right)$$

Condider the function

$$f \;=\; \frac{x^4 - 3\,x^2 + 6}{x^6 - 5\,x^4 + 5\,x^2 + 4}$$

Using the Trager-Rothstein algorithm, the following result involving complex logarithms is obtained.

$$\frac{i}{2}\ln\left(x^3 + ix^2 - 3\,x - 2\,i\right) \;-\; \frac{i}{2}\ln\left(x^3 - ix^2 - 3\,x + 2\,i\right)$$

An improvement due to Lazard and Rioboo [6] is implemented in Release 2. It seeks to obtain a real result from a real integrand with no new poles.

```
> f := (x^4-3*x^2+6)/(x^6-5*x^4+5*x^2+4):
> int(f,x);
```

$$\arctan\left(-\frac{3}{2}x^3 + \frac{1}{2}x^5 + \frac{1}{2}x\right) + \arctan(x^3) + \arctan(x)$$

Examples from definite integration

Maple V includes a new method for computing definite integrals based on evaluating the derivatives of special functions. See [10]. For example, it can compute integrals of the form

$$\int_a^\infty t^n \, \exp(-ut^s) \, \ln(bt)^m \quad \begin{matrix} \sin \\ \text{or} \quad \cos \end{matrix} (ct^r) \, \mathrm{d}t$$

where $r \in \{0, s, 2s, s/2\}$, and other restrictions on the constants apply. In general these integrals can be expressed in terms of the incomplete Gamma function, the Riemann zeta function, the Meijer G function, and other special functions. For example

```
> int( exp(-t)*cos(t)*ln(t), t=0..infinity );
```

$$-\frac{1}{4}\ln(2) - \frac{1}{2}\gamma - \frac{1}{8}\pi$$

where γ is Euler's constant. The calculation below illustrates a new family of definite integrals that Release 2 knows about and also the ability to make assumptions about symbolic parameters. The answer to the integral below depends on the parameters p and q, in this case whether they are positive or negative. With no additional information, Maple cannot solve the problem. The assume facility in Release 2 allows users to state assumptions about symbolic parameters e.g. here that p and q are real and positive

```
> f := x^(a-1)*exp(-p*x^s-q*x^(-s));
```

$$f := x^{a-1} e^{-p\tilde{}\, x^s - q\tilde{}\, x^{-s}}$$

```
> assume(p>0); assume(q>0);
> int(f,x=0..infinity);
```

$$2\frac{\mathrm{signum}(s) \sqrt{q\tilde{}^{\frac{a}{s}}} \, \mathrm{BesselK}(\frac{a}{s}, 2\sqrt{p\tilde{}}\sqrt{q\tilde{}})}{\sqrt{p\tilde{}^{\frac{a}{s}}}\, s}$$

Examples of asymptotic series and limits

A simple but powerful method for computing limits analytically is to develop a series expansion about the limiting point. For example, to compute the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

we can try to expand the function as a Taylor series about the origin. In Maple

```
> series(sin(x)/x,x=0);
```

$$1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + O\left(x^5\right)$$

¿From this Taylor series, one can immediately deduce that the limit is 0. Limits at $+\infty$ can be computed by computing asymptotic series. Maple includes an implementation of a new model of "hierarchical" series in which various essential singularities in addition to simple poles are allowed e.g. $\ln(x)$ and $e^{\frac{1}{x}}$ about $x = 0$. For example, the asymptotic series for the error function $\mathrm{erf}(x)$ is

```
> series(erf(x),x=+infinity);
```

$$1 + \frac{-\frac{1}{\sqrt{\pi}x} + \frac{1}{2\sqrt{\pi}x^3} - \frac{3}{4\sqrt{\pi}x^5} + O\left(\frac{1}{x^7}\right)}{e^{x^2}}$$

¿From this series expansion, Maple can determine the following limit

```
> Limit(exp(x^2)*(1-erf(x)),x=infinity) =
> limit(exp(x^2)*(1-erf(x)),x=infinity);
```

$$\lim_{x \to \infty} e^{x^2}\left(1 - \mathrm{erf}(x)\right) = 0$$

Release 2 includes a more powerful model of series for nested functions. The idea behind the new algorithm is the following: first locate the

most varying part of the expression. Treat this indeterminate as a symbol and compute a series expansion in this symbol alone. If there is no constant coefficient in the resulting series, you are done, otherwise, recurse on that constant coefficient. See [5] for details of the algorithm. Here is a difficult limit involving nested functions

```
> f := exp(exp(exp(Psi(Psi(Psi(n))))))/n:
> Limit(f,n=infinity) = limit(f,n=infinity);
```

$$\lim_{n\to\infty} \frac{e^{e^{e^{\Psi(\Psi(\Psi(n)))}}}}{n} = 0$$

Here is a difficult example, involving nested Γ functions

```
> f := (GAMMA(n+1/GAMMA(n))-GAMMA(n))/ln(n):
> Limit(f,n=infinity) = limit(f,n=infinity);
```

$$\lim_{n\to\infty} \frac{\Gamma\left(n + \frac{1}{\Gamma(n)}\right) - \Gamma(n)}{\ln(n)} = 1$$

References

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