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A note on the Kirichenko–Uskorev structure

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Abstract. In terms of Cartan structural equations, a necessary and sufficient condition for an arbitrary almost contact metric structure to belong to the class of Kirichenko–Uskorev structures is established.

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Dedicated to the memory of
Professor Vadim Fedorovich Kirichenko (1947–2021)

1 INTRODUCTION

The most important and significant examples of almost contact metric (acm-) structures are the cosymplectic structure, the nearly cosymplectic structure and structures of Sasaki and Kenmotsu [7]. These structures, as well as their diverse generalizations, are the subject of a wide variety of studies carried out by specialists in the fields of differential geometry and theoretical physics.

As is known, a quadruple $\{\Phi, \xi, \eta, g\}$ of tensor fields is called an almost contact metric structure on an orientable odd-dimensional manifold N^{2n+1} if the following conditions are fulfilled [7]:

$$\eta(\xi) = 1;\ \Phi(\xi) = 0;\ \eta \circ \Phi = 0;\ \Phi^2 = -id + \xi \otimes \eta;$$

$$\langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),\ X, Y \in \aleph(N^{2n+1}).$$

Here, Φ is a field of a tensor of type (1, 1), ξ is a vector field, η is a covector field, $g = \langle \cdot, \cdot\rangle$ is a Riemannian metric, and $\aleph(N^{2n+1})$ is the module of smooth vector fields on the manifold N^{2n+1}. Usually the vector fields ξ, η and Φ are called the characteristic vector field, the contact form, and the structural endomorphism, respectively.

The cosymplectic acm-structure is defined by the condition

$$\nabla\eta = \nabla\Phi = 0,$$

where ∇ is the Riemannian connection of the metric $g = \langle \cdot, \cdot\rangle$. Manifolds equipped with such a structure are locally equivalent to the product of a Kählerian manifold and the real line [9].

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It is also known that the Kenmotsu structure, introduced in the early 1970s, is defined by the following equality [6]:

$$\nabla_X(\Phi)Y = \langle \Phi X, Y\rangle\, \xi - \eta(Y)\Phi X,\ \ X, Y \in \aleph(N^{2n+1}).$$

In 2008 V.F. Kirichenko and I.V. Uskorev have introduced into consideration a new kind of almost contact metric structure, a structure that they called a structure of cosymplectic type [10]. It is defined as an acm-structure with a closed contact form. The main property of an almost contact metric structure of cosymplectic type is its invariance under the so-called canonical conformal transformations [10]. The simplest example of a structure of cosymplectic type is, of course, the cosymplectic structure, and the most important non-trivial example, which largely determines its value in contact geometry, is the Kenmotsu structure. Recently, acm-structures of cosymplectic type are most often studied under the name of Kirichenko–Uskorev structures [2, 3]. We note that examples of Kirichenko–Uskorev structures that differ from both of the cosymplectic structure and the Kenmotsu structure have been presented [1, 3].

It is known [7] that the equalities

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b, \\ d\omega_a &= -\omega^b_a \wedge \omega_b, \\ d\omega &= 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + \omega \wedge \omega^a, \\ d\omega_a &= -\omega^b_a \wedge \omega_b + \omega \wedge \omega_a, \\ d\omega &= 0 \end{aligned} \tag{2}$$

are the first group of Cartan structural equations of cosymplectic and Kenmotsu structures, respectively. V.F. Kirichenko assumed that, most probably, the condition $d\omega = 0$ (it is evidently present in both (1) and (2)) completely characterizes acm-structures of cosymplectic type.

It turned out that V.F. Kirichenko was correct: the above condition is a criterion for an arbitrary acm-structure to belong to the class of Kirichenko–Uskorev almost contact metric structures. We present the following result.

Theorem 1. *Equality $d\omega = 0$ in the first group of Cartan structural equations of an almost contact metric structure is a necessary and sufficient condition for this almost contact metric structure to be a Kirichenko–Uskorev structure.*

The second section of our note contains the proof of this theorem, and the third section contains comments on this result.

2 PROOF OF THE THEOREM

Let us use the first group of structural equations of the Riemannian connection on the space of the associated G-structure written in the frame adapted to the acm-structure [4]:

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^a_b \omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b, \\ d\omega_a &= -\omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B^b_a \omega \wedge \omega_b + B_{ab}\omega \wedge \omega^b, \\ d\omega &= C_{bc}\omega^b \wedge \omega^c + C^{bc}\omega_b \wedge \omega_c + C^b_c \omega^c \wedge \omega_b + C_b \omega \wedge \omega^b + C^b \omega \wedge \omega_b \end{aligned} \tag{3}$$

Here and below, the components of the displacement forms are denoted by $\{\omega^\alpha\}$ and $\{\omega_\alpha\}$ $(\omega_\alpha = \omega^{\widehat{a}},\ \omega^0 = \omega)$; $\{\omega^k_j\}$ are the components of the forms of the Riemannian connection; $k,\ j = 1, \ldots, 2n$; $a,\ b, c = 1, \ldots, n$; $\hat{a} = a + n$; the symbol $[\cdot,\cdot]$ means alternation,

$$B^{ab}_c = -\frac{i}{2}\Phi^a_{\hat{b},c};\ B^{abc} = \frac{i}{2}\Phi^a_{[\hat{b},\hat{c}]};\ B^a_b = i\ \Phi^a_{0,b};$$

$$B^c_{ab} = \frac{i}{2}\Phi^{\hat{a}}_{b,\hat{c}};\ B_{abc} = -\frac{i}{2}\Phi^{\hat{a}}_{[b,c]};\ B^b_a = -i\ \Phi^{\hat{a}}_{0,\hat{b}};$$

$$B^{ab} = i\ \left(\Phi^a_{0,\hat{b}} - \frac{1}{2}\Phi^a_{\hat{b},0}\right);\ B_{ab} = -i\ \left(\Phi^{\hat{a}}_{0,b} - \frac{1}{2}\Phi^{\hat{a}}_{b,0}\right);$$

$$C^a_b = -i\ \left(\Phi^0_{\hat{a},b} + \Phi^0_{b,\hat{a}}\right);\ C^{ab} = i\ \Phi^0_{[\hat{a},\hat{b}]};\ C_{ab} = -i\ \Phi^0_{[a,b]};$$

$$C^a = -i\ \Phi^0_{\hat{a},0};\ C_a = i\ \Phi^0_{a,0}.$$

As noted above, equalities (3) are usually called the first group of structural equations of the acm-structure. Their sometimes more, sometimes less detailed derivation is contained in several papers related to almost contact metric manifolds; in particular, such computation can be found in the article by V.F. Kirichenko and I.V. Uskorev [10] and in the monograph [7].

Let us introduce the notations:

$$C^{abc} = \frac{i}{2}\Phi^a_{\hat{b},\hat{c}};\ C_{abc} = -\frac{i}{2}\Phi^{\hat{a}}_{b,c};\ F^{ab} = i\ \Phi^0_{\hat{a},\hat{b}};\ F_{ab} = -i\ \Phi^0_{a,b}$$

Recall that the systems of functions 1) $F = \left\{F^k_j\right\}$, where $F^a_{\hat{b}} = F^{ab}$, $F^{\hat{a}}_b = F_{ab}$, and all other components of this family are zero; 2) $G = \left\{G^j\right\}$, where $G^a = C^a$, $G^{\hat{a}} = C_a$, $G^0 = 0$ define tensors on the manifold N^{2n+1}. In the V.F. Kirichenko's terminology[7], F and G are the fifth and sixth structural tensors of an almost contact metric structure, respectively.

The contact form η is closed if and only if the following equalities hold:

$$\Phi^0_{[a,b]} = \Phi^0_{[\hat{a},\hat{b}]} = 0;\ \Phi^0_{\hat{a},b} = \Phi^0_{a,\hat{b}} = 0;\ \Phi^0_{a,0} = \Phi^0_{\hat{a},0} = 0.$$

Hence,

$$C^{ab} = 0;\ C_{ab} = 0;\ C^a_b = 0;\ C^a = 0;\ C_a = 0.$$

From this, among other things, it follows that

$$F^{ab} = 0;\ F_{ab} = 0,$$

that is why the fifth and sixth structure tensors of the considered acm-structure of cosymplectic type vanish. We obtain:

$$d\omega = C_{bc}\omega^b \wedge \omega^c + C^{bc}\omega_b \wedge \omega_c + C^b_c\omega^c \wedge \omega_b + C_b\omega \wedge \omega^b + C^b\omega \wedge \omega_b = 0.$$

Conversely, the equality $d\omega = 0$, due to the linear independence of the basic forms, entails the vanishing of the fifth and sixth structure tensors of an almost contact metric structure, which means that the considered almost contact metric structure belongs to the class of Kirichenko–Uskorev structures, Q.E.D.

3 COMMENTS

Remark that since the end of the last century the structural equations of the most important types of almost contact metric structures have been known. Equations (1) and (2) mentioned in the first part of this note, are the structural equations of cosymplectic and Kenmotsu structures (as we noted above, they correspond to the condition for acm-structure to be Kirichenko–Uskorev). We also note that the authors of [10], of course, have proved that the Kenmotsu structure is an acm-structure of cosymplectic type structures in a different way.

Let us now consider the first groups of Cartan structural equations of other most important kinds of acm-structures [4, 7]. We present equations for a nearly cosymplectic structure (also known as Endo structure):

$$\begin{aligned} d\,\omega^a &= \omega^a_b \wedge \omega^b \ + B^{abc}\,\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\,\omega_b \wedge \omega\ ; \\ d\omega_a &= -\omega^b_a \wedge \omega_b \ + B_{abc}\,\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\,\omega^b \wedge \omega\ ; \\ d\omega &= C_{bc}\,\omega^b \wedge \omega^c + C^{bc}\,\omega_b \wedge \omega_c\,, \end{aligned}$$

for Sasakian structure:

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b - i\omega \wedge \omega^a; \\ d\omega_a &= -\omega^b_a \wedge \omega_b + i\omega \wedge \omega_a; \\ d\omega &= -2i\omega^a \wedge \omega_a, \end{aligned}$$

for a structure homothetic to a Sasakian structure (a new approach to the construction of which is contained in the interesting paper [5]):

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b - i\,\lambda\,\omega \wedge \omega^a; \\ d\omega_a &= -\omega^b_a \wedge \omega_b + i\,\lambda\,\omega \wedge \omega_a; \\ d\omega &= -2i\,\lambda\omega^a \wedge \omega_a, \end{aligned}$$

where λ is a nonzero constant, and also for quasi-Sasakian structure:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^a_b \omega \wedge \omega^b\,;$$

$$d\omega_a = -\omega^b_a \wedge \omega_b - B^b_a \omega \wedge \omega_b\,;$$

$$d\omega = 2\,B^a_b\,\omega^b \wedge \omega_a\,.$$

Analyzing these equations, it is easy to come to the conclusion that a nearly cosymplectic and quasi-Sasakian structure belong to the class of Kirichenko–Uskorev structures if and only if these structures are cosymplectic; the other two of these structures cannot be Kirichenko–Uskorev structures.

At the end of our note, we remark that in the last chapter of the monograph [7] devoted to almost contact metric manifolds (in our opinion, the most profound and interesting chapter of this fundamental work), there is no paragraph on manifolds with an acm-structure of cosymplectic type. The explanation is the following: the article [10] was published in 2008, and the monograph of 2013 [7] is just a reprint (with some corrections and minor additions) of the book of 2002 [8].

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