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Exact solutions to differential equations with different arguments

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Abstract. Various linear and non-linear first-order differential equations with different arguments are considered. Exact solutions to these equations are provided. Systems of two coupled linear first-order differential equations are also solved explicitly and exactly.

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1 Introduction

In [1] (see also [2]), the authors considered the following linear *delay differential equation* (DDE):

$$y'(t) = \frac{1}{2}e^{t/2}y(t/2) + \frac{1}{2}y(t) \tag{1}$$

for $0 \leq t \leq 1$, subject to the initial condition $y(0) = 1$. Its exact solution is $y(t) = e^t$.

Similarly, the exact solution of the non-linear DDE

$$y'(t) = 1 - 2y^2(t/2) \quad \text{for } 0 \leq t \leq 1, \tag{2}$$

subject to the initial condition $y(0) = 0$, is $y(t) = \sin(t)$.

Equations (1) and (2) are indeed considered as delay differential equations, which are very important in various fields, in particular in epidemiology and in mathematical biology; see, for instance, Smith [4] and Rihan [3]. Equation (1) can be rewritten as follows:

$$y'(t) = \frac{1}{2}e^{t/2}y\left(t - \frac{t}{2}\right) + \frac{1}{2}y(t). \tag{3}$$

Thus, the delay τ is equal to $t/2$ and is therefore dependent on t. Notice that, contrary to an equation of the form

$$y'(t) = y(t - \tau), \tag{4}$$

we only need the value of $y(t)$ at time $t = 0$, rather than in the interval $[-\tau, 0]$. Similarly for Eq. (2).

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To solve DDEs, various authors have proposed algorithms that sometimes enable one to obtain the exact solutions to certain equations. However, in general these algorithms yield either approximate or numerical solutions to DDEs. Therefore, it is important to have exact solutions to a number of DDEs in order to check whether the algorithms provide accurate solutions.

Currently, the mathematical software packages *Mathematica* and *Maple* are unable to give exact solutions to differential equations with different arguments. Using the NDSolve function in *Mathematica*, one can obtain numerical solutions to DDEs with constant delays. *Maple* can handle variable delays and also systems of DDEs.

In the next section, Eqs. (1) and (2) will be respectively generalized to

$$y'(t) = c_1 e^{c_2 t} y(c_0 t) + c_3 y(t), \tag{5}$$

where c_i is a constant for $i = 0, 1, 2, 3$, and to

$$y'(t) = c_0 + c_1 y^2(c_2 t). \tag{6}$$

Explicit and exact solutions to these equations will be obtained.

In Section 3, various differential equations with different arguments will be considered and exact solutions will be provided. We will also solve systems of two coupled linear first-order differential equations with different arguments.

2 Generalized equations

First, we consider Eq. (5). We look for a solution of the form $y(t) = e^{kt}$. Substituting into Eq. (5), we find that the above function is indeed a solution of this DDE if and only if

$$k c_0 + c_2 - k = 0 \quad \text{and} \quad k = c_1 + c_3. \tag{7}$$

Therefore, we can state the following proposition.

Proposition 2.1. *If $c_0 \neq 1$, then the function $y(t) = e^{kt}$ is an exact solution of the linear DDE (5) if and only if*

$$k = \frac{c_2}{1 - c_0} \quad \textit{and} \quad k = c_1 + c_3. \tag{8}$$

Moreover, $y(t)$ satisfies the initial condition $y(0) = 1$.

Remarks. (i) In the case considered in [1], $c_0 = c_1 = c_2 = c_3 = 1/2$. We can check that the two conditions in Proposition 2.1 are indeed satisfied with $k = 1$.

(ii) Equation (1) can be transformed into an approximate ordinary differential equation (ODE) by using Taylor's formula:

$$y\left(\frac{t}{2}\right) = y\left(t - \frac{t}{2}\right) \approx y(t) - \frac{t}{2} y'(t). \tag{9}$$

Equation (1) becomes

$$y'(t) \approx \frac{1}{2} e^{t/2} \left(y(t) - \frac{t}{2} y'(t) \right) + \frac{1}{2} y(t). \tag{10}$$

The solution that satisfies the initial condition $y(0) = 1$ is

$$y_{appr}(t) = \exp \left\{ \int_0^t \frac{2\,(e^{z/2} + 1)}{z\, e^{z/2} + 4} \, \mathrm{d}z \right\}. \tag{11}$$

As can be seen in Figure 1, the functions $y(t) = e^t$ and $y_{appr}(t)$ are very similar in the interval $[0, 1]$.

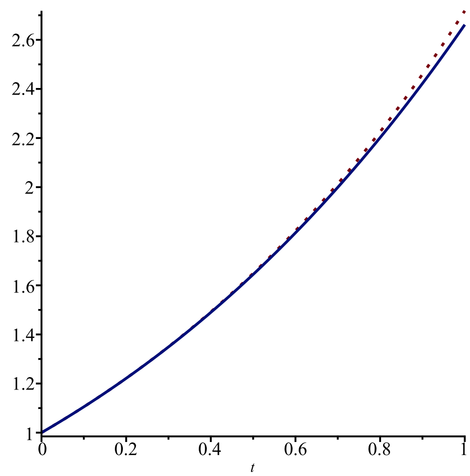

Figure 1. Functions $y(t) = e^t$ (solid line) and $y_{appr}(t)$ defined in Eq. (11) in the interval $[0, 1]$.

Next, we turn to Eq. (6). This time, we look for a solution of the form $y(t) = \sin(kt)$. Substituting $y(t)$ into Eq. (6), we find that we must have

$$\cos^2(k c_2 t) c_1 + k \cos(kt) - c_0 - c_1 = 0. \tag{12}$$

Assume that $c_2 = 1/2$. Then, using the identity

$$\cos(kt) = -1 + 2 \cos^2 \left(\frac{kt}{2} \right), \tag{13}$$

we find that Eq. (12) reduces to

$$\cos^2 \left(\frac{kt}{2} \right) (c_1 + 2k) - k - c_0 - c_1 = 0. \tag{14}$$

Proposition 2.2. *The function $y(t) = \sin(kt)$ is an exact solution of the non-linear DDE (6) with $c_2 = 1/2$ if and only if*

$$k = c_0 = -\frac{c_1}{2}. \tag{15}$$

Moreover, $y(t)$ satisfies the initial condition $y(0) = 0$.

For instance, suppose that $c_1 = -1$. Then, we must have $k = c_0 = 1/2$. Equation (6) becomes

$$y'(t) = \frac{1}{2} - y^2\left(\frac{t}{2}\right). \tag{16}$$

One can check that the function $y(t) = \sin(t/2)$ is indeed a solution of the above equation.

3 Other differential equations with different arguments

1. First, we consider the non-linear first-order differential equation with constant coefficients

$$[y'(t)]^2 = c_1 + c_2\, y(2t) \tag{17}$$

for $0 \leq t \leq 1$, such that $y(0) = 1$. We can write that $y(2t) = y(t + t)$. Hence, Eq. (17) is a DDE with negative delay $\tau = -t$.

We try a solution of the form $y(t) = \cos(kt)$. We have

$$\cos(2kt) = \cos^2(kt) - \sin^2(kt) = 1 - 2\,\sin^2(kt), \tag{18}$$

so that the constants c_1 and c_2 must be such that

$$k^2 \sin^2(kt) = c_1 + c_2 - 2c_2 \sin^2(kt). \tag{19}$$

Hence, we must set $c_1 = -c_2 > 0$ and $k = \sqrt{-2c_2}$. Moreover, the solution $y(t) = \cos(kt)$ satisfies the initial condition $y(0) = 1$.

2. Next, we look for a solution of the non-linear first-order differential equation with non-constant coefficients

$$[c_1 + c_2\, t y'(t)]^2\, y(c_3 t) = 0 \tag{20}$$

for $t \geq 1$, with $y(1) = c \in \mathbb{R}$. Let $y(t) = \ln(kt)$, where $k > 0$. We find that this function is indeed a solution of Eq. (20) for any constant $c_3 > 0$, iff $c_1 + c_2 = 0$. Moreover, the constant k must be equal to e^c.

Assume that $c_3 = 1/2$. Then, using Taylor's formula, we can write that

$$[c_1 + c_2\, t y'(t)]^2 \left[y(t) - \frac{t}{2} y'(t)\right] \simeq 0. \tag{21}$$

This equation, when $c_2 = -c_1 \neq 0$, has two solutions that satisfy the initial condition $y(1) = c$:

$$y(t) = \ln(t) + c \quad \text{and} \quad y(t) = ct^2. \tag{22}$$

The first solution is actually the exact solution, since $\ln(e^c t) = c + \ln(t)$, while the second one is a very poor approximation to the exact solution, unless c is small and t is not too large.

3. Assume now that

$$y'(t) = c_1\, t\, y(c_2 t)\, e^{c_3 t^2} \quad \text{for } t \geq 0, \tag{23}$$

where $c_2 \neq 1$, and $y(0) = 1$. If we let $y(t) = e^{kt^2}$, we obtain that

$$2kt e^{kt^2} = c_1 t e^{(kc_2^2 + c_3)t^2}. \tag{24}$$

Hence, we must have

$$k = \frac{c_1}{2} \quad \text{and} \quad k = \frac{c_3}{1 - c_2^2}. \tag{25}$$

Suppose that $c_1 = 1$, $c_2 = 1/2$ and $c_3 = 3/8$. An exact solution to Eq. (23) is then $y(t) = e^{t^2/2}$. The approximate equation deduced from Taylor's formula is

$$y_1'(t) \approx t \left[y_1(t) - \frac{t}{2} y_1'(t) \right] e^{3t^2/8}. \tag{26}$$

The solution that satisfies the initial condition $y_1(0) = 1$ is

$$y_1(t) = \exp\left\{ \int_0^t \frac{2w e^{3w^2/8}}{w^2 e^{3w^2/8} + 2} \mathrm{d}w \right\}. \tag{27}$$

We see in Figure 2 that, as in the case of the first equation considered in Section 2, the function $y_1(t)$ is a good approximation to the exact solution $y(t) = e^{t^2/2}$ in the interval $[0, 1]$.

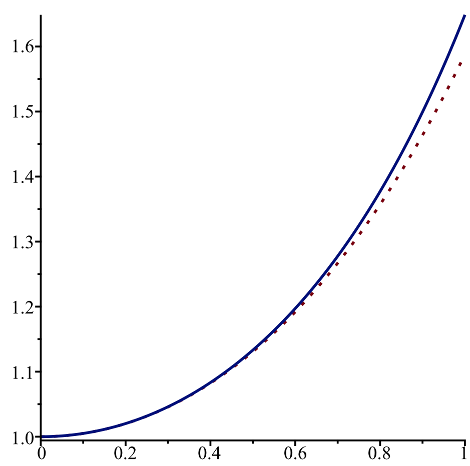

Figure 2. Functions $y(t) = e^{t^2/2}$ (solid line) and $y_1(t)$ defined in Eq. (27) in the interval $[0, 1]$.

4. We consider the system

$$x'(t) = c_1 x(t) + c_2 y(c_3 t), \tag{28}$$
$$y'(t) = c_4 x(c_5 t) + c_6 y(t), \tag{29}$$

for $t \geq 0$, with $x(0) = y(0) = 1$. We assume that c_3 and c_5 are not equal to 1 and are different from 0. If we define $x(t) = e^{k_1 t}$ and $y(t) = e^{k_2 t}$, then we find that we must have

$$k_1 = c_1 + c_2 \quad \text{and} \quad k_2 = c_4 + c_6. \tag{30}$$

Moreover, the relations $k_1 = c_3 k_2$ and $c_3 = 1/c_5$ must hold.

5. Finally, we look for a solution of the system (where c_3 and $c_6 \neq 1$ and $c_3 c_6 \neq 0$)

$$x'(t) = c_1 x(c_3 t) + c_2 y(c_3 t), \tag{31}$$
$$y'(t) = c_4 x(c_6 t) + c_5 y(c_6 t) \tag{32}$$

for $t \geq 0$, such that $x(0) = y(0) = 1$. As in the previous case, we define $x(t) = e^{k_1 t}$ and $y(t) = e^{k_2 t}$. We then find that we must have $c_1 = c_5 = 0$, $k_1 = c_2$ and $k_2 = c_4$, as well as $c_3 = 1/c_6$.

For example, if $c_2 = c_3 = 1/2$, $c_4 = 1$ and $c_6 = 2$, so that

$$x'(t) = \frac{1}{2} y(t/2), \tag{33}$$
$$y'(t) = x(2t), \tag{34}$$

one sees at once that $x(t) = e^{t/2}$ and $y(t) = e^t$ satisfy both equations and the initial conditions.

4 Conclusion

The aim of this paper was to provide exact solutions to differential equations with different arguments. Although the calculations are straightforward, these solutions should be useful to measure the accuracy of the algorithms that many authors have proposed to obtain numerical solutions to delay differential equations.

After having generalized two equations that appear in various papers, we gave explicit solutions to other differential equations with different arguments. We were also able to solve two systems of coupled first-order differential equations with different arguments.

We also saw that by making use of Taylor's formula, we sometimes obtain approximate solutions that are quite accurate (or even exact), at least within a short interval after the initial time.

All the solutions obtained in this paper were elementary functions. It would be interesting to find solutions that are expressed in terms of special functions, such as Bessel functions and hypergeometric functions. We could also consider higher-order equations and/or partial differential equations.

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