Some subsets in bitopological spaces and bioperations continuity associated

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Abstract. The aim of this paper is to introduce and study some new types of sets via bioperations and to define and characterize some new forms of bioperations continuity using these sets and their relationships.

Mathematics subject classification: 54C05, 54C08, 54E55.
Keywords and phrases: Topological spaces, (γ, γ')-α-open set.

1 Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. Kasahara [1] defined the concept of an operation on topological spaces. Umehara et al.[6] introduced the notion of $\tau_{(\gamma,\gamma')}$, which is the collection of all (γ, γ')-open sets in a topological space (X, τ). Carpintero et al. (see [3, 4] and [5]) obtained new decomposition forms of bioperation continuity. The aim of this paper is to introduce and study some new types of sets via bioperations and introducing new associated bioperation continuity with these new sets and relationships between them.

2 Preliminaries

The closure and the interior of a subset A of (X, τ) are denoted by (γ, γ')-Cl(A) and (γ, γ')-Int(A), respectively.

Definition 2.1. *[1] Let (X, τ) be a topological space. An operation γ on the topology τ is a function from τ onto power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of τ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.*

Definition 2.2. *[6] A topological space (X, τ) equipped with two operations, say γ and γ', defined on τ is called a bioperation topological space, it is denoted by $(X, \tau, \gamma, \gamma')$.*

Definition 2.3. *[6] A subset A of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be (γ, γ')-open set if for each $x \in A$ there exist open neighborhoods U and V of x such that $U^\gamma \cup V^{\gamma'} \subset A$. The complement of a (γ, γ')-open set is called a (γ, γ')-closed set. $\tau_{(\gamma, \gamma')}$ denotes the set of all (γ, γ')-open sets in $(X, \tau, \gamma, \gamma')$.*

Definition 2.4. *[6] For a subset A of a bioperation topological space $(X, \tau, \gamma, \gamma')$, (γ, γ')-Cl(A) denotes the intersection of all (γ, γ')-closed sets containing A, that is, (γ, γ')-Cl$(A) = \cap \{F : A \subset F, X \backslash F \in \tau_{(\gamma, \gamma')}\}$.*

Definition 2.5. *Let A be any subset of X. The (γ, γ')-Int(A) is defined as (γ, γ')-Int$(A) = \cup \{U : U$ is a (γ, γ')-open set and $U \subset A\}$.*

Definition 2.6. *Let (X, τ) be a topological space and A be a subset of X and γ and γ' be operations on τ. Then A is said to be*

1. *(γ, γ')-α-open if $A \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(A)))$,*

2. *(γ, γ')-preopen [2] if $A \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(A))$,*

3. *(γ, γ')-preclosed if (γ, γ')-Cl$((\gamma, \gamma')$-Int$(A)) \subset A$,*

4. *(γ, γ')-semi-open if $A \subset (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(A))$,*

5. *(γ, γ')-semi-closed if (γ, γ')-Int$((\gamma, \gamma')$-Cl$(A)) \subset A$,*

6. *(γ, γ')-semi-preopen if $A \subset (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(A)))$,*

7. *(γ, γ')-semi-preclosed if (γ, γ')-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(A))) \subset A$,*

8. *(γ, γ')-regular open if $A = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(A))$,*

9. *(γ, γ')-regular closed if $A = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(A))$,*

10. *(γ, γ')-t-set if (γ, γ')-Int$((\gamma, \gamma')$-Cl$(S)) = (\gamma, \gamma')$-Int(S),*

11. *(γ, γ')-α^*-set if (γ, γ')-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(S))) = (\gamma, \gamma')$-Int$(S)$,*

12. *(γ, γ')-$\mathcal{A}$-set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and H is a (γ, γ')-regular closed set,*

13. *(γ, γ')-$\mathcal{B}$-set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and H is a (γ, γ')-t-set,*

14. *(γ, γ')-$\mathcal{C}$-set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and H is a (γ, γ')-α^*-set.*

15. *(γ, γ')-locally closed set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and H is a (γ, γ')-closed set.*

The family of all (γ, γ')-α-open (resp. (γ, γ')-semiopen, (γ, γ')-preopen, (γ, γ')-semi-preopen, (γ, γ')-regular open, (γ, γ')-regular closed, (γ, γ')-$\mathcal{A}$, (γ, γ')-$\mathcal{B}$, (γ, γ')-$\mathcal{C}$, (γ, γ')-locally closed) sets of $(X, \tau, \gamma, \gamma')$ is denoted by $\alpha O(X)$ (resp. $SO(X)$, $PO(X)$, $SPO(X)$, $RO(X)$, $RC(X)$, $\mathcal{A}(X)$, $\mathcal{B}(X)$, $\mathcal{C}(X)$, $\mathcal{LC}(X)$).

Definition 2.7. *A function* $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ *is said to be*

1. (γ, γ')-(β, β')-*continuous if* $f^{-1}(V)$ *is* (γ, γ')-*open in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

2. (γ, γ')-(β, β')-α-*continuous if* $f^{-1}(V)$ *is* (γ, γ')-α-*open in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

3. (γ, γ')-(β, β')-*precontinuous if* $f^{-1}(V)$ *is* (γ, γ')-*preopen in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

4. (γ, γ')-(β, β')-*semi-continuous if* $f^{-1}(V)$ *is* (γ, γ')-*semi-open in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

5. (γ, γ')-(β, β')-*semi-precontinuous if* $f^{-1}(V)$ *is* (γ, γ')-*semi-preopen in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

6. (γ, γ')-(β, β')-*completely continuous if* $f^{-1}(V)$ *is* (γ, γ')-*regular open in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

7. (γ, γ')-(β, β')-$\mathcal{A}$-*continuous if* $f^{-1}(V)$ *is* (γ, γ')-$\mathcal{A}$-*set in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

8. (γ, γ')-(β, β')-$\mathcal{B}$-*continuous if* $f^{-1}(V)$ *is* (γ, γ')-$\mathcal{B}$-*set in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

9. (γ, γ')-(β, β')-$\mathcal{C}$-*continuous if* $f^{-1}(V)$ *is* (γ, γ')-$\mathcal{C}$-*set in* X *for every* $V \in \sigma_{(\beta,\beta)}$,

10. (γ, γ')-(β, β')-$\mathcal{LC}$-*continuous if* $f^{-1}(V)$ *is* (γ, γ')-$\mathcal{LC}$-*set in* X *for every* $V \in \sigma_{(\beta,\beta)}$.

3   Some subsets in bioperation spaces

Definition 3.1. *A subset* F *of a bioperation topological space* $(X, \tau, \gamma, \gamma')$ *is said to be:*

1. *a* (γ, γ')-αA_1-*set if* $F = G \cap H$, *where* G *is a* (γ, γ')-α-*open set and* H *is a* (γ, γ')-*regular closed set,*

2. *a* (γ, γ')-αA_2-*set if* $F = G \cap H$, *where* G *is a* (γ, γ')-α-*open set and* H *is a* (γ, γ')-*locally closed set,*

3. *a* (γ, γ')-αA_3-*set if* $F = G \cap H$, *where* G *is a* (γ, γ')-α-*open set and* H *is a* (γ, γ')-t-*set,*

4. *a* (γ, γ')-αA_4-*set if* $F = G \cap H$, *where* G *is a* (γ, γ')-α-*open set and* H *is a* (γ, γ')-*pre-closed set,*

5. *a* (γ, γ')-αA_5-*set if* $F = G \cap H$, *where* G *is a* (γ, γ')-α-*open set and* H *is a* (γ, γ')-*semi-preclosed set.*

The families of all (γ, γ')-αA_i subsets of $(X, \tau, \gamma, \gamma')$ are denoted by $\alpha A_i(X)$, where $i = 1, 2, 3, 4, 5$.

Proposition 3.2. *For a bioperation topological space $(X, \tau, \gamma, \gamma')$, we have the following:*

1. *Every (γ, γ')-αA_1-set is a (γ, γ')-αA_2-set.*

2. *Every (γ, γ')-αA_2-set is a (γ, γ')-αA_3-set.*

3. *Every (γ, γ')-αA_3-set is a (γ, γ')-αA_5-set.*

4. *Every (γ, γ')-αA_2-set is a (γ, γ')-αA_4-set.*

5. *Every (γ, γ')-αA_4-set is a (γ, γ')-αA_5-set.*

6. *The notions of (γ, γ')-αA_3-sets and (γ, γ')-αA_4-sets are independent.*

Proof. (1). Let $F = G \cap H \in \alpha A_1(X)$, where $G \in \alpha O(X)$ and (γ, γ')-Cl$((\gamma, \gamma')$-Int$(H)) = H$. Since (γ, γ')-Cl$((\gamma, \gamma')$-Int$(H)) = (\gamma, \gamma')$-Cl$(H) = H$, we obtain $F \in \alpha A_2(X)$.
The proofs of items (2), (3), (4) and (5) are similar. In order to prove (6) it is necessary to find counterexamples and we will write them later. $\square$

The converses of Proposition 3.2 are not true as shown in the following examples.

Example 3.3. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. We define the operations $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ as follows*
$$A^\gamma = \begin{cases} A & \text{if } a \in A, \\ A \cup \{a\} & \text{if } a \notin A, \end{cases} \quad \text{and} \quad A^{\gamma'} = \begin{cases} A & \text{if } a \in A, \\ Cl(A) & \text{if } a \notin A. \end{cases}$$
Observe that:

1. $\tau_{(\gamma, \gamma')} = \{\emptyset, \{a\}, \{a, b\}, X\}$.

2. (γ, γ')-αA_1-set $= \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$.

3. (γ, γ')-αA_2-set $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Clearly, $\{b\} \in \alpha A_2(X) \backslash \alpha A_1(X)$.

Example 3.4. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. We define the operations $\gamma, \gamma' : \tau \longrightarrow \mathcal{P}(X)$ as follows*
$$A^\gamma = \begin{cases} A & \text{if } b \notin A, \\ Cl(A) & \text{if } b \in A, \end{cases} \quad \text{and} \quad A^{\gamma'} = \begin{cases} Cl(A) & \text{if } b \notin A, \\ A & \text{if } b \in A. \end{cases}$$
Observe that:

1. $\tau_{(\gamma, \gamma')} = \{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}$.

2. (γ, γ')-αA_2-set $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.

3. (γ, γ')-αA_4-set $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$

Clearly, $\{b, c\} \in \alpha A_4(X) \backslash \alpha A_2(X)$.

Example 3.5. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\},$
$\{a, b, c\}, X\}$, γ *and* γ' *be the identity operations. Then observe that:*

1. $\tau_{(\gamma, \gamma')} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$.

2. $(\gamma, \gamma')\text{-}\alpha A_3\text{-}set = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$
   $\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

3. $(\gamma, \gamma')\text{-}\alpha A_4\text{-}set = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$
   $\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$.

4. $(\gamma, \gamma')\text{-}\alpha A_5\text{-}set = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$
   $\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Clearly:

1. $\{a, c, d\} \in \alpha A_3(X) \backslash \alpha A_4(X)$.

2. $\{a, c, d\} \in \alpha A_5(X) \backslash \alpha A_4(X)$.

Example 3.6. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\},$
$\{a, b, c\}, \{a, b, d\}, X\}$. *We define the operations* $\gamma = \gamma' : \tau \to \mathcal{P}(X)$ *as follows*
$A^\gamma = \begin{cases} A & \text{if } c \notin A, \\ X & \text{if } c \in A, \end{cases}$ *and* $A^{\gamma'} = \begin{cases} A & \text{if } A \neq \{a\}, \\ X & \text{if } A = \{a\}, \end{cases}$
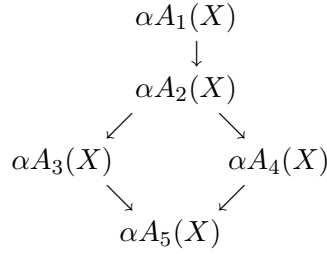*Observe that:*

1. $\tau_{(\gamma, \gamma')} = \{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}, X\}$.

2. $(\gamma, \gamma')\text{-}\alpha A_3\text{-}set = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, X\}$.

3. $(\gamma, \gamma')\text{-}\alpha A_4\text{-}set = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$
   $\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

4. $(\gamma, \gamma')\text{-}\alpha A_5\text{-}set = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$
   $\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Clearly,

1. $\{b, c, d\} \in \alpha A_4(X) \backslash \alpha A_3(X)$.

2. $\{b, c, d\} \in \alpha A_5(X) \backslash \alpha A_3(X)$.

We have the following implication diagram.

Remark 3.7. *For a bioperation topological space* $(X, \tau, \gamma, \gamma')$, *we have the following implication diagram:*

$$\alpha A_1(X)$$
$$\downarrow$$
$$\alpha A_2(X)$$
$$\swarrow \qquad \searrow$$
$$\alpha A_3(X) \qquad\qquad \alpha A_4(X)$$
$$\searrow \qquad \swarrow$$
$$\alpha A_5(X)$$

Definition 3.8. *A subset A of bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be (γ, γ')-dense if (γ, γ')-Cl$(A) = X$.*

Definition 3.9. *A bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be*

1. *(γ, γ')-submaximal if each (γ, γ')-dense set is a (γ, γ')-open set.*

2. *(γ, γ')-extremally disconnected if the (γ, γ')-closure of every (γ, γ')-open set is (γ, γ')-open.*

Theorem 3.10. *If $(X, \tau, \gamma, \gamma')$ is a (γ, γ')-extremally disconnected space, then $\alpha A_5(X) = \alpha A_4(X)$.*

Proof. Let $F = G \cap H \in \alpha A_5(X)$, where $G \in \alpha O(X)$ and (γ, γ')-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) \subset H$. Since $(X, \tau, \gamma, \gamma')$ is extremally disconnected, (γ, γ')-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))$ and so (γ, γ')-Cl$((\gamma, \gamma')$-Int$(H)) \subset H$. As a consequence, we obtain $\alpha A_5(X) \subset \alpha A_4(X)$ and hence $\alpha A_5(X) = \alpha A_4(X)$. $\qquad\square$

Proposition 3.11. *Let $(X, \tau, \gamma, \gamma')$ be a bioperation topological space. Then*

1. *$RO(X) \subset \mathcal{LC}(X)$.*

2. *$\mathcal{LC}(X) \subset \mathcal{B}(X)$.*

3. *$\mathcal{B}(X) \subset \mathcal{C}(X)$.*

Proof. (1). Let $F = G \cap H \in \mathcal{A}(X)$, where G is (γ, γ')-open and (γ, γ')-Cl$((\gamma, \gamma')$-Int$(H)) = H$. Then we obtain (γ, γ')-Cl$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = (\gamma, \gamma')$-Cl$(H) = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))$ and so $H = (\gamma, \gamma')$-Cl(H). As a consequence, F is (γ, γ')-locally closed.
(2). It is obvious, as every (γ, γ')-closed set is a (γ, γ')-t-set.
(3). It is clear, as every (γ, γ')-t-set is a (γ, γ')-t^*-set. $\qquad\square$

Proposition 3.12. *For a bioperation topological space $(X, \tau, \gamma, \gamma')$, we have the following:*

1. *Every (γ, γ')-regular open set is a (γ, γ')-αA_1-set.*

2. *Every (γ, γ')-locally closed set is a (γ, γ')-αA_2-set.*

3. *Every (γ, γ')-$\mathcal{C}$-set is a (γ, γ')-αA_5-set.*

Proof. It is obvious since every (γ, γ')-open set is a (γ, γ')-α-open set. $\qquad\square$

The following examples show that the converse of Proposition 3.12, in general is not true.

Example 3.13. *In Example 3.3, $RO(X) = \{\emptyset, X\}$ and then $\{a\} \in \alpha A_1(X) \backslash RO(X)$.*

Example 3.14. *In Example 3.3, the (γ, γ')-locally closed set $= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$, then $\{a, c\}$ is a (γ, γ')-$\alpha A_2(X)$-set that is not a (γ, γ')-locally closed set.*

Example 3.15. *In Example 3.5, $\mathcal{C} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$, and then $\{a, b, d\} \in \alpha A_5(X) \backslash \mathcal{C}(X)$.*

Theorem 3.16. *Let $(X, \tau, \gamma, \gamma')$ be a (γ, γ')-submaximal space. Then the following hold:*

1. *$\mathcal{A}(X) = \alpha A_1(X)$.*

2. *$\mathcal{LC}(X) = \alpha A_2(X)$.*

3. *$\mathcal{B}(X) = \alpha A_3(X)$.*

4. *$\mathcal{C}(X) = \alpha A_5(X)$.*

Proof. It is clear that in a (γ, γ')-submaximal space, $\tau_{(\gamma, \gamma')} = \alpha O(X)$. $\qquad\square$

Proposition 3.17. *Let $(X, \tau, \gamma, \gamma')$ be a bioperation topological space.*

1. *If $F \in SO(X)$ and $G \in \alpha O(X)$, then $F \cap G \in SO(X)$.*

2. *If $F \in PO(X)$ and $G \in \alpha O(X)$, then $F \cap G \in PO(X)$.*

Theorem 3.18. *A subset F of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is a (γ, γ')-regular open set if and only if it is a (γ, γ')-semi-open set and a (γ, γ')-locally closed set.*

Proof. Let $F = G \cap H \in A(X)$, where G is (γ, γ')-open and (γ, γ')-Cl$((\gamma, \gamma')$-Int$(H)) = H$. Then we obtain (γ, γ')-Cl$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H)) = (\gamma, \gamma')$-Cl$(H) = H$. Also (γ, γ')-Int$(F) = G \cap (\gamma, \gamma')$-Int$(H)$ and $F = G \cap (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H)) = (\gamma, \gamma')$-Cl$(G \cap (\gamma, \gamma')$-Int$(H))$. Hence $F \subset (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(F))$. As a consequence, F is (γ, γ')-semi-open set. Conversely, let F be (γ, γ')-semi-open set and (γ, γ')-locally closed. Then $F = G \cap H$, where G is (γ, γ')-open and (γ, γ')-Cl$(H) = H$. Since F is (γ, γ')-semi-open, $F \subset (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(F)) \subset (\gamma, \gamma')$-Cl$(F)$ and so (γ, γ')-Cl$(F) \subset (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(F)) \subset (\gamma, \gamma')$-Cl$(F)$. Thus we obtain (γ, γ')-Cl$(F) = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(F))$. Also since $F = G \cap H$ and $F \subset G \subset (\gamma, \gamma')$-Cl$(F)$, we obtain $F \subset G \cap (\gamma, \gamma')$-Cl$(F) \subset G \cap (\gamma, \gamma')$-Cl$(G \cap H) \subset G \cap [(\gamma, \gamma')$-Cl$(G \cap (\gamma, \gamma')$-Cl$(H)] = [G \cap (\gamma, \gamma')$-Cl$(G)] \cap (\gamma, \gamma')$-Cl$(H) = G \cap (\gamma, \gamma')$-Cl$(H) = G \cap H = F$. As a consequence, $F = G \cap (\gamma, \gamma')$-Cl$(F) = G \cap (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(F))$ and so F is a (γ, γ')-regular open set. $\qquad\square$

Theorem 3.19. *For a bioperation topological space $(X, \tau, \gamma, \gamma')$,*
$\alpha O(X) = PO(X) \cap \alpha A_3(X)$.

Proof. Let F be a (γ, γ')-α-open set. Put $G = F$ and $H = X \in \tau_{(\gamma, \gamma')}$. Then $F = G \cap H$, where $G \in \alpha O(X)$ and (γ, γ')-Int$((\gamma, \gamma')$-Cl$(H)) = (\gamma, \gamma')$-Int(H). Since F is (γ, γ')-preopen, $\alpha O(X) \subset PO(X) \cap \alpha A_3(X)$. Let F be a (γ, γ')-preopen set and a (γ, γ')-αA_3-set. Since F is a (γ, γ')-preopen set, $F \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(F))$. Also we have $F = G \cap H$, where G is a (γ, γ')-α-open set and (γ, γ')-Int$((\gamma, \gamma')$-Cl$(H)) = (\gamma, \gamma')$-Int(H) since F is a (γ, γ')-αA_3-set. Then $F \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(F)) = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((G) \cap H)) \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(G) \cap (\gamma, \gamma')$-Cl$(H)) \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(G)))) \cap (\gamma, \gamma')$-Cl$(H)) = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(G)) \cap (\gamma, \gamma')$-Cl$(H)) = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(((\gamma, \gamma')$-Int$(G))) \cap (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$(H)) = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(G))) \cap (\gamma, \gamma')$-Int$(H)$ and so $F \subset (\gamma, \gamma')$-Int$[((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(G))) \cap (\gamma, \gamma')$-Int$(H)] \subset (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$[(\gamma, \gamma')$-Int$(G) \cap (\gamma, \gamma')$-Int$(H)] = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(G) \cap H))) = (\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(F)))$. As a consequence, $F \in \alpha O(X)$. □

Lemma 3.20. *A subset of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is (γ, γ')-α-open if and only if it is (γ, γ')-semi-open and (γ, γ')-preopen.*

Theorem 3.21. *For a bi0peration topological space $(X, \tau, \gamma, \gamma')$, the following holds:*
$\alpha O(X) = PO(X) \cap \alpha A_2(X)$.

Proof. By Lemma 3.20, $\alpha O(X) \subset PO(X)$ since $\alpha O(X) = PO(X) \cap SO(X)$. Also since $\alpha O(X) \subset \alpha A_2(X)$, we obtain that $\alpha O(X) \subset PO(X) \cap \alpha A_2(X)$. By Theorem 3.19, we have $\alpha O(X) = PO(X) \cap \alpha A_3(X)$. Also since $\alpha A_2(X) \subset \alpha N A_3(X)$, we obtain that $PO(X) \cap \alpha A_2(X) \subset PO(X) \cap \alpha A_3(X) = \alpha O(X)$.

□

4   Decomposition of bioperation continuity

Definition 4.1. *A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is said to be (γ, γ')-(β, β')-αA_i-continuous if $f^{-1}(G) \in \alpha A_i(X)$ for every $G \in \sigma_{(\beta, \beta')}$, where $i = 1, 2, 3, 4, 5$.*

Proposition 4.2. *For a function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$, we have the following:*

1. *Every (γ, γ')-(β, β')-αA_1-continuous function is a (γ, γ')-(β, β')-αA_2-continuous function.*

2. *Every (γ, γ')-(β, β')-αA_2-continuous function is a (γ, γ')-(β, β')-αA_3-continuous function.*

3. *Every (γ, γ')-(β, β')-αA_3-continuous function is a (γ, γ')-(β, β')-αA_5-continuous function.*

4. Every (γ, γ')-(β, β')-αA_2-continuous function is a (γ, γ')-(β, β')-αA_4-continuous function.

5. Every (γ, γ')-(β, β')-αA_4-continuous function is a (γ, γ')-(β, β')-αA_5-continuous function.

Proof. The proof follows from Proposition 3.2. $\qquad \square$

The converse of the above Proposition is easily to construct using the Examples 3.3, 3.4, 3.5, 3.6.

Proposition 4.3. *For a function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$, we have the following*

1. *Every (γ, γ')-(β, β')-completely continuous function is (γ, γ')-(β, β')-αA_1-continuous.*

2. *Every (γ, γ')-(β, β')-$\mathcal{LC}$-continuous function is (γ, γ')-(β, β')-αA_2-continuous.*

3. *Every (γ, γ')-(β, β')-$\mathcal{B}$-continuous function is (γ, γ')-(β, β')-αA_3-continuous.*

4. *Every (γ, γ')-(β, β')-$\mathcal{C}$-continuous function is (γ, γ')-(β, β')-αA_5-continuous.*

Proof. It is a direct consequence of Proposition 3.12. $\qquad \square$

Theorem 4.4. *For a function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$, where $(X, \tau, \gamma, \gamma')$ is a (γ, γ')-submaximal space. Then the following hold:*

1. *(γ, γ')-(β, β')-$\mathcal{A}$-continuous if and only if (γ, γ')-(β, β')-αA_1-continuous.*

2. *(γ, γ')-(β, β')-$\mathcal{LC}$-continuous if and only if (γ, γ')-(β, β')-αA_2-continuous.*

3. *(γ, γ')-(β, β')-$\mathcal{B}$-continuous if and only if (γ, γ')-(β, β')-αA_3-continuous.*

4. *(γ, γ')-(β, β')-$\mathcal{C}$-continuous if and only if (γ, γ')-(β, β')-αA_5-continuous.*

Proof. The proof follows from Theorem 3.16. $\qquad \square$

Theorem 4.5. *A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is (γ, γ')-(β, β')-α-continuous if and only if f is (γ, γ')-(β, β')-precontinuous and (γ, γ')-(β, β')-αA_3-continuous.*

Proof. The proof follows from Theorem 3.19. $\qquad \square$

Theorem 4.6. *A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is (γ, γ')-(β, β')-α-continuous if and only if f is (γ, γ')-(β, β')-precontinuous and (γ, γ')-(β, β')-αA_2-continuous.*

Proof. The proof follows from Theorem 3.21. $\qquad \square$

Conclusions

1. The (γ, γ')-αA_i-sets, for $i \in \{1, 2, 3, 4, 5\}$, are characterized and the relationships between them and also with other well known in the literature (γ, γ')-sets are studied.

2. Under what conditions some of the studied sets are the same.

3. The (γ, γ')-(β, β')-αA_i-continuous functions, for $i \in \{1, 2, 3, 4, 5\}$, are studied and the relationships between them are studied.

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