BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(103), 2023, Pages 67–79
ISSN 1024–7696, E-ISSN 2587–4322

Approximation of fixed points in convex G-metric spaces

Nora Fetouci

Abstract. In this paper, we extend some fixed point results for various classes of mappings to approximating fixed points, using Mann iterative process in the context of convex G-metric spaces.

Mathematics subject classification: 47H10, 46B20, 54H25.
Keywords and phrases: fixed point, convex G-metric space, Mann iterative process.

1 Introduction and preliminaries

The concept of G-metric space was introduced by Mustafa and Sims [5] as an extension of the notion of metric spaces in which to every triplet of elements a non-negative real number is assigned. These authors developed the well known Banach contraction principle, and some fixed point results for various classes of mappings in this new context.
Later on, many authors obtained some fixed point results under different conditions, in the setting of G-metric spaces, but without approximating these fixed points. The notion of convex metric spaces was first introduced in 1970 by Takahashi [8], and was investigated by several authors in approximation of fixed points in convex spaces.
Our aim in this paper is to use Mann iterative process in convex G-metric spaces to approximate fixed points for some types of mappings.

Definition 1. [6] Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

DOI: https://doi.org/10.56415/basm.y2023.i3.p67

Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.
Every G-metric on X defines a metric d_G on X by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \text{ for all } x,y \in X.$$

Example 1. [4] Let (X,d) be a metric space. The function $G : X \times X \times X \to [0,+\infty[$, defined as

$$G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$$

or

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\},$$

for all $x,y,z \in X$, is a G-metric on X.

Definition 2. [6] Let (X,G) be a G-metric space. Let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$; that is, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geq k$. We call x the limit of the sequence and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1. *[6] Let (X,G) be a G-metric space. Then the following are equivalent:*

1. *(x_n) is G-convergent to x,*
2. *$G(x_n,x_n,x) \to 0$ as $n \to \infty$,*
3. *$G(x_n,x,x) \to 0$ as $n \to \infty$,*
4. *$G(x_m,x_n,x) \to 0$ as $m,n \to \infty$.*

Definition 3. [6] Let (X,G) be a G-metric space. A sequence (x_n) is called $G-$Cauchy if for each $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $m,n,l \geq k$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2. *[6] Let (X,G) be a G-metric space, then the following conditions are equivalent:*

1. *The sequence (x_n) is G-Cauchy.*
2. *For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$, for all $n,m \geq k$.*

Definition 4. [6] A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Proposition 3. *[6] Let (X,G) be a G-metric space. Then for any x,y,z, and $a \in X$, it follows that*

1. *if* $G(x,y,z)=0$, *then* $x=y=z$
2. $G(x,y,z)\leq G(x,x,y)+G(x,x,z)$,
3. $G(x,y,y)\leq 2G(y,x,x)$,
4. $G(x,y,z)\leq G(x,a,z)+G(a,y,z)$,
5. $G(x,y,z)\leq \frac{2}{3}(G(x,y,a)+G(x,a,z)+G(a,y,z)$,
6. $G(x,y,z)\leq G(x,a,a)+G(y,a,a)+G(z,a,a)$.

Proposition 4. *[6] Let (X,G) be a G-metric space. Then $T:X\to X$ is G-continuous at $x\in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, (Tx_n) is G-convergent to Tx.*

Proposition 5. *[6] Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all the three of its variables*

Theorem 1. *[5] Let (X,G) be a complete G-metric space, and let $T:X\to X$, be a mapping satisfying one of these conditions*

$$G(Tx,Ty,Ty)\leq a[G(x,Ty,Ty)+G(y,Tx,Tx)] \tag{1}$$

or

$$G(Tx,Ty,Ty)\leq a[G(x,x,Ty)+G(y,y,Tx)], \tag{2}$$

for all $x,y\in X$, where $a\in[0,\frac{1}{2}[$, then T has a unique fixed point (say u), and T is G-continuous at u.

Theorem 2. *[5] Let (X,G) be a complete G-metric space, and let $T:X\to X$, be a mapping which satisfies the following conditions, for all $x,y,z\in X$,*

$$G(Tx,Ty,Tz)\leq k\max\left\{\begin{array}{c} G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),\\ G(z,Tz,Tz),G(x,Ty,Ty),\\ G(y,Tz,Tz),G(z,Tx,Tx)\end{array}\right\}, \tag{3}$$

where $k\in[0,\frac{1}{2}[$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Theorem 3. *[5] Let (X,G) be a complete G-metric space, and let $T:X\to X$, be a mapping which satisfies the following conditions, for all $x,y,z\in X$,*

$$G(Tx,Ty,Tz)\leq k\max\left\{\begin{array}{c} G(x,Ty,Ty)+G(y,Tx,Tx),\\ G(y,Tz,Tz)+G(z,Ty,Ty),\\ G(x,Tz,Tz)+G(z,Tx,Tx)\end{array}\right\}, \tag{4}$$

where $k\in[0,\frac{1}{2}[$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Theorem 4. *[5] Let (X,G) be a complete G-metric space, and let $T : X \to X$, be a mapping which satisfies the following conditions, for all $x, y, z \in X$,*

$$G(Tx,Ty,Ty) \leq k \max \left\{ \begin{array}{c} G(y,Ty,Ty) + G(x,Ty,Ty), \\ 2G(y,Tx,Tx) \end{array} \right\}, \tag{5}$$

where $k \in [0, \frac{1}{3}[$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Theorem 5. *[2] Let (X,G) be a complete G-metric space, let $T : X \to X$, be a mapping such that*

$$G(Tx,Ty,Tz) \leq kM(x,y,z) \tag{6}$$

for all x, y, z, where $k \in [0, \frac{1}{2}[$ and

$$M(x,y,z) = \max \left\{ \begin{array}{c} G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ G(y,Tx,Ty), G(x,Tx,z), G(z,T^2x,Tz), \\ G(Tx,T^2x,Tz), G(z,Tx,Ty), G(x,Tx,Tx), \\ G(x,y,z), G(y,Ty,Ty), G(z,Tz,Tz), \\ G(z,Tx,Tx), G(x,Ty,Ty), G(y,Tz,Tz) \end{array} \right\}. \tag{7}$$

Then there is a unique $x \in X$ such that $Tx = x$.

In what follows, we recall Suzuki-type fixed point results in G-metric spaces.

Theorem 6. *[1] Let (X,G) be a complete G-metric space and let T be a mapping on X. Define a strictly decreasing function, η from $[0,1[$ onto $]\frac{1}{2},1]$ by $\eta(r) = \frac{1}{1+r}$. Assume that there exists $r \in [0,1[$ such that for every $x, y \in X$,*

$$\eta(r)G(x,Tx,Tx) \leq G(x,y,y) \text{ implies } \quad G(Tx,Ty,Ty) \leq rG(x,y,y). \tag{8}$$

Then there exists a unique fixed point z of T and $\lim_n T^n(x) = z$ for all $x \in X$. Moreover, T is G-continuous at z.

Definition 5. [8] A convex structure in a metric space (X,d) is a mapping $S_C : X^2 \times [0,1] \to X$ satisfying, for all $x, y, u \in X$ and all $\beta \in [0,1]$,

$$d(S_C(x,y;\beta),u) \leq \beta d(x,u) + (1-\beta)d(y,u). \tag{9}$$

Convex structure was introduced by Modi et al.[3], in the context of G-metric spaces.

Definition 6. [3] Let (X,G) be a G-metric space. A mapping $S_C : X^3 \times]0,1] \to X$ is said to be a convex structure on (X,G) if for each $(x,y,z,\lambda) \in X^3 \times]0,1]$ and for all $u, v \in X$ the condition

$$G(u,v,S_C(x,y,z,\lambda)) \leq \frac{\lambda}{3}G(u,v,x) + \frac{\lambda}{3}G(u,v,y) + \frac{\lambda}{3}G(u,v,z) \tag{10}$$

holds. If S_C is a convex structure on a G-metric space (X,G), then the triplet (X,G,S_C) is called a convex G-metric space.

A more appropriate definition of convex structure in G-metric spaces was given by Yildirim and Khan [9] as follows:

Definition 7. [9] Let (X, G) be a G-metric space. A mapping $S_C : X^2 \times I^2 \to X$ is termed as a convex structure on X if

$$G(S_C(x, y; \lambda, \gamma), u, v) \leq \lambda G(x, u, v) + \gamma G(y, u, v) \tag{11}$$

for real numbers λ and γ in $I = [0, 1]$ satisfying $\lambda + \gamma = 1$ and x, y, u and $v \in X$.
A G-metric space (X, G) with a convex structure S_C is called a convex G-metric space and denoted as (X, G, S_C).
A nonempty subset C of a convex G-metric space (X, G, S_C) is said to be convex if $S_C(x, y; \lambda; \gamma) \in C$ for all $x, y \in C$ and $\lambda, \gamma \in I$.

In the end of this section we give the definition of Mann iterative process in the setting of G-metric space.

Definition 8. [9] Let (X, G, S_C) be a convex G-metric space with a convex structure S_C and
$T : X \to X$ be a mapping. Let (β_n) be a sequence in $[0, 1]$ for $n \in \mathbb{N}$. Then for any given $x_0 \in X$, the iterative process defined by the sequence

$$x_{n+1} = S_C(x_n, Tx_n; 1 - \beta_n, \beta_n), \; n \in \mathbb{N}, \tag{12}$$

is called Mann iterative process in the convex G-metric space (X, G, S_C).
It follows from the structure of convex G-metric space that

$$\begin{aligned} G(x_{n+1}, u, v) &= G(S_C(x_n, Tx_n; 1 - \beta_n, \beta_n), u, v) \\ &\leq (1 - \beta_n)G(x_n, u, v) + \beta_n G(Tx_n, u, v). \end{aligned} \tag{13}$$

2 Main results

Our first main result is the following:

Theorem 7. *Let (X, G, S_C) be a convex G-metric space with a convex structure S_C, and let*
$T : X \to X$, be a mapping satisfying one of these conditions

$$G(Tx, Ty, Ty) \leq a[G(x, Ty, Ty) + G(y, Tx, Tx)] \tag{14}$$

or

$$G(Tx, Ty, Ty)) \leq a[G(x, x, Ty) + G(y, y, Tx)], \tag{15}$$

for all $x, y \in X$, where $a \in [0, \frac{1}{3}[$. Let (x_n) be defined iteratively by (12), $x_0 \in X$, and $(\beta_n) \subset [0, 1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$. Then (x_n) converges strongly to a fixed point of T.

Proof. Using (13) and the fact that u is a fixed point of the mapping T, we have

$$\begin{aligned} G(x_{n+1},u,u) &= G(S_C(x_n,Tx_n;1-\beta_n,\beta_n),u,u) \\ &\leq (1-\beta_n)G(x_n,u,u)+\beta_n G(Tx_n,u,u) \\ &= (1-\beta_n)G(x_n,u,u)+\beta_n G(Tx_n,Tu,Tu). \end{aligned} \tag{16}$$

Applying the inequality (14) to $G(Tx_n,Tu,Tu)$,

$$\begin{aligned} G(Tx_n,Tu,Tu) &\leq a[G(x_n,Tu,Tu)+G(u,Tx_n,Tx_n)] \\ &\leq a[G(x_n,Tu,Tu)+2G(Tx_n,u,u)], \end{aligned} \tag{17}$$

hence from (17) we have:

$$G(Tx_n,u,u) \leq \frac{a}{1-2a}G(x_n,u,u), \tag{18}$$

and so from the inequalities (16) and (18) we obtain

$$\begin{aligned} G(x_{n+1},u,u) &\leq (1-\beta_n)G(x_n,u,u)+\beta_n G(Tx_n,Tu,Tu) \\ &\leq (1-\beta_n)G(x_n,u,u)+\beta_n \frac{a}{1-2a}G(x_n,u,u) \\ &\leq [(1-\beta_n)+\beta_n \frac{a}{1-2a}]G(x_n,u,u) \\ &\leq [1-\beta_n[1-\frac{a}{1-2a}]G(x_n,u,u)] \\ &= [1-\beta_n(1-\delta)]G(x_n,u,u), \end{aligned} \tag{19}$$

where $\delta = \frac{a}{1-2a}$.
Note that $0 \leq \delta < 1$ and

$$G(x_{n+1},u,u) \leq [1-\beta_n(1-\delta)]G(x_n,u,u).$$

Indeed

$$a < \frac{1}{3} \Longrightarrow \frac{a}{1-2a} < 1,$$

furthermore, if $1-2a=0$, then $a<1-2a$, means that $a<0$, which is a contradiction since $a \geq 0$. By induction, we get

$$G(x_{n+1},u,u) \leq \prod_{k=0}^{n}[1-\beta_k(1-\delta)]G(x_0,u,u). \tag{20}$$

Since $\delta < 1$, $\beta_k \in [0,1]$ and $\sum_{n=0}^{\infty}\beta_n = \infty$, we deduce that

$$\lim_{n\to\infty}\prod_{k=0}^{n}[1-\beta_k(1-\delta)] = 0, \tag{21}$$

and (16) implies that

$$\lim_{n\to\infty} G(x_n, u, u) = 0. \tag{22}$$

Thus the sequence (x_n) defined iteratively by (12) converges strongly to the fixed point of T.
Now, applying the inequality (15) to $G(Tx_n, Tu, Tu)$, and using the same method as above, one obtains

$$\begin{aligned} G(x_{n+1}, u, u) &= G(S_C(x_n, Tx_n; 1-\beta_n, \alpha_n), u, u) \\ &\leq (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, u, u) \\ &= (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu) \end{aligned} \tag{23}$$

and

$$\begin{aligned} G(Tx_n, Tu, Tu) &\leq a[G(x_n, x_n, Tu) + G(u, u, Tx_n)] \\ &\leq a[G(x_n, Tu, Tu) + 2G(Tx_n, u, u)], \end{aligned} \tag{24}$$

which is the case of (14).

Our second result is the following

Theorem 8. *Let (X, G, S_C) be a convex G-metric space with a convex structure S_C, and let*
$T : X \to X$, be a mapping which satisfies the following conditions, for all $x, y, z \in X$,

$$G(Tx, Ty, Tz) \leq k \max \left\{ \begin{array}{c} G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), \\ G(z, Tz, Tz), G(x, Ty, Ty), \\ G(y, Tz, Tz), G(z, Tx, Tx) \end{array} \right\}, \tag{25}$$

where $k \in [0, \frac{1}{3}[$. Let (x_n) be defined iteratively by (12), $x_0 \in X$, and $(\beta_n) \subset [0, 1]$ satisfying
$\sum_{n=0}^{\infty} \beta_n = \infty$. Then (x_n) converges strongly to a fixed point of T.

Proof. Regarding (13) and the fact that u is a fixed point of the mapping T, one has

$$\begin{aligned} G(x_{n+1}, u, u) &= G(S_C(x_n, Tx_n; 1-\beta_n, \beta_n), u, u) \\ &\leq (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, u, u) \\ &= (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu). \end{aligned} \tag{26}$$

In view of the inequality (25) and the rectangle inequality, we obtain

$$G(Tx_n, Tu, Tu) \leq k[G(x_n, u, u) + G(u, Tx_n, Tx_n)], \tag{27}$$

then from Proposition 3 we get

$$G(Tx_n, Tu, Tu) \leq kG(x_n, u, u) + 2kG(Tx_n, u, u), \tag{28}$$

so

$$G(Tx_n, Tu, Tu) \leq \frac{k}{1-2k} G(x_n, u, u). \tag{29}$$

By inequalities (26) and (29), we obtain

$$\begin{aligned} G(x_{n+1}, u, u) &\leq (1-\beta_n)G(x_n, u, u) + \beta_n \frac{k}{1-2k} G(x_n, u, u) \\ &= [1-\beta_n(1-\frac{k}{1-2k})]G(x_n, u, u) \\ &= [1-\beta_n(1-\delta)]G(x_n, u, u), \end{aligned} \tag{30}$$

where $\delta = \frac{k}{1-2k}$, one can see that $0 \leq \delta < 1$ and $1-2k \neq 0$. Indeed,

$$k \leq \frac{1}{3} \Longrightarrow k < 1-2k \Longrightarrow \frac{k}{1-2k} < 1,$$

in addition, if $1-2k \neq 0$, then according to above, $k < 1-2k$ leads to $k < 0$ which is a contradiction.
Next, similarly to the proof of theorem above, we obtain

$$\lim_{n\to\infty} G(x_n, u, u) = 0,$$

which completes the proof. □

Our third result is given by

Theorem 9. *Let (X, G, S_C) be a convex G-metric space with a convex structure S_C, and let*
$T : X \to X$, be a mapping which satisfies the following conditions, for all $x, y, z \in X$,

$$G(Tx, Ty, Tz) \leq k \max \left\{ \begin{array}{c} G(x, Ty, Ty) + G(y, Tx, Tx), \\ G(y, Tz, Tz) + G(z, Ty, Ty), \\ G(x, Tz, Tz) + G(z, Tx, Tx) \end{array} \right\}, \tag{31}$$

where $k \in [0, \frac{1}{2}[$. Let (x_n) be defined iteratively by (12), $x_0 \in X$, and $(\beta_n) \subset [0, 1]$ satisfying
$\sum_{n=0}^{\infty} \beta_n = \infty$. Then (x_n) converges strongly to a fixed point of T.

Proof. By (13) and the fact that u is a fixed point of the mapping T, one has

$$\begin{aligned} G(x_{n+1}, u, u) &= G(S_C(x_n, Tx_n; 1-\beta_n, \beta_n), u, u) \\ &\leq (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, u, u) \\ &= (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu). \end{aligned} \tag{32}$$

By inequality (31), we obtain

$$G(Tx_n, Tu, Tu) \leq k \max \left\{ \begin{array}{l} G(x_n, Tu, Tu) + G(u, Tx_n, Tx_n), \\ G(u, Tu, Tu) + G(u, Tu, Tu), \\ G(x_n, Tu, Tu) + G(u, Tx_n, Tx_n) \end{array} \right\},$$

so

$$G(Tx_n, Tu, Tu) \le k[G(x_n, u, u) + G(u, Tx_n, Tx_n)], \tag{33}$$

and hence the result follows from the same argument as in Theorem 8. ☐

We give by the sequel our fourth result.

Theorem 10. *Let (X, G, S_C) be a convex G-metric space with a convex structure S_C, and let*
$T : X \to X$, be a mapping which satisfies the following conditions, for all $x, y, z \in X$,

$$G(Tx, Ty, Ty) \le k \max \left\{ \begin{array}{c} G(y, Ty, Ty) + G(x, Ty, Ty), \\ 2G(y, Tx, Tx) \end{array} \right\} \tag{34}$$

where $k \in [0, \frac{1}{4}[$. Let (x_n) be defined iteratively by (12), $x_0 \in X$, and $(\beta_n) \subset [0, 1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$. Then (x_n) converges strongly to a fixed point of T.

Proof. In view of (13) and the fact that u is a fixed point of the mapping T, one has

$$\begin{aligned} G(x_{n+1}, u, u) &= G(S_C(x_n, Tx_n; 1 - \beta_n, \beta_n), u, u) \\ &\le (1 - \beta_n)G(x_n, u, u) + \beta_n G(Tx_n, u, u) \\ &= (1 - \beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu). \end{aligned} \tag{35}$$

Using inequality (34), we obtain

$$G(Tx_n, Tu, Tu) \le k \max \left\{ \begin{array}{l} G(u, Tu, Tu) + G(x_n, Tu, Tu), \\ 2G(u, Tx_n, Tx_n) \end{array} \right\}, \tag{36}$$

it follows that

$$G(Tx_n, Tu, Tu) \le k \max \left\{ \begin{array}{l} G(x_n, Tu, Tu), \\ 4G(Tx_n, u, u) \end{array} \right\}. \tag{37}$$

There are two distinct cases: for the first case, assume that

$$G(Tx_n, Tu, Tu) \le 4kG(Tx_n, u, u).$$

It is a contradiction since $0 \le k < \frac{1}{4}$.
For the second case, suppose that $G(Tx_n, Tu, Tu) \le 4kG(x_n, u, u)$.
Then, the expression (35) turns into

$$\begin{aligned} G(x_{n+1}, u, u) &\le (1 - \beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu) \\ &\le (1 - \beta_n)G(x_n, u, u) + \beta_n 4kG(x_n, u, u) \\ &= [1 - \beta_n(1 - 4k)]G(x_n, u, u) \\ &= [1 - \beta_n(1 - \delta)]G(x_n, u, u), \end{aligned} \tag{38}$$

it is easily seen that $0 \le \delta < 1$. Indeed, we have $k < \frac{1}{4} \Longrightarrow 4k < 1$.
Using the same argument as in the proof of Theorem 8, one can deduce that the sequence (x_n) defined iteratively by (12) converges strongly to the fixed point of T. ☐

Theorem 11. *Let (X, G, S_C) be a convex G-metric space with a convex structure S_C, and let*
$T : X \to X$, be a mapping which satisfies the following conditions,

$$G(Tx, Ty, Tz) \leq kM(x, y, z) \tag{39}$$

for all x, y, z, where $k \in [0, \frac{1}{6}[$ and

$$M(x,y,z) = \max\left\{\begin{array}{c} G(x,Tx,y), G(y,T^2x,Ty), G(Tx,T^2x,Ty), \\ G(y,Tx,Ty), G(x,Tx,z), G(z,T^2x,Tz), \\ G(Tx,T^2x,Tz), G(z,Tx,Ty), G(x,Tx,Tx), \\ G(x,y,z), G(y,Ty,Ty), G(z,Tz,Tz), \\ G(z,Tx,Tx), G(x,Ty,Ty), G(y,Tz,Tz) \end{array}\right\}. \tag{40}$$

Let (x_n) be defined iteratively by (12), $x_0 \in X$, and $(\beta_n) \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$. Then (x_n) converges strongly to a fixed point of T.

Proof. Employing (13) and the fact that u is a fixed point of the mapping T, one has

$$\begin{aligned} G(x_{n+1}, u, u) &= G(S_C(x_n, Tx_n; 1-\beta_n, \beta_n), u, u) \\ &\leq (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, u, u) \\ &= (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu). \end{aligned} \tag{41}$$

Applying inequality (40), we obtain

$$G(Tx_n, Tu, Tu) \leq 2k \max\left\{\begin{array}{c} G(u,Tu,x_n), G(x_n,T^2u,Tx_n), G(Tu,T^2u,Tx_n), \\ G(x_n,Tu,Tx_n), G(u,Tu,x_n), G(x_n,T^2u,Tx_n), \\ G(Tu,T^2u,Tx_n), G(x_n,Tu,Tx_n), G(u,x_n,x_n), \\ G(x_n,Tx_n,Tx_n), G(u,Tu,Tu), G(x_n,Tx_n,Tx_n), \\ G(u,Tx_n,Tx_n), G(x_n,Tu,Tu), G(x_n,Tx_n,Tx_n) \end{array}\right\}, \tag{42}$$

this entails

$$G(Tx_n, Tu, Tu) \leq 2k \max\left\{\begin{array}{c} G(u,Tx_n,x_n), G(u,u,Tx_n), G(u,x_n,x_n) \\ G(x_n,Tx_n,Tx_n), G(u,Tx_n,Tx_n), G(x_n,u,u) \end{array}\right\}. \tag{43}$$

Now, we have to examine six cases:

1. $G(Tx_n, u, u) \leq 2kG(Tx_n, u, u)$, leads to a contradiction, since $0 \leq k < \frac{1}{6}$,
2. $G(Tx_n, u, u) \leq 2kG(Tx_n, x_n, u)$, from the rectangle inequality we get $G(Tx_n, u, u) \leq 2k[G(Tx_n, u, u) + G(u, u, x_n)]$ which implies that

$$G(Tx_n, u, u) \leq \frac{2k}{1-2k} G(u, u, x_n).$$

Then, the expression (41) turns into

$$\begin{aligned} G(x_{n+1},u,u) &\leq (1-\beta_n)G(x_n,u,u)+\beta_n G(Tx_n,Tu,Tu) \\ &\leq (1-\beta_n)G(x_n,u,u)+\beta_n \frac{2k}{1-2k}G(x_n,u,u) \\ &= [(1-\beta_n(1-\frac{2k}{1-2k})]G(x_n,u,u) \\ &= [1-\beta_n(1-\delta)]G(x_n,u,u), \end{aligned} \tag{44}$$

3. $G(Tx_n,u,u) \leq 2kG(x_n,Tx_n,Tx_n)$, by the rectangle inequality and Proposition 3 we get

$$\begin{aligned} G(Tx_n,u,u) &\leq 2k[G(x_n,u,u)+G(u,Tx_n,Tx_n)] \\ &\leq 2k[G(x_n,u,u)+2G(Tx_n,u,u)] \\ &= 2kG(x_n,u,u)+4kG(Tx_n,u,u), \end{aligned}$$

so

$$G(Tx_n,u,u) \leq \frac{2k}{1-4k}G(x_n,u,u),$$

hence

$$\begin{aligned} G(x_{n+1},u,u) &\leq (1-\beta_n)G(x_n,u,u)+\beta_n G(Tx_n,Tu,Tu) \\ &\leq (1-\beta_n)G(x_n,u,u)+\beta_n \frac{2k}{1-4k}G(x_n,u,u) \\ &= [1-\beta_n(1-\frac{2k}{1-4k})]G(x_n,u,u) \\ &= [1-\beta_n(1-\delta)]G(x_n,u,u). \end{aligned} \tag{45}$$

4. $G(Tx_n,u,u) \leq 2kG(u,Tx_n,Tx_n)$, using Proposition 3 we obtain

$$G(Tx_n,u,u) \leq 4kG(Tx_n,u,u),$$

leads to a contradiction.

5. $G(Tx_n,u,u) \leq 2kG(x_n,u,u)$, thus

$$\begin{aligned} G(x_{n+1},u,u) &\leq (1-\beta_n)G(x_n,u,u)+\beta_n G(Tx_n,Tu,Tu) \\ &\leq (1-\beta_n)G(x_n,u,u)+\beta_n 2kG(x_n,u,u) \\ &= [1-\beta_n(1-2k)]G(x_n,u,u) \\ &= [1-\beta_n(1-\delta)]G(x_n,u,u), \end{aligned} \tag{46}$$

it is easily seen that in above cases $0 \leq \delta < 1$, because $k < \frac{1}{6}$.

6. $G(Tx_n, u, u) \leq 2kG(u, x_n, x_n)$, using Proposition 3 we obtain $G(Tx_n, u, u) \leq 4kG(x_n, u, u)$, thus

$$\begin{aligned} G(x_{n+1}, u, u) &\leq (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu) \\ &\leq (1-\beta_n)G(x_n, u, u) + \beta_n 4kG(x_n, u, u) \\ &= [1-\beta_n(1-4k)]G(x_n, u, u) \\ &= [1-\beta_n(1-\delta)]G(x_n, u, u), \end{aligned} \tag{47}$$

it is easily seen that in above cases $0 \leq \delta < 1$, because $k < \frac{1}{6}$.

Using the same argument as in the proof of Theorem 8, one can conclude that the sequence (x_n) defined iteratively by (12) converges strongly to the fixed point of T. □

Our last result is the following

Theorem 12. *Let (X, G, S_C) be a convex G-metric space with a convex structure S_C, and let*
$T : X \to X$, be a mapping on X. Define a strictly decreasing function η from $[0, \frac{1}{4}[$ onto $]\frac{4}{5}, 1]$ by $\eta(r) = \frac{1}{1+r}$. Assume that there exists $r \in [0, 1[$ such that for every $x, y \in X$,

$$\eta(r)G(x, Tx, Tx) \leq G(x, y, y) \text{ implies } \quad G(Tx, Ty, Ty) \leq rG(x, y, y). \tag{48}$$

Let (x_n) be defined iteratively by (12), $x_0 \in X$, and $(\beta_n) \subset [0, 1]$ satisfying $\sum_{n=0}^{\infty} \beta_n = \infty$. Then (x_n) converges strongly to a fixed point of T.

Proof. Using (13), and since u is a fixed point of the mapping T, we have

$$\begin{aligned} G(x_{n+1}, u, u) &= G(S_C(x_n, Tx_n; 1-\beta_n, \beta_n), u, u) \\ &\leq (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, u, u) \\ &= (1-\beta_n)G(x_n, u, u) + \beta_n G(Tx_n, Tu, Tu) \\ &\leq (1-\beta_n)G(x_n, u, u) + 2\beta_n G(Tu, Tx_n, Tx_n), \end{aligned} \tag{49}$$

since

$$\eta(r)G(u, Tu, Tu) \leq G(u, x_n, x_n),$$

then by hypothesis we get

$$G(Tu, Tx_n, Tx_n) \leq rG(u, x_n, x_n),$$

thus, (49) implies that

$$\begin{aligned} G(x_{n+1}, u, u) &\leq (1-\beta_n)G(x_n, u, u) + 2\alpha_n G(Tu, Tx_n, Tx_n) \\ &\leq (1-\beta_n)G(x_n, u, u) + 2\beta_n rG(u, x_n, x_n) \\ &\leq (1-\beta_n)G(x_n, u, u) + 4r\beta_n G(x_n, u, u) \end{aligned} \tag{50}$$

$$= \quad [1-\beta_n(1-4r)]G(x_n,u,u),$$

as $r < \frac{1}{4}$, so $0 \leq 4r < 1$. Using the same method as in the proof of Theorem 8, one can conclude that the sequence (x_n) defined iteratively by (12) converges strongly to the fixed point of T.

□

References

[1] M. Aggarwal, R. Chugh and R. Kamal, *Suzuki-fixed point results in G-metric spaces and applications*, Int. J. Comp. Appl., **47**(12), (2012), 14-17.

[2] E. Karapinar and R. Agarwal, *Further fixed point results on G-metric spaces*, Fixed Point Theory. Appl., **154**(2013), 1-14.

[3] G. Modi and B. Bhatt, *Fixed point results for weakly compatible mappings in convex G-metric space*, Int. J. Maths. Stat. Inven., **2**(11) (2014), 34-38.

[4] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory*, PHD Thesis, the University of Newcastle, Australia, 2005.

[5] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory. Appl., **2008**(2008), 1-12.

[6] Z. Mustafa, and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear and Convex Anal., **7**(2) (2006), 289-297.

[7] Z. Mustafa and B. Sims *Fixed point theorems for contractive mappings in G-metric spaces*, Fixed Point Theory. Appl., **2009**(2009), 1-10.

[8] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, Kodai Math. Sem. Rep., **22** (1970) 142-149.

[9] I. Yildirim and S. H. Khan, *Convexity in G-metric spaces and approximation of fixed points by Mann iterative process*, Int. J. Nonlinear Anal. Appl., **13**(2022) 1, 1957-1964.

Nora Fetouci
LMPA Laboratory, Department of
Mathematics, Jijel University, Algeria
E-mail: *n.fetouci@univ-jijel.dz*

Received October 24, 2022