BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(103), 2023, Pages 37–56
ISSN 1024–7696, E-ISSN 2587–4322

Existence of solutions to multi-point boundary value problem of fractional order on the half-line

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Abstract. The purpose of this paper is to establish the existence of solutions to multi-point fractional boundary value problem on an infinite interval. Using the fixed point theory, sufficient conditions are obtained that guarantee the existence of at least one solution. At the end, an example is presented to illustrate the main results

Mathematics subject classification: 34A08, 34A12, 34B10.
Keywords and phrases: Boundary value problem, Infinite interval, Fractional differential equation, Nonlocal condition, Fixed point theorem.

1 Introduction

In this paper, we will consider the boundary value problem (bvp for short)

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t, u(t), D^{\alpha-1}_{0^+}u(t), D^{\alpha-2}_{0^+}u(t)), \qquad t \in (0, +\infty), \\ I^{3-\alpha}_{0^+}u(0) = 0, \quad D^{\alpha-2}_{0^+}u(0) = 0, \quad \lim\limits_{t\to+\infty} D^{\alpha-1}_{0^+}u(t) = \sum\limits_{i=1}^{i=p} c_i u(\xi_i), \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $c_1, c_2, \ldots, c_p$ are given constants with $p \in \mathbb{N}^*$ and $0 < \xi_1 < \xi_2 < \ldots < \xi_p < +\infty$, $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given function. $D^{\alpha}_{0^+}$ refers to the standard Riemann-Liouville fractional derivative and $I^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional integral.

Fractional equations are a natural generalization of the classical integer-order differential equations. They turn out to be very adequate for modeling dynamics of many processes involving complex systems that can be found in science, engineering, aerodynamics, etc. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, electrical circuits, biology, and so on. Fractional derivatives turn out to be an excellent tool for the description of memory and hereditary properties of various materials and processes.

This is the main advantage of fractional differential equations in comparison with classical integer-order models. Further, the concept of nonlocal boundary conditions has been introduced to extend the study of classical boundary value problems. This notion is more precise for describing natural phenomena than the classical notion because additional information is taken into account.

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DOI: https://doi.org/10.56415/basm.y2023.i3.p37

Recently, several papers have studied questions of existence of solutions for some classes of bvps to fractional differential equations on finite intervals, see, e.g., [2,6,7,9,15,16] and references therein. Different methods have been employed. However, research works on the existence of multiple solutions to fractional differential equations with nonlocal boundary condition on infinite intervals are few, we refer to [3–5,10,13,14] and references therein.

By using famous Leray-Schauder Nonlinear Alternative theorem, Y. Gholami [3] obtained an unbounded solution to the following multi-point bvp in unbounded interval

$$\begin{cases} D_{0^+}^{\alpha} u(t) + a(t) f(t, u(t), u'(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) = u'(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \sum\limits_{i=1}^{m} \beta_i D_{0^+}^{\alpha-1} u(\xi_i), \end{cases}$$

where $2 < \alpha < 3,\ f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ a \in C([0, +\infty), [0, +\infty)),\ 0 < \xi_1 < \xi_2 < \ldots < \xi_m < +\infty$, $\beta_i \in \mathbb{R}$ with $\sum\limits_{i=1}^{m} \beta_i < 1$.

In [13], Shen, Zhou and Yang established the existence results of positive solutions to the bvp

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t), D_{0^+}^{\alpha-1} u(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D_{0^+}^{\alpha-1} u(+\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha \leq 3$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Gamma(\alpha) - \sum\limits_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} \neq 0$. Using the Schauder fixed point theorem, they have shown the existence of one solution with suitable growth conditions imposed on the nonlinear term.

K. Ghanbari, Y. Gholami [4] discussed the existence and multiplicity of positive solutions to an m-point nonlinear fractional bvp on an infinite interval

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda a(t) f(t, u(t)) = 0, \qquad t \in (0, +\infty), \\ u(0) + u'(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \sum\limits_{i=1}^{m-2} \beta_i u'(\xi_i), \end{cases}$$

where $2 < \alpha < 3,\ f \in C([0, +\infty) \times [0, +\infty), [0, +\infty)),\ a \in C([0, +\infty), [0, +\infty)),\ \lambda$ is a positive parameter and $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < +\infty$, $\beta_i \in [0, +\infty)$ with $0 < \sum\limits_{i=1}^{m-2} (\alpha - 1)\beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$.

Z. Bai, Y. Zhang [2] studied the multi-point bvp on bounded interval

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t)), \qquad t \in (0\,,\,1), \\ I_{0^+}^{3-\alpha} u(0) = 0, \quad D_{0^+}^{\alpha-1} u(0) = D_{0^+}^{\alpha-1} u(\eta), \quad u(1) = \sum\limits_{i=1}^{m} \alpha_i u(\eta_i), \end{cases}$$

where $2 < \alpha \leq 3$, $0 < \eta \leq 1$, $0 < \eta_1 < \eta_2 < \ldots < \eta_m < 1$, $m \geq 2$, $f : [0\,,\,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfying the Caratheodory conditions and $\sum\limits_{i=1}^{m} \alpha_i \eta_i^{\alpha-1} = \sum\limits_{i=1}^{m} \alpha_i \eta_i^{\alpha-2} = 1$, that makes the above problem at resonance. They established an existence theorem under a nonlinear growth restriction on f. Their method is based upon the coincidence degree theory of Mawhin.

Motivated by the above works and by recent studies of nonlocal boundary value problems of fractional order, we consider a more general problem of fractional differential equations of arbitrary order with nonlocal boundary conditions. Precisely, we investigate the problem (1.1).

The work presented in this paper is a continuation of previous works and is concerned with a bvp of fractional order set on the half-axis. The main difficulty in treating this class of the fractional differential equations is the possible lack of compactness due to the infinite interval. In order to overcome these difficulties, we use a special Banach space in which similar inequalities as finite interval can be established. The main tool used in this paper is Krasnosel'skii' s fixed point theorem (nonlinear alternative). Under a compactness criterion, the existence of solutions is established.

The plan of the paper is as follows. In Section 2, we outline some basic concepts of fractional calculus. In Section 3, we prove some technical lemmas which we use in the main results. Section 4 is devoted to our main existence results. In Section 5, an example of applications is supplied to illustrate our theoretical results.

2 Preliminaries

We start with some definitions and lemmas on the fractional calculus (see [8,11]). One of the basic tools of the fractional calculus is the Gamma function which extends the factorial to positive real numbers.

Definition 2.1. *For $\alpha > 0$, the Euler Gamma function is defined by*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Proposition 2.1. *Let $\alpha > 0,\ p > 0,\ q > 0$ and n be a positive integer. Then*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\Gamma(\alpha+n) = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\Gamma(\alpha).$$

In particular

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}.$$

Definition 2.2. *The fractional integral of order $\alpha > 0$ for function h is defined by*

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided the right hand side is point-wise defined on $(0, +\infty)$.

Definition 2.3. *For a given function h defined on the interval $[0, +\infty)$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$D_{0^+}^{\alpha} h(t) = \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$.

Lemma 2.1. *[8] Let $\alpha > 0$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\alpha] + 1$.

Proposition 2.2. *[11] The following composition relations hold:*
(a) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} h(t) = h(t), \quad \alpha > 0, \quad h \in L^1[0, +\infty).$
(b) $D_{0^+}^{\alpha} I_{0^+}^{\gamma} h(t) = I_{0^+}^{\gamma-\alpha} h(t), \quad \gamma > \alpha > 0, \quad h \in L^1[0, +\infty).$
(c) $I_{0^+}^{\alpha} I_{0^+}^{\gamma} h(t) = I_{0^+}^{\alpha+\gamma} h(t), \quad \alpha > 0, \quad \gamma > 0, \quad h \in L^1[0, +\infty).$
(d) $D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$, *for $\lambda > -1$, in particular for* $D_{0^+}^{\alpha} t^{\alpha-m} = 0$,
$m = 1, 2, \ldots, N$, *where N is the smallest integer greater than or equal to α.*
(e) $I_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)} t^{\alpha+\lambda}, \quad \alpha > 0, \quad \lambda > -1.$

The following result is needed to prove our main existence result. This is a nonlinear alternative for Krasnosel'skii' s fixed point theorem [1].

Theorem 2.1. *[1] Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$, $F(\overline{U})$ bounded and $F : \overline{U} \to C$ is given by $F = F_1 + F_2$, where $F_1 : \overline{U} \to E$ is continuous and completely continuous and $F_2 : \overline{U} \to E$ is a nonlinear contraction (i.e., there exists a constant $0 < l < 1$, such that $\|F_2(x) - F_2(y)\| \leqslant l\|x - y\|$, for all $x, y \in \overline{U}$). Then either*
(A1) F has a fixed point in $\overline{U}$, or
(A2) there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

3 Related Lemmas

Consider the Banach spaces X, Y defined by

$$X = \left\{ u \in C([0, +\infty), \mathbb{R}), \quad \sup_{t \geq 0} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty \right\}$$

with the norm

$$\|u\|_X = \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}}$$

and

$$Y = \Big\{ u \in X,\ D_{0^+}^{\alpha-2}u,\ D_{0^+}^{\alpha-1}u \in C([0,\,+\infty),\mathbb{R}),$$
$$\sup_{t\geq 0} \frac{|D_{0^+}^{\alpha-2}u(t)|}{1+t} < +\infty,\ \sup_{t\geq 0} |D_{0^+}^{\alpha-1}u(t)| < +\infty \Big\}$$

with the norm

$$\|u\|_Y = \max\left\{ \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}},\ \sup_{t\geq 0} \frac{|D_{0^+}^{\alpha-2}u(t)|}{1+t},\ \sup_{t\geq 0} |D_{0^+}^{\alpha-1}u(t)| \right\}.$$

Now, we list some conditions in this paper for convenience:

$(H1)$ The function $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(H2)$ There exist nonnegative functions $(1+t^{\alpha-1})\varphi(t),\ \psi(t),\ (1+t)\mu(t),\ \phi(t) \in L^1[0,\,+\infty)$ such that
$|f(t,x,y,z)| \leqslant \varphi(t)|x| + \psi(t)|y| + \mu(t)|z| + \phi(t)$ for all $x,\ y,\ z \in \mathbb{R}$ and $t \in [0,+\infty)$.

$(H3)$ $0 < \sum\limits_{i=1}^{i=p} c_i \left(1+\xi_i^{\alpha-1}\right) < \Gamma(\alpha)$.

Lemma 3.1. *Let $h \in L^1[0,+\infty)$, then the bvp*

$$\begin{cases} D_{0^+}^{\alpha}u(t) = h(t), \qquad t \in (0,+\infty), \\ I_{0^+}^{3-\alpha}u(0) = 0, \quad D_{0^+}^{\alpha-2}u(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \sum\limits_{i=1}^{i=p} c_i u(\xi_i), \end{cases} \tag{3.1}$$

has a unique solution given by

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{i=p} c_i u(\xi_i).$$

Proof. By Lemma 2.1 and from $D_{0^+}^{\alpha}u(t) = h(t)$, we have

$$u(t) = I_{0^+}^{\alpha}h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},\ \text{for some constants } c_1,\ c_2,\ c_3 \in \mathbb{R}.$$

So the solution of (3.1) can be written as

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Moreover

$$I_{0^+}^{3-\alpha}u(t) \quad = \quad I_{0^+}^{3}h(t) + c_1\frac{\Gamma(\alpha)}{2}t^2 + c_2\Gamma(\alpha-1)t + c_3\Gamma(\alpha-2)$$

$$= \frac{1}{2}\int_0^t (t-s)^2 h(s)ds + c_1\frac{\Gamma(\alpha)}{2}t^2 + c_2\Gamma(\alpha-1)t + c_3\Gamma(\alpha-2),$$

together with $I_{0^+}^{3-\alpha}u(0)=0$, $c_3=0$.
On the other hand, we have

$$\begin{aligned} D_{0^+}^{\alpha-2}u(t) &= D_{0^+}^{\alpha-2}I_{0^+}^{\alpha}h(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \\ &= I_{0^+}^{2}h(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \\ &= \int_0^t (t-s)h(s)ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1). \end{aligned}$$

From $D_{0^+}^{\alpha-2}u(0)=0$ we known that $c_2=0$.
Furthermore

$$D_{0^+}^{\alpha-1}u(t) = D_{0^+}^{\alpha-1}I_{0^+}^{\alpha}h(t) + c_1\Gamma(\alpha)$$

i.e.,

$$\begin{aligned} D_{0^+}^{\alpha-2}u(t) &= I_{0^+}^{1}h(t) + c_1\Gamma(\alpha) \\ &= \int_0^t h(s)ds + c_1\Gamma(\alpha). \end{aligned}$$

From $\lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \sum\limits_{i=1}^{i=p} c_i u(\xi_i)$, we get $c_1 = \frac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{i=p} c_i u(\xi_i) - \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds$.
Therefore, the unique solution of fractional bvp (3.1) is

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{i=p} c_i u(\xi_i).$$

□

Now, define the following operators $T_1,\ T_2,\ T$ on Y by

$$\begin{aligned} (T_1u)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \\ &\quad -\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds, \\ (T_2u)(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{i=p} c_i u(\xi_i), \\ (Tu)(t) &= (T_1u)(t) + (T_2u)(t). \end{aligned}$$

We will prove the existence of a fixed point of T. For this we verify that the operator T satisfies all conditions of Theorem 2.1.
Since the Arzela-Ascoli theorem fails to work in the space Y, we need a modified compactness criterion to prove T_1 is compact.

Lemma 3.2. *[12] Let* $Z = \{u \in Y,\ \|u\|_Y < l\}$ *be such that* $l > 0$, $Z_1 = \{\frac{u(t)}{1+t^{\alpha-1}},\ u \in Z\}$, $Z_2 = \{D_{0^+}^{\alpha-1}u(t),\ u \in Z\}$ *and* $Z_3 = \left\{\frac{D_{0^+}^{\alpha-2}u(t)}{1+t},\ u \in Z\right\}$. *Then* Z *is relatively compact on* Y *if* Z_1, Z_2 *and* Z_3 *are equicontinuous on any compact interval of* $[0, +\infty)$ *and are equiconvergent at infinity.*

Definition 3.1. Z_1, Z_2 *and* Z_3 *are called equiconvergent at infinity if and only if for all* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon,\ \left|D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)\right| < \varepsilon \text{ and}$$

$$\left|\frac{D_{0^+}^{\alpha-2}u(t_1)}{1+t_1} - \frac{D_{0^+}^{\alpha-2}u(t_2)}{1+t_2}\right| < \varepsilon,$$

for any $t_1,\ t_2 > \delta$ *and* $u \in Z$.

Let $\Omega_r = \{u \in Y, \quad \|u\|_Y < r\},\ (r > 0)$ be the open ball of radius r in Y.

Lemma 3.3. *If* $(H1) - (H4)$ *hold, then* $T(\overline{\Omega}_r)$ *is a bounded set.*

Proof. We have

$$\begin{aligned}
\sup_{t\geq 0}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \leqslant\ & \frac{1}{\Gamma(\alpha)}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\
&+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\left|\sum_{i=1}^{i=p}c_i u(\xi_i)\right|\right)\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right.\\
&\left.+r\sum_{i=1}^{i=p}c_i\left(1+\xi_i^{\alpha-1}\right)\right).
\end{aligned}$$

In addition

$$\begin{aligned}
\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \leqslant\ & 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\left|\sum_{i=1}^{i=p}c_i u(\xi_i)\right|\\
\leqslant\ & 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\\
+\ & \frac{r\sum_{i=1}^{i=p}c_i\left(1+\xi_i^{\alpha-1}\right)}{\Gamma(\alpha)}.
\end{aligned}$$

Also

$$\begin{aligned}\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\left|\sum_{i=1}^{i=p}c_iu(\xi_i)\right| \\ &\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds \\ &+ \frac{r\sum\limits_{i=1}^{i=p}c_i\left(1+\xi_i^{\alpha-1}\right)}{\Gamma(\alpha)}.\end{aligned}$$

So

$$\|Tu\|_Y<+\infty,\ \text{for } u\in\overline{\Omega}_r.$$

□

Lemma 3.4. *If* $(H1)$, $(H2)$ *hold, then* $T_1:\overline{\Omega}_r\to Y$ *is completely continuous.*

Proof. We firstly verify that the set $T_1(\overline{\Omega}_r)$ is bounded.
By definition of the operator T_1 we have, for any $u\in\overline{\Omega}_r$,

$$\begin{aligned}\left|\frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| &\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\right. \\ &\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right) \\ &\leqslant \frac{2}{\Gamma(\alpha)}\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.\end{aligned}$$

In addition

$$\begin{aligned}\left|D_{0^+}^{\alpha-1}T_1u(t)\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\ &\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.\end{aligned}$$

Also

$$\begin{aligned}\left|\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\ &\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.\end{aligned}$$

Hence

$$\|T_1u\|_Y\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds,\ \text{for } u\in\overline{\Omega}_r.$$

Now, we divide the proof into three steps.

Claim 1. *We show that T_1 is continuous.*
Let $u_n \to u$ as $n \to +\infty$ in $\overline{\Omega}_r$, we have

$$\begin{aligned}
\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \leqslant \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} &|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} &|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s))|ds \\
+\frac{2}{\Gamma(\alpha)}\int_0^{+\infty} &|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} &(\|u_n\|_Y((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds \\
+\frac{2}{\Gamma(\alpha)}\int_0^{+\infty} &(\|u\|_Y((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds \\
\leqslant \frac{4}{\Gamma(\alpha)}\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right) &< +\infty.
\end{aligned}$$

Using the continuity of f, we obtain that

$$|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))| \to 0, \quad \text{as } n \to +\infty,$$

which implies

$$\|T_1u_n - T_1u\|_X = \sup_{t\geq 0}\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \to 0,$$

uniformly as $n \to +\infty$.
Moreover

$$\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_1u_n(t) - D_{0^+}^{\alpha-1}T_1u(t)\right| \leqslant 2\int_0^{+\infty} &|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant 4\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right) &< +\infty.
\end{aligned}$$

Also

$$\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_1u_n(t)}{1+t} - \frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right| \leqslant 2\int_0^{+\infty} &|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant 4\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right) &< +\infty.
\end{aligned}$$

Using again the continuity of f, we get

$$\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}T_1u_n(t)-D_{0^+}^{\alpha-1}T_1u(t)\right|\to 0,\ \sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}T_1u_n(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right|\to 0,$$

uniformly as $n\to+\infty$. We conclude

$$\|T_1u_n-T_1u\|_Y\to 0,\ \text{uniformly as } n\to+\infty,\ \text{as claimed.}$$

Claim 2. *We show that $T_1:\overline{\Omega}_r\to X$ is relatively compact.*
According to the above $T_1(\overline{\Omega}_r)$ is uniformly bounded. We show that functions from $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$, functions from $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equicontinuous on any compact interval of $[0,+\infty)$.
Let $I\subset[0,+\infty)$ be a compact interval. Then, for any $t_1,\ t_2\in I$ such that $t_1<t_2$, and for $u\in\overline{\Omega}_r$, we have

$$\begin{aligned}
\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\quad-\int_0^{+\infty}\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\quad-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\quad\left.+\int_0^{+\infty}\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\leqslant\frac{1}{\Gamma(\alpha)}\left(\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\right.\\
&\quad\left.-\int_0^{t_2}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\quad+\left|\int_0^{t_2}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\quad\left.-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\quad\left.+\int_0^{+\infty}\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\
&\leqslant\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\
&\quad+\int_0^{t_2}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\quad\left.+\int_0^{+\infty}\frac{\left|t_1^{\alpha-1}-t_2^{\alpha-1}\right|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)
\end{aligned}$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\Big(\int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds$$
$$+\int_0^{t_2}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|$$
$$(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds$$
$$+\int_0^{+\infty}\frac{\left|t_1^{\alpha-1}-t_2^{\alpha-1}\right|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}$$
$$(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\Big).$$

The last term converges to 0 *uniformly as* $|t_1-t_2|\to 0$.
Moreover

$$\left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right| = \left|\int_0^{t_1} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.$$
$$\left.-\int_0^{t_2} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right|$$
$$\leqslant \int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds,$$

which converges to 0 *uniformly as* $|t_1-t_2|\to 0$. *Also*

$$\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right| = \left|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.$$
$$-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds$$
$$\left.+\frac{t_2-t_1}{(1+t_1)(1+t_2)}\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|$$
$$\leqslant \left|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.$$
$$\left.-\int_0^{t_2}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|$$
$$+\left|\int_0^{t_2}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.$$
$$\left.-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|$$
$$+\frac{|t_2-t_1|}{(1+t_1)(1+t_2)}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds$$
$$\leqslant \int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds$$
$$+\frac{2|t_1-t_2|}{(1+t_1)(1+t_2)}\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))$$

$$+\phi(s))ds,$$

which converges to 0 *uniformly as* $|t_1 - t_2| \to 0$.
Then, for any $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}} - \frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon,\ \left|D_{0^+}^{\alpha-1}T_1u(t_1) - D_{0^+}^{\alpha-1}T_1u(t_2)\right| < \varepsilon$$

and

$$\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1} - \frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right| < \varepsilon,$$

for all $u \in \overline{\Omega}_r$, *if* $|t_1 - t_2| < \delta$, $t_1,\ t_2 \in I$.
Showing that, the functions belonging to $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$ *and the functions belonging to* $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ *and to* $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ *are locally equicontinuous on* $[0,+\infty)$.

Claim 3. *We show that the functions from* $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$, $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ *and from* $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ *are equiconvergent at infinity.*
For any $u \in \overline{\Omega}_r$, *we have*

$$\int_0^{+\infty} \left|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))\right| ds < +\infty.$$

Considering condition $(H2)$, *for given* $\varepsilon > 0$, *there exists a constant* $L > 0$ *such that*

$$\int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds < \varepsilon.$$

On the other hand, since $\lim\limits_{t\to+\infty} \frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}} = 1$ *and* $\lim\limits_{t\to+\infty} \frac{t-L}{1+t} = 1$, *there exists a constant* $\delta > L > 0$ *such that for any* t_1, $t_2 \geq \delta$ *and* $0 \leq s \leq L$, *we have*

$$\begin{aligned}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| &= \left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - 1 + 1 - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| \\ &\leqslant \left|1 - \frac{(t_1-L)^{\alpha-1}}{1+t_1^{\alpha-1}}\right| + \left|1 - \frac{(t_2-L)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\left|\frac{t_1-s}{1+t_1} - \frac{t_2-s}{1+t_2}\right| &= \left|\frac{t_1-s}{1+t_1} - 1 + 1 - \frac{t_2-s}{1+t_2}\right| \\ &\leqslant \left|1 - \frac{t_1-L}{1+t_1}\right| + \left|1 - \frac{t_2-L}{1+t_2}\right| < \varepsilon.\end{aligned}$$

Thus, for any t_1, $t_2 \geq \delta > L > 0$, we get

$$\begin{aligned}
\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}} - \frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right. \\
&\quad - \int_0^{+\infty} \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \\
&\quad - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds \\
&\quad \left. + \int_0^{+\infty} \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^{L}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\right. \\
&\quad |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\quad + \int_L^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\quad + \int_L^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\quad \left. +2\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right) \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^{L}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\right. \\
&\quad |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\quad +2\int_0^{L} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\quad \left. +4\int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right) \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(\sup_{s\in[0,L],\,u\in\overline{\Omega}_r} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L\varepsilon\right. \\
&\quad \left. +2\sup_{s\in[0,L],\,u\in\overline{\Omega}_r} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L + 4\varepsilon\right).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_1u(t_1) - D_{0^+}^{\alpha-1}T_1u(t_2)\right| &= \left|\int_{t_1}^{t_2} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right| \\
&\leqslant \int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right| &= \left|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\quad -\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\quad -\frac{t_1}{1+t_1}\int_0^{+\infty}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\quad \left.+\frac{t_2}{1+t_2}\int_0^{+\infty}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\leqslant \int_0^{L}\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\\
&\quad +\int_L^{t_1}\frac{t_1-s}{1+t_1}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\quad +\int_L^{t_2}\frac{t_2-s}{1+t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\quad +2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\leqslant \int_0^{L}\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\quad +2\int_0^{L}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\quad +4\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\leqslant \sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L\varepsilon\\
&\quad +2\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L+4\varepsilon.
\end{aligned}$$

Which yields that the functions from $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$, $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ *and from* $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ *are equiconvergent at infinity.*

According to Lemma 3.2, it follows that $T_1(\overline{\Omega}_r)$ is relatively compact, ending the proof of the Lemma. □

Lemma 3.5. *If* $(H3)$ *holds, then* $T_2:\overline{\Omega}_r\to Y$ *is a contraction mapping.*

Proof. We have

$$\left|\frac{T_2u(t)}{1+t^{\alpha-1}}-\frac{T_2v(t)}{1+t^{\alpha-1}}\right| \leqslant \frac{1}{\Gamma(\alpha)}\left|\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\right|\left|\sum_{i=1}^{i=p}c_iu(\xi_i)-\sum_{i=1}^{i=p}c_iv(\xi_i)\right|$$

$$\begin{aligned}
&\leqslant \frac{1}{\Gamma(\alpha)}\left|\sum_{i=1}^{i=p} c_i u(\xi_i) - \sum_{i=1}^{i=p} c_i v(\xi_i)\right| \\
&\leqslant \frac{\sum\limits_{i=1}^{i=p} c_i\left(1+\xi_i^{\alpha-1}\right)}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}$$

Moreover

$$\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_2u(t) - D_{0^+}^{\alpha-1}T_2v(t)\right| &= \left|\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\right|\left|\sum_{i=1}^{i=p} c_i u(\xi_i) - \sum_{i=1}^{i=p} c_i v(\xi_i)\right| \\
&\leqslant \frac{\sum\limits_{i=1}^{i=p} c_i\left(1+\xi_i^{\alpha-1}\right)}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}$$

Also

$$\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_2u(t)}{1+t} - \frac{D_{0^+}^{\alpha-2}T_2v(t)}{1+t}\right| &= \left|\frac{t}{1+t}\left(\sum_{i=1}^{i=p} c_i u(\xi_i) - \sum_{i=1}^{i=p} c_i v(\xi_i)\right)\right| \\
&\leqslant \frac{\sum\limits_{i=1}^{i=p} c_i\left(1+\xi_i^{\alpha-1}\right)}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}$$

We conclude

$$\|T_2u - T_2v\|_Y \leqslant \frac{\sum\limits_{i=1}^{i=p} c_i\left(1+\xi_i^{\alpha-1}\right)}{\Gamma(\alpha)}\|u-v\|_Y.$$

From $(H3)$, we infer that T_2 is a contraction mapping. □

4 Main results

Theorem 4.1. *Further to assumptions $(H1)-(H3)$, assume that*
$(H4)$ *there exists $\rho > 0$ such that*

$$\frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\right)ds + \frac{\rho\sum\limits_{i=1}^{i=p} c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}} > 1.$$

Then, the problem (1.1) has at least one solution.

Proof. Consider the parameterized bvp

$$\begin{cases}
D_{0^+}^{\alpha}u(t) = \lambda f(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha-2}u(t)), & t \in (0,+\infty), \\
I_{0^+}^{3-\alpha}u(0) = 0, \quad D_{0^+}^{\alpha-2}u(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \sum\limits_{i=1}^{i=p} c_i u(\xi_i),
\end{cases} \tag{4.1}$$

for $\lambda \in (0,1)$.
To solve problem (4.1) is equivalent to finding the fixed point of equation $u = \lambda T u$.
Let

$$\Omega_\rho = \{u \in Y, \quad \|u\|_Y < \rho\}.$$

From Lemma 3.3, the set $T(\overline{\Omega}_\rho)$ is bounded and by Lemma 3.4, the operator $T_1 : \overline{\Omega}_\rho \to Y$ is completely continuous, while Lemma 3.5 implies that the operator $T_2 : \overline{\Omega}_\rho \to Y$ is contractive. So it remains to prove that $u \neq \lambda T u$ for $u \in \partial\Omega_\rho$ and $\lambda \in (0,1)$.
Arguing by contradiction, if there exists $u \in \partial\Omega_\rho$ with $u = \lambda T u$, then for $\lambda \in (0,1)$ we have

$$\begin{aligned}
\sup_{t\geq 0}\left|\frac{u(t)}{1+t^{\alpha-1}}\right| &= \sup_{t\geq 0}\left|\frac{(\lambda T u)(t)}{1+t^{\alpha-1}}\right| \\
&\leqslant \sup_{t\geq 0}\left|\frac{(T u)(t)}{1+t^{\alpha-1}}\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right. \\
&\quad + \frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\quad \left. + \frac{t^{\alpha-1}}{1+t^{\alpha-1}}\left|\sum_{i=1}^{i=p}c_i u(\xi_i)\right|\right) \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right. \\
&\quad \left. + \left|\sum_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})\frac{1}{1+\xi_i^{\alpha-1}}u(\xi_i)\right|\right) \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds\right. \\
&\quad \left. + \frac{\rho\sum_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}\right).
\end{aligned}$$

In addition

$$\begin{aligned}
\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}u(t)\right| &= \sup_{t\geq 0}\left|\lambda D_{0^+}^{\alpha-1}Tu(t)\right| \\
&\leqslant \sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \\
&\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds + \left|\sum_{i=1}^{i=p}c_i u(\xi_i)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds\\
&+\frac{\rho\sum\limits_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}u(t)}{1+t}\right| &= \sup_{t\geq 0}\left|\lambda\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right|\\
&\leqslant \sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right|\\
&\leqslant \int_0^t\frac{t-s}{1+t}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&+\frac{t}{1+t}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\left|\sum_{i=1}^{i=p}c_iu(\xi_i)\right|\\
&\leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds\\
&+\frac{\rho\sum\limits_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}.
\end{aligned}
$$

So

$$
\begin{aligned}
\|u\|_Y \leqslant 2\int_0^{+\infty}&\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds\\
&+\frac{\rho\sum\limits_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}
\end{aligned}
$$

and thus

$$
\rho\leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{\rho\sum\limits_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}.
$$

This implies that

$$
\frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\right)ds+\frac{\rho\sum_{i=1}^{i=p}c_i(1+\xi_i^{\alpha-1})}{\Gamma(\alpha)}}\leqslant 1,
$$

contradicting condition $(H4)$. With Theorem 2.1 we conclude that bvp (1.1) has at least one solution. $\square$

5 Example

Example 5.1. *Consider the bvp on infinite interval*

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t)=\frac{e^{-30t}}{1+\sqrt{t^3}}u(t)+\frac{D_{0^+}^{\frac{3}{2}}u(t)}{(60+t)^2}+\frac{D_{0^+}^{\frac{1}{2}}u(t)}{60(1+t)^3}+e^{-t}, \quad t\in(0,+\infty),\\ I_{0^+}^{\frac{1}{2}}u(0)=0, \quad D_{0^+}^{\frac{1}{2}}u(0)=0, \quad \lim\limits_{t\to+\infty}D_{0^+}^{\frac{3}{2}}u(t)=\frac{1}{10}u(1)+\frac{1}{20}u(4)+\frac{1}{60}u(9).\end{cases} \tag{5.1}$$

In this case, $\alpha=\frac{5}{2},\ \Gamma(\frac{5}{2})\approx 1.329340388,\ c_1=\frac{1}{10},\ c_2=\frac{1}{20},\ c_3=\frac{1}{60},\ \xi_1=1,\ \xi_2=4,\ \xi_3=9.$

We will apply Theorem 4.1 to show that the problem (5.1) has at least one solution. Let

$$f(t,x,y,z)=\frac{e^{-30t}}{1+\sqrt{t^3}}x+\frac{y}{(60+t)^2}+\frac{z}{60(1+t)^3}+e^{-t}.$$

Choose

$$\rho>\frac{120\Gamma(\frac{5}{2})}{52\Gamma(\frac{5}{2})-67}.$$

Then
$(H1)$ $f:[0,+\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\times\mathbb{R}$ *is continuous.*
$(H2)$ $|f(t,x,y,z)|\leqslant\frac{e^{-30t}}{1+\sqrt{t^3}}|x|+\frac{1}{(60+t)^2}|y|+\frac{1}{60(1+t)^3}|z|+e^{-t}$. *So we may take*

$$\varphi(t)=\frac{e^{-30t}}{1+\sqrt{t^3}},\ \psi(t)=\frac{1}{(60+t)^2},\ \mu(t)=\frac{1}{60(1+t)^3},\ \phi(t)=e^{-t}$$

and note that $(1+\sqrt{t^3})\varphi(t),\ \psi(t),\ (1+t)\mu(t),\ \phi(t)\in L^1[0,+\infty)$ *such that*

$$\begin{aligned}\int_0^{+\infty}\left(1+s^{\frac{3}{2}}\right)\varphi(s)ds &= \frac{1}{30},\ \int_0^{+\infty}\psi(s)ds=\frac{1}{60},\\ \int_0^{+\infty}(1+s)\mu(s)ds &= \frac{1}{60},\ \int_0^{+\infty}\phi(s)ds=1.\end{aligned}$$

$(H3)$ *We have* $\gamma=c_1(1+\sqrt{\xi_1^3})+c_2(1+\sqrt{\xi_2^3})+c_3(1+\sqrt{\xi_3^3})=\frac{67}{60}$ *verify* $0<\gamma<\Gamma(\frac{5}{2})$ *with* $\left|\sum\limits_{i=1}^{i=3}c_iu(\xi_i)-\sum\limits_{i=1}^{i=3}c_iv(\xi_i)\right|\leqslant\frac{\gamma}{\Gamma(\frac{5}{2})}\|u-v\|_Y$ *for all* $u,\ v\in Y.$
$(H4)$

$$\frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\right)ds+\frac{\gamma\rho}{\Gamma(\alpha)}} = \frac{\rho}{2+\frac{8\Gamma(\frac{5}{2})+67}{60\Gamma(\frac{5}{2})}\rho}.$$

Thus,

$$\frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\right)ds+\frac{\gamma\rho}{\Gamma(\alpha)}}=$$

$$= \frac{60\Gamma(\frac{5}{2})}{\frac{120\Gamma(\frac{5}{2})}{\rho} + 8\Gamma(\frac{5}{2}) + 67} > 1.$$

Hence, all conditions of Theorem 4.1 are satisfied, we deduce that the bvp (5.1) has at least one solution.

Acknowledgement. The author is grateful to the referee for his/her valuable comments which led to the improvement of this paper.

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Received June 5, 2020